Year 12 Mathematics Extension 2

Topic: The Nature of Proof (3 of 5)
Review Questions: Proof of inequalities
Term 3, 2026

How to use this review set

These are real past-paper questions, grouped by question pattern. The grey label on each question shows where it is from and its marks.

- The questions are grouped by the pattern that keeps coming up in exams, a few of each in a row, running from the plainer patterns into the harder ones near the end. Work through them and you will meet every type an exam can throw at you.
- This set **is your diagnostic**. Working through it shows you exactly which patterns are solid and which need work. Attempt every question properly in your own book, and go back to your notes to help you work through tough questions if needed.
- Every question you get wrong, need help with or struggle with goes in your **Mistakes Log**, which you should be working on daily to get you exam-ready.
- Questions marked ★ draw on another Extension 2 topic (here, statistical distributions): they are genuine cross-topic exam questions, flagged so you know.

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Foundational inequality proofs

1. Inequalities from squares and differences

Q1 2025 Caringbah Trial Q13b · **1 mark**

Show that $\dfrac{a^4 + a^2 + 1}{a^2} \geq 3$ for all a.

Q2 2023 HSC Q12b · **2 marks**

Prove that for all real numbers x and y, where $x^2 + y^2 \neq 0$,

$$\frac{(x+y)^2}{x^2+y^2} \leq 2.$$

Q3 2025 Trinity Grammar Trial Q14a · **2 marks**

Prove that $a^3 + b^3 \geq ab(a+b)$ for positive real numbers a and b with $a \neq b$.

Q4 2025 Sydney Girls Trial Q15b · **2 marks**

Prove that if $x > 0$, $y > 0$ and $x + y = 1$, then

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9.$$

Q5 2025 Ascham Trial Q12e · **3 marks**

Prove that for all $x, y \in \mathbb{R}$,

$$\frac{(x-1)^2 + (y+2)^2}{2} \geq x + y.$$

Q6 2025 Gosford Trial Q13d · **3 marks**

If $a > 0$ and $b > 0$ and $a + b = 1$, prove that the minimum value of $\dfrac{1}{a} + \dfrac{1}{b}$ is 4.

The AM–GM inequality and its applications

2. Proving and applying the AM–GM inequality

Q7 2022 HSC Q12a · **2 marks**

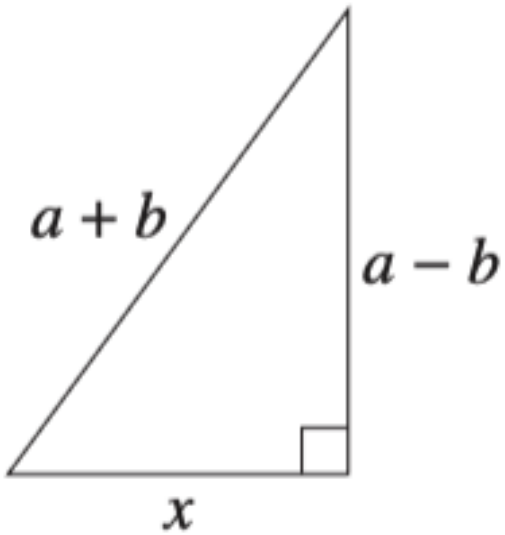
For real numbers $a, b \geq 0$ prove that

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Q8 2020 HSC Q13(c(i)) · **2 marks**

By considering the right-angled triangle below, or otherwise, prove that

$$\frac{a+b}{2} \geq \sqrt{ab}, \qquad \text{where } a > b \geq 0.$$



Q9 2021 HSC Q15a · **4 marks**

For all non-negative real numbers x and y,

$$\sqrt{xy} \leq \frac{x+y}{2}. \qquad \text{(Do NOT prove this.)}$$

(i) Using this fact, show that for all non-negative real numbers a, b and c,

$$\sqrt{abc} \leq \frac{a^2+b^2+2c}{4}.$$

(ii) Using part (i), or otherwise, show that for all non-negative real numbers a, b and c,

$$\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}.$$

Q10 2025 North Sydney Girls Trial Q12e · **3 marks**

(i) Prove that $\forall x, y \in \mathbb{R}^+$,

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

(ii) Let a, b, c and d be positive real numbers. Use the result from (i) to prove that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16.$$

Q11 2025 Ascham Trial Q14a · **3 marks**

(i) Prove that $x + y \geq 2\sqrt{xy}$ for positive real numbers x, y.

(ii) Hence for positive real numbers a, b, c and d, prove that

$$a^2 + 4b^2 + 4c^2 + d^2 \geq 8\sqrt{abcd}.$$

Q12 2022 HSC Q16c · **4 marks**

It is given that for positive numbers $x_1, x_2, x_3, \ldots, x_n$ with arithmetic mean A,

$$\frac{x_1 \times x_2 \times x_3 \times \cdots \times x_n}{A^n} \leq 1. \qquad \text{(Do NOT prove this.)}$$

Suppose a rectangular prism has dimensions a, b, c and surface area S.

(i) Show that

$$abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}.$$

(ii) Using part (i), show that when the rectangular prism with surface area S is a cube, it has maximum volume.

Q13 2025 Kinross Wolaroi Trial Q16d · **6 marks**

It is known that the arithmetic mean of three numbers is greater than or equal to the geometric mean of those numbers, that is,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

(i) Given $a + b + c = t$ and $a, b, c > 0$, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{t}.$$

(ii) Hence, given $a, b, c > 0$ and $a + b + c = 1$, prove that

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8.$$

Q14 2024 HSC Q13d · **3 marks**

It is known that for all positive real numbers x, y,

$$x + y \geq 2\sqrt{xy}. \qquad \text{(Do NOT prove this.)}$$

Show that if a, b, c are positive real numbers with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ then

$$a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab} \leq abc.$$

Q15 2025 Sydney Grammar Trial Q16c · **6 marks**

(i) Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers which satisfy

$$x_1 + x_2 + \cdots + x_n = M,$$

for some fixed value of M. Prove by contradiction that the product $x_1 \times x_2 \times \cdots \times x_n$ is maximised when $x_1 = x_2 = \cdots = x_n$.

(ii) Hence prove the AM/GM inequality for n terms, that is,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \times x_2 \times \cdots \times x_n}.$$

(iii) By considering the graph of $y = \sqrt{x}$ and the result in part (ii), prove that

$$\sqrt{x} + \sqrt{1 + x^2} + \sqrt{2 + x^3} + \cdots + \sqrt{(n-1) + x^n} > \frac{2n}{3}\sqrt{nx},$$

for all $x > 0$ and positive integers n.

Advanced inequality techniques

3. Building inequalities from an earlier result

Q16 2025 Hurlstone Trial Q13b · 3 marks

If x and y are real and unequal, then $x^2 + y^2 > 2xy$. **Do not prove this.** Deduce that for a, b and c, which are real and unequal:

(i) $a^2 + b^2 + c^2 > ab + bc + ac$.

(ii) If $a + b + c = 6$, show that $ab + bc + ac < 12$.

Q17 2025 Girraween Trial Q15b · 3 marks

(i) Prove for all real numbers a and b that $a^2 + b^2 \geq 2ab$.

(ii) Hence prove $a^2 + b^2 + c^2 \geq ab + ac + bc$ for all real numbers a, b, c.

(iii) Hence prove $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} \geq \dfrac{a + b + c}{abc}$ for all real numbers a, b, c.

Q18 2025 Gosford Trial Q15b · 3 marks

For positive x, y and z, prove that

$$x + y + z + \frac{1}{4x} + \frac{1}{4y} + \frac{1}{4z} \geq 3.$$

Q19 2025 Hills Grammar Trial Q15a · **4 marks**

For the positive real numbers a and b, $a \neq b$, it is given that

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \qquad \text{and} \qquad (a-b)^2 > 0.$$

(i) Using these facts, or otherwise, prove

$$a^3 + b^3 > ab(a+b).$$

(ii) Hence, or otherwise, show that if c is also a positive real number that is not equal to a or b, that

$$2\left(a^3 + b^3 + c^3\right) > a^2(b+c) + b^2(a+c) + c^2(a+b).$$

Q20 2025 Normanhurst Boys Trial Q15a · **5 marks**

Let a, b and c be positive real numbers.

(i) Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

(ii) Hence or otherwise, prove that $a^3 + b^3 + c^3 \geq 3abc$.

(iii) Hence prove that $ac(a+c) + ab(a+b) + bc(b+c) \geq 6abc$.

Q21 2025 Riverview Trial Q13d · **3 marks**

(i) Prove that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

(ii) Hence, or otherwise, prove that $a^2b^2 + a^2c^2 + b^2c^2 \geq abc(a+b+c)$.

4. Absolute value, triangle and geometric inequalities

Q22 2025 Sydney Grammar Trial Q9 · **1 mark**

Given that x and y are real numbers, which of the following is true?

(A) If $x < y$ then $x - y < |x| - |y|$

(B) If $x < y$ then $x - y > |x| - |y|$

(C) If $x - y < |x| - |y|$ then $x < y$

(D) If $x - y < |x| - |y|$ then $x > y$

Q23 2025 Caringbah Trial Q15b · **2 marks**

(i) Show that $|\sin(\alpha+\beta)| \leq |\sin\alpha| + |\sin\beta|$.

(ii) Under what conditions will equality hold in part (i)?

Q24 2025 Normanhurst Boys Trial Q13b · **2 marks**

Prove that $|a-b| + |c-b| + |c-d| \geq a-d$.

Q25 2025 Penrith Trial Q12d · **3 marks**

For all real numbers x and y, with $x^2+y^2 \neq 0$,

(i) prove that $\left(\dfrac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\dfrac{y}{\sqrt{x^2+y^2}}\right)^2 = 1$.

(ii) Hence, or otherwise, prove that $\left|\dfrac{x}{\sqrt{x^2+y^2}}\right| \leq 1$ and $\left|\dfrac{y}{\sqrt{x^2+y^2}}\right| \leq 1$.

Q26 2025 Sydney Girls Trial Q16a · **3 marks**

Suppose that a, b and c are the lengths of a right-angled triangle, where c is the hypotenuse. Show that if $n \leq 2$, then $c^n \leq a^n + b^n$.

5. Inequalities using calculus and area or integral bounds

Q27 2025 North Sydney Girls Trial Q14d · **3 marks**

Consider $f(x) = x^m - mx$ where $m > 1$. By using calculus, or otherwise, show that

$$x^m - 1 \geq m(x-1) \quad \text{for all } x > 0.$$

Q28 2025 Gosford Trial Q16a · **4 marks**

(i) Show that $\sin x \geq \dfrac{2x}{\pi}$ for $0 < x < \dfrac{\pi}{2}$.

(ii) Explain why

$$\int_0^{\pi/2} e^{-\sin x}\, dx < \int_0^{\pi/2} e^{-2x/\pi}\, dx.$$

Q29 2023 HSC Q16b · **5 marks**

(i) Prove that $x > \ln x$, for $x > 0$.

(ii) Using part (i), or otherwise, prove that for all positive integers n,

$$e^{n^2+n} > (n!)^2.$$

Q30 2025 North Sydney Boys Trial Q15a · **3 marks**

By considering the function $y = \sqrt{x}$ from $x = 0$ to $x = n + 1$, use $n + 1$ applications of the trapezoidal rule to show that

$$\sum_{k=1}^{n} \sqrt{k} < \frac{(4n+1)\sqrt{n+1}}{6}.$$

Q31 ★ 2025 Merewether Trial Q14c · **4 marks**

★ *Part (ii) is framed with independent Bernoulli random variables (statistical distributions); the proof itself needs only the inequality in part (i) and independence.*

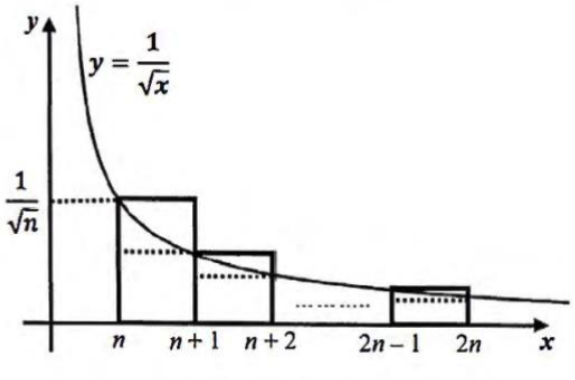
(i) Prove that $1 + x \leq e^x$ for $x \geq 0$.

(ii) $X_1, X_2, X_3, \ldots, X_n$ are independent Bernoulli random variables with parameters $p_1, p_2, p_3, \ldots, p_n$ respectively. The random variable T is given by $T = X_1 + X_2 + X_3 + \cdots + X_n$ and $E(T) = \mu$ denotes the mean of T. Prove that

$$P(T = 0) \leq e^{-\mu}.$$

Q32 2025 Riverview Trial Q16a · **4 marks**

In the diagram below, the graph of $y = \dfrac{1}{\sqrt{x}}$ has been drawn, and n rectangles have been constructed between $x = n$ and $x = 2n$, each of width 1 unit.



(i) Show that

$$\int_n^{2n} \frac{dx}{\sqrt{x}} = 2\sqrt{n}\left(\sqrt{2}-1\right).$$

(ii) Let

$$S_n = \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{2n}}.$$

Show that

$$2\sqrt{n}\left(\sqrt{2}-1\right) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\left(\sqrt{2}-1\right).$$

Q33 2025 Caringbah Trial Q16c · **8 marks**

(i) Given that for $k > 0$, $2k + 3 > 2\sqrt{(k+1)(k+2)}$, use mathematical induction to show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right)$$

for all positive integers n.

(ii) Use the graph of $y = \dfrac{1}{\sqrt{x}}$ to show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_{1}^{n} \frac{1}{\sqrt{x}}\, dx.$$

(iii) Hence show that

$$198 < \sum_{r=1}^{10000} \frac{1}{\sqrt{r}} < 199.$$