Year 12 Mathematics Extension 2

Topic: The Nature of Proof (4 of 5)

Review Questions: Further proof by mathematical induction

Term 3, 2026

How to use this review set

These are real past-paper questions, grouped by question pattern. The grey label on each question shows where it is from and its marks.

- The questions are grouped by the pattern that keeps coming up in exams, a few of each in a row, running from the plainer patterns into the harder ones near the end. Work through them and you will meet every type an exam can throw at you.
- This set **is your diagnostic**. Working through it shows you exactly which patterns are solid and which need work. Attempt every question properly in your own book, and go back to your notes to help you work through tough questions if needed.
- Every question you get wrong, need help with or struggle with goes in your **Mistakes Log**, which you should be working on daily to get you exam-ready.

Contents

Further proof by mathematical induction

1. Proof by induction of identities and general results
2. Induction on recurrence relations and closed forms
3. Proving inequalities by induction

Further proof by mathematical induction

1. Proof by induction of identities and general results

Q1 2025 Cheltenham Girls Trial Q13b · 3 marks

Prove by mathematical induction that $(1+\sqrt{2})^n = p_n + q_n\sqrt{2}$, where p_n and q_n are unique positive integers, for n an integer, $n \geq 2$.

Q2 2025 HSC Q12b · 3 marks

Given the function $y = xe^{2x}$, use mathematical induction to prove that

$$\frac{d^n y}{dx^n} = \left(2^n x + n\,2^{n-1}\right) e^{2x}$$

for all positive integers n, where $\dfrac{d^n y}{dx^n}$ is the nth derivative of y and $\dfrac{d}{dx}\dfrac{d^n y}{dx^n} = \dfrac{d^{n+1} y}{dx^{n+1}}$.

Q3 2025 Trinity Grammar Trial Q13e · 3 marks

Let $z_2 = 1 + i$ and, for $n > 2$, let $z_n = z_{n-1}\left(1 + \dfrac{i}{|z_{n-1}|}\right)$.

Use mathematical induction to prove that $|z_n| = \sqrt{n}$ for all integers $n \geq 2$.

Q4 2025 Penrith Trial Q14d · 3 marks

It is given that, for $k \geq 0$,

$$\int_0^{\pi/2} \sin^{2k}\theta \cos 2\theta \, d\theta = -\frac{k}{k+1}\int_0^{\pi/2} \sin^{2k}\theta \, d\theta.$$

Hence prove by mathematical induction that

$$\int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{(2n)!}{(n!)^2} \times \frac{\pi}{2^{2n+1}}$$

for all integers $n \geq 0$.

Q5 2025 Abbotsleigh Trial Q15b · 4 marks

Let $S_n = \displaystyle\sum_{p=1}^{n} \tan^{-1}\left(\frac{1}{2p^2}\right)$.

Prove by induction that $\tan(S_n) = \dfrac{n}{n+1}$.

Q6 2025 Newington Trial Q15c · **5 marks**

Consider the sequence of propositions P_n, $n = 1, 2, 3, \ldots$ defined by

$$P_n : \ 14^n \text{ can be expressed as the sum of the squares of three different positive integers.}$$

(i) Show that P_1 and P_2 are true.

(ii) Use mathematical induction to prove that P_n is true for all positive integers $n \geq 1$.

2. Induction on recurrence relations and closed forms

Q7 2025 North Sydney Girls Trial Q11e · **3 marks**

Consider the sequence u_n where $u_1 = 0$, $u_{n+1} = 2n - u_n$ for $n = 1, 2, \ldots$. Use mathematical induction to show that

$$2u_n = 2n - 1 + (-1)^n \qquad \text{for } n = 1, 2, \ldots.$$

Q8 2025 Hills Grammar Trial Q13c · **3 marks**

Given $a_{n+1} = \sqrt{2a_n}$ and $a_1 = 1$, prove by mathematical induction that for all integers $n \geq 1$,

$$a_n = 2^{\left(1-2^{(1-n)}\right)}.$$

Q9 2025 St George Girls Trial Q14b · **3 marks**

Given a series defined by $T_0 = 0$, and $T_n = 2T_{n-1} + 1$ for $n \geq 1$, use mathematical induction to prove that

$$T_n = 2^n - 1 \qquad \text{for all positive integers } n.$$

Q10 2025 Barker Trial Q13a · **3 marks**

Given that $u_1 = 1$ and $u_{n+1} = 2u_n + (-1)^{n+1}$, prove that

$$u_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right), \qquad \text{where } n \in \mathbb{Z}^+,$$

by mathematical induction.

Q11 2022 HSC Q13b · **4 marks**

The numbers a_n, for integers $n \geq 1$, are defined as

$$a_1 = \sqrt{2}, \qquad a_2 = \sqrt{2+\sqrt{2}}, \qquad a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \text{ and so on.}$$

These numbers satisfy the relation $a_{n+1}^2 = 2 + a_n$, for $n \geq 1$. (Do NOT prove this.)

Use mathematical induction to prove that $a_n = 2\cos\dfrac{\pi}{2^{n+1}}$, for all integers $n \geq 1$.

Q12 2025 Sydney Grammar Trial Q14c · **5 marks**

The Fermat numbers are a sequence of (not necessarily prime) integers which satisfy the equation

$$F_n = 2^{2^n} + 1 \qquad \text{for } n \geq 0.$$

(i) Use induction to prove that $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$ for $n \geq 1$.

(ii) Hence prove using contradiction that no two Fermat numbers share a common factor other than 1.

3. Proving inequalities by induction

Q13 2025 North Sydney Girls Trial Q14a · **3 marks**

Prove by mathematical induction that $n^3 - 1 > n^2 + n$, for integers $n \geq 2$.

Q14 2021 HSC Q12d · **3 marks**

Prove by mathematical induction that $n! > 2^n$, for integers $n \geq 9$.

Q15 2025 St George Girls Trial Q12c · **3 marks**

Use mathematical induction to prove that $n^2 \geq 2n + 1$, for all integers $n \geq 3$.

Q16 2025 Kinross Wolaroi Trial Q15b · **3 marks**

Use the principle of mathematical induction to prove that

$$(1+x)^n > 1 + nx \qquad \text{for } n > 1 \text{ and } x > 0.$$

Q17 2024 HSC Q14b · **3 marks**

Use mathematical induction to prove that $\binom{2n}{n} < 2^{2n-2}$, for all integers $n \geq 5$.

Q18 2023 HSC Q13b · **4 marks**

(i) Show that $k^2 - 2k - 3 \geq 0$ for $k \geq 3$.

(ii) Hence, or otherwise, use mathematical induction to prove that

$$2^n \geq n^2 - 2, \qquad \text{for all integers } n \geq 3.$$

Q19 2025 Ascham Trial Q16b · **7 marks**

(i) Show that for $k \geq 0$, $2k + 3 > 2\sqrt{(k+1)(k+2)}$.

(ii) Hence show that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right).$$

(iii) Is the statement $\displaystyle\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$ true for all positive integers N? Justify your answer.

Q20 2020 HSC Q14(c) · **4 marks**

Prove by mathematical induction that, for $n \geq 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{n-1}{n}.$$