Year 12 Mathematics Extension 2

Topic: The Nature of Proof (5 of 5)
Review Questions: Mixed one-off exam questions
Term 3, 2026

How to use this set

These are real past-paper questions on the nature of proof that do not fit any one recurring pattern: genuine one-offs. They are here so you also meet the rarer, less predictable exam questions. Work through them once you are solid on the patterns in the main review docs. Every question you get wrong or struggle with goes in your **Mistakes Log**.

Mixed one-off exam questions

Q1 2022 HSC Q2 · **1 mark**

The following proof aims to establish that $-4 = 0$.

$$\begin{aligned} \text{Let} \quad a &= -4 \\ a^2 = 16 \text{ and } 4a + 4 &= -12 \quad \text{Line 1} \\ a^2 + 4a + 4 &= 4 \quad \text{Line 2} \\ (a+2)^2 &= 2^2 \quad \text{Line 3} \\ a + 2 &= 2 \quad \text{Line 4} \\ a &= 0 \end{aligned}$$

At which line is the implication incorrect?

(A) Line 1

(B) Line 2

(C) Line 3

(D) Line 4

Q2 2024 HSC Q14d · **2 marks**

The following argument attempts to prove that $0 = 1$.

We evaluate $\int \frac{1}{x}\,dx$ using the method of integration by parts.

$$\int \frac{1}{x}\,dx = \int \frac{1}{x} \cdot 1\,dx = \frac{1}{x} \cdot x - \int x \cdot \left(-\frac{1}{x^2}\right) dx = 1 + \int \frac{1}{x}\,dx.$$

So we have

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

We may now subtract $\int \frac{1}{x}\,dx$ from both sides to show that $0 = 1$.

Explain what is wrong with this argument.

Q3 2024 HSC Q12d · **2 marks**

Explain why there is no integer n such that

$$(n+1)^{41} - 79n^{40} = 2.$$

Q4 2025 Fort St Trial Q13b · **2 marks**

Explain why there is no integer p such that $p^{11} - (p+2)^7 = 2025$.

Q5 2025 Cheltenham Girls Trial Q16c(i) · **2 marks**

Consider the following definite integral.

$$I = \int_2^3 \tan^{-1}\left(\frac{1}{x^2 - 5x + 7}\right) dx$$

Show that

$$\tan^{-1}\left(\frac{1}{x^2 - 5x + 7}\right) = \tan^{-1}(x-2) - \tan^{-1}(x-3).$$

Q6 2025 Caringbah Trial Q16b · **2 marks**

Let $P(x)$ be a monic polynomial of degree n with rational coefficients and zeroes $\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n$.

Show that

$$P'(x) = P(x)\left(\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \frac{1}{x-\alpha_3} + \cdots + \frac{1}{x-\alpha_n}\right).$$

Q7 2025 North Sydney Boys Trial Q16c · **3 marks**

Coprime integers are integers which share no common factors other than 1.

(i) If a and b are coprime and $ax = by$, show that a is a factor of y, for integers a, b, x and y.

(ii) Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

be a polynomial of degree n with integer coefficients.

Let the rational roots of $P(x)$ be expressed in the form $\frac{p}{q}$ such that p and q are coprime integers.

Prove that q is a factor of a_n.

(If a and b are coprime you may assume without proof that a^m and b^n are also coprime for positive integers m and n.)

Q8 2025 Trinity Grammar Trial Q15c · **3 marks**

Double factorials are defined as follows:

$$n!! = n(n-2)(n-4)\cdots 6\times 4\times 2, \qquad \text{where } n \text{ is an even positive integer}$$

(e.g. $10!! = 10\times 8\times 6\times 4\times 2$), and

$$n!! = n(n-2)(n-4)\cdots 5\times 3\times 1, \qquad \text{where } n \text{ is an odd positive integer}$$

(e.g. $11!! = 11\times 9\times 7\times 5\times 3\times 1$).

(i) Show that $(2k)!! = k!\,2^k$ for even integers $n = 2k$.

(ii) Hence, prove that $(2k+1)!! = \dfrac{(2k+1)!}{k!\,2^k}$ for odd integers $n = 2k+1$.

Q9 2021 HSC Q15b · **3 marks**

For integers $n \geq 1$, the triangular numbers t_n are defined by $t_n = \dfrac{n(n+1)}{2}$, giving $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$ and so on.

For integers $n \geq 1$, the hexagonal numbers h_n are defined by $h_n = 2n^2 - n$, giving $h_1 = 1$, $h_2 = 6$, $h_3 = 15$, $h_4 = 28$ and so on.

(i) Show that the triangular numbers t_1, t_3, t_5, and so on, are also hexagonal numbers.

(ii) Show that the triangular numbers t_2, t_4, t_6, and so on, are not hexagonal numbers.

Q10 2025 Trinity Grammar Trial Q15a · **3 marks**

For positive real numbers p and q with $p \neq 2q$, explain why

$$\frac{1}{p-2q} \neq \frac{1}{p} - \frac{1}{2q}.$$

Q11 2025 Trinity Grammar Trial Q16b(i) · **3 marks**

Let

$$f(x) = \frac{1+x-x^2}{1-x+x^2}, \qquad \text{where } 0 < x < 1.$$

Prove that

$$1 < f(x) \leq \tfrac{5}{3}.$$

Q12 2022 HSC Q14b · **3 marks**

Let

$$J_n = \int_0^1 x^n e^{-x}\, dx,$$

where n is a non-negative integer. You are given that $J_0 = 1 - \dfrac{1}{e}$ and, for $n \geq 1$,

$$J_n = n\, J_{n-1} - \frac{1}{e}. \qquad \text{(from part (iii))}$$

You are also given that $J_n \leq \dfrac{1}{n+1}$.

(i) Using the two given results $J_0 = 1 - \dfrac{1}{e}$ and $J_n = n\, J_{n-1} - \dfrac{1}{e}$, show by mathematical induction, or otherwise, that for all $n \geq 0$,

$$J_n = n! - \frac{n!}{e} \sum_{r=0}^{n} \frac{1}{r!}.$$

(ii) Using the result $J_n \leq \dfrac{1}{n+1}$ together with part (i), prove that

$$e = \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!}.$$

Q13 2025 James Ruse Trial Q13b · **4 marks**

(i) Prove that if a and b have no common factors other than 1, then a^2 and b also have no common factors other than 1.

(ii) Let x be a positive rational number different from 1. Using part (i), or otherwise, prove that $x + \dfrac{1}{x}$ can never be an integer.

Q14 2025 Baulkham Hills Trial Q16b · **6 marks**

Let $S = \left(\sqrt{50} + 7\right)^{2000} + \left(\sqrt{50} - 7\right)^{2000}$.

(i) Prove that S is an even integer.

(ii) Without the use of a calculator, show that $0 < \sqrt{50} - 7 < 0.1$. (You may use the fact that $71^2 = 5041$.)

(iii) Hence, or otherwise, prove that the first two thousand digits after the decimal point in the decimal expansion of $\left(\sqrt{50} + 7\right)^{2000}$ are all the digit 9.