Year 12 Mathematics Extension 2

Topic: Complex Numbers (1 of 5)

Review Questions (Worked Solutions): Arithmetic, Conjugates & Forms

Term 3, 2026

Arithmetic in the complex plane

1. Adding, subtracting, multiplying & powers of i

Q1 2025 Abbotsleigh Trial Q1 · 1 mark

What is the value of i^{2025}?

(A) 1

(B) -1

(C) i

(D) $-i$

Criteria	Marks
correct option (C)	1

Solution.

Powers of i cycle with period 4: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. Since $2025 = 4 \times 506 + 1$,

$$i^{2025} = \left(i^4\right)^{506} \cdot i = 1 \cdot i = i.$$

The answer is (C).

Q2 2025 Sydney Tech Trial Q1 · **1 mark**

Let $z = 3 + 2i$. What is the value of $i\overline{z}$, expressed in the form $x + iy$?

(A) $2 + 3i$

(B) $-2 + 3i$

(C) $-2 - 3i$

(D) $2 - 3i$

Criteria	Marks
correct option (A)	1

Solution.

The conjugate is $\overline{z} = 3 - 2i$, so

$$i\overline{z} = i(3 - 2i) = 3i - 2i^2 = 3i + 2 = 2 + 3i.$$

The answer is (A).

Q3 2023 HSC Q1 · **1 mark**

Which of the following is equal to $(a + ib)^3$?

(A) $(a^3 - 3ab^2) + i(3a^2b + b^3)$

(B) $(a^3 + 3ab^2) + i(3a^2b + b^3)$

(C) $(a^3 - 3ab^2) + i(3a^2b - b^3)$

(D) $(a^3 + 3ab^2) + i(3a^2b - b^3)$

Criteria	Marks
correct option (C)	1

Solution.

Expanding with $i^2 = -1$ and $i^3 = -i$:

$$(a + ib)^3 = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3 = a^3 + 3a^2b\,i - 3ab^2 - b^3i.$$

Collecting real and imaginary parts,

$$(a + ib)^3 = (a^3 - 3ab^2) + i(3a^2b - b^3).$$

The answer is (C).

Q4 2024 HSC Q11(b) · **2 marks**

Let $z = 2 + 3i$ and $w = 1 - 5i$.

(i) Find $\overline{z + w}$.

(ii) Find z^2.

Criteria	Marks
(i) correct value of $\overline{z + w}$	1
(ii) correct value of z^2	1

Solution.

(i) $z + w = (2 + 3i) + (1 - 5i) = 3 - 2i$, so $\overline{z + w} = 3 + 2i$.

(ii) $z^2 = (2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$.

Q5 2025 HSC Q11(c) · **3 marks**

The complex number z is given by $x + iy$. Find, in Cartesian form:

(i) z^2

(ii) $\dfrac{1}{z}$.

Criteria	Marks
(i) correct value of z^2	1
(ii) rationalises by multiplying by $\dfrac{\overline{z}}{\overline{z}}$	1
(ii) correct Cartesian form of $\dfrac{1}{z}$	1

Solution.

(i) $z^2 = (x + iy)^2 = x^2 + 2ixy + i^2y^2 = (x^2 - y^2) + 2xy\,i$.

(ii) Multiply numerator and denominator by the conjugate $\overline{z} = x - iy$:

$$\frac{1}{z} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}\,i.$$

Q6 2025 Hills Grammar Trial Q12(a) · **2 marks**

Consider the complex number $z = a + ib$.

- The product of z and $\overline{z}$ is 25.
- The sum of z and $\overline{z}$ is 8.

Find z.

Criteria	**Marks**
finds a, or forms a quadratic with z and $\overline{z}$ as its roots	1
correct value(s) of z	1

Solution.

With $\overline{z} = a - ib$: $z\overline{z} = a^2 + b^2 = 25$ and $z + \overline{z} = 2a = 8$. From the second equation $a = 4$. Substituting, $16 + b^2 = 25$, so $b^2 = 9$ and $b = \pm 3$. Therefore

$$z = 4 + 3i \quad \text{or} \quad z = 4 - 3i.$$

Q7 2025 Penrith Trial Q11(a) · **3 marks**

Given $z = 1 + i$, find $z^2 + \dfrac{1}{z^2}$ in the form $a + ib$, where a and b are real numbers.

Criteria	**Marks**
correct value of z^2	1
correct value of $\dfrac{1}{z^2}$ in Cartesian form	1
correct final answer $\dfrac{3}{2}i$	1

Solution.

First, $z^2 = (1 + i)^2 = 1 + 2i + i^2 = 2i$. Then

$$\frac{1}{z^2} = \frac{1}{2i} = \frac{-i}{2i \cdot (-i)} = \frac{-i}{2} = -\frac{1}{2}i.$$

Adding,

$$z^2 + \frac{1}{z^2} = 2i - \frac{1}{2}i = \frac{3}{2}i = 0 + \frac{3}{2}i.$$

2. Division & realising denominators

Q8 2022 HSC Q11(a) · **2 marks**

Express $\dfrac{3-i}{2+i}$ in the form $x+iy$, where x and y are real numbers.

Criteria	Marks
multiplies by the conjugate of the denominator	1
correct answer $1-i$	1

Solution.

Multiply numerator and denominator by the conjugate $2-i$:

$$\frac{3-i}{2+i}=\frac{(3-i)(2-i)}{(2+i)(2-i)}=\frac{6-3i-2i+i^2}{4+1}=\frac{5-5i}{5}=1-i.$$

So $x=1$ and $y=-1$.

Q9 2021 HSC Q11(e) · **2 marks**

The complex numbers $z=5+i$ and $w=2-4i$ are given. Find $\dfrac{\overline{z}}{w}$, giving your answer in Cartesian form.

Criteria	Marks
forms $\overline{z}=5-i$ and multiplies by the conjugate of w	1
correct answer $\frac{7}{10}+\frac{9}{10}i$	1

Solution.

Here $\overline{z}=5-i$. Multiply numerator and denominator by the conjugate $2+4i$:

$$\frac{\overline{z}}{w}=\frac{5-i}{2-4i}=\frac{(5-i)(2+4i)}{(2-4i)(2+4i)}=\frac{10+20i-2i-4i^2}{4+16}=\frac{14+18i}{20}=\frac{7}{10}+\frac{9}{10}i.$$

Q10 2025 Normanhurst Boys Trial Q11(a) · **2 marks**

Find $\dfrac{-3+i}{1-2i}$, showing all working out.

Criteria	Marks
writes an equivalent fraction using the conjugate of the denominator	1
correct answer $-1-i$	1

Solution.

Multiply numerator and denominator by the conjugate $1+2i$:

$$\frac{-3+i}{1-2i} = \frac{(-3+i)(1+2i)}{(1-2i)(1+2i)} = \frac{-3-6i+i+2i^2}{1+4} = \frac{-5-5i}{5} = -1-i.$$

Q11 2025 Caringbah Trial Q2 · **1 mark**

If $z = \dfrac{3+4i}{1+2i}$, the imaginary component of z is:

(A) -2

(B) $\dfrac{2}{5}$

(C) $-\dfrac{2}{5}$

(D) 2

Criteria	Marks
correct option (C)	1

Solution.

Multiply numerator and denominator by the conjugate $1-2i$:

$$z = \frac{(3+4i)(1-2i)}{(1+2i)(1-2i)} = \frac{3-6i+4i-8i^2}{1+4} = \frac{11-2i}{5} = \frac{11}{5} - \frac{2}{5}i.$$

So the imaginary component is $-\dfrac{2}{5}$, which is option (C).

Q12 2025 Pymble Trial Q12(a)(i) · **2 marks**

The complex number z is such that $z = \dfrac{1+i}{1-ki}$, where k is a real number. Find the real part of z and the imaginary part of z, giving your answers in terms of k.

Criteria	Marks
multiplies by the conjugate $1+ki$ and expands	1
correct $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$	1

Solution.

Multiply numerator and denominator by the conjugate $1+ki$:

$$z = \frac{(1+i)(1+ki)}{(1-ki)(1+ki)} = \frac{1+ki+i+ki^2}{1+k^2} = \frac{(1-k)+(1+k)i}{1+k^2}.$$

Therefore

$$\mathrm{Re}(z) = \frac{1-k}{1+k^2}, \qquad \mathrm{Im}(z) = \frac{1+k}{1+k^2}.$$

Q13 2025 Gosford Trial Q12(c) · **2 marks**

If $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, find θ where $-\pi < \theta \leq \pi$.

Criteria	Marks
realises the denominator and simplifies the numerator	1
correct values $\theta = \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}$	1

Solution.

Multiply numerator and denominator by the conjugate $1+2i\sin\theta$:

$$\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)} = \frac{(3-4\sin^2\theta)+8i\sin\theta}{1+4\sin^2\theta}.$$

The expression is purely imaginary when the real part is zero:

$$3-4\sin^2\theta = 0 \quad\Longrightarrow\quad \sin^2\theta = \frac{3}{4} \quad\Longrightarrow\quad \sin\theta = \pm\frac{\sqrt{3}}{2}.$$

For $-\pi < \theta \leq \pi$ this gives $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, -\dfrac{\pi}{3}, -\dfrac{2\pi}{3}$.

3. Square roots of a complex number

Q14 2025 HSC Q3 · **1 mark**

What are the square roots of $3 - 4i$?

(A) $1 - 2i$ and $-1 + 2i$

(B) $1 + 2i$ and $-1 - 2i$

(C) $2 - i$ and $-2 + i$

(D) $-2 - i$ and $2 + i$

Criteria	Marks
correct option (C)	1

Solution.

Let $\sqrt{3-4i} = x + iy$. Then $(x+iy)^2 = 3 - 4i$, so $x^2 - y^2 = 3$ and $2xy = -4$. From $y = -2/x$: $x^2 - \dfrac{4}{x^2} = 3$, so $x^4 - 3x^2 - 4 = 0$, i.e. $(x^2-4)(x^2+1) = 0$. Hence $x = \pm 2$, $y = \mp 1$. The square roots are $2 - i$ and $-2 + i$. The answer is (C).

Q15 2025 James Ruse Trial Q11(b)(i) · **2 marks**

Find the square roots of $2i$, giving the answers in the form $x + iy$, where x and y are real numbers.

Criteria	Marks
obtains $x^2 - y^2 = 0$ and $2xy = 2$ (or equivalent merit)	1
both square roots correct	1

Solution.

Let $\sqrt{2i} = x + iy$. Then $x^2 - y^2 = 0$ and $2xy = 2$. From $x^2 = y^2$ and $xy = 1$ (same sign), $x = y$ and $x^2 = 1$, so $x = \pm 1$. The square roots are $1 + i$ and $-1 - i$.

Q16 2025 Caringbah Trial Q11(e)(i) · **2 marks**

Find the square roots of $3 + 4i$.

Criteria	Marks
obtains $x^2 - y^2 = 3$ and $2xy = 4$ (or equivalent merit)	1
both square roots correct	1

Solution.

Let $\sqrt{3+4i} = x+iy$. Then $x^2-y^2 = 3$ and $2xy = 4$. From $y = 2/x$: $x^4-3x^2-4 = 0$, i.e. $(x^2-4)(x^2+1) = 0$, so $x = \pm 2$, $y = \pm 1$. The square roots are $2 + i$ and $-2 - i$.

Q17 2025 Gosford Trial Q11(d) · **2 marks**

Determine the square roots of $-15 + 8i$. Show all relevant working.

Criteria	Marks
obtains a pair of simultaneous equations (or equivalent merit)	1
correct solution	1

Solution.

Let $\sqrt{-15+8i} = x + iy$. Then $x^2 - y^2 = -15$ and $2xy = 8$. From $y = 4/x$: $x^4 + 15x^2 - 16 = 0$, i.e. $(x^2 - 1)(x^2 + 16) = 0$, so $x = \pm 1$, $y = \pm 4$. The square roots are $1 + 4i$ and $-1 - 4i$.

Q18 2025 Manly Trial Q11(c) · **2 marks**

Find the square roots of $12 - 16i$, giving your answers in the form $a + bi$, where a and b are real numbers.

Criteria	Marks
correct equation in x or y (or equivalent merit)	1
both square roots correct	1

Solution.

Let $\sqrt{12-16i} = x + iy$. Then $x^2 - y^2 = 12$ and $2xy = -16$. From $y = -8/x$: $x^4 - 12x^2 - 64 = 0$, i.e. $(x^2 - 16)(x^2 + 4) = 0$, so $x = \pm 4$, $y = \mp 2$. The square roots are $4 - 2i$ and $-4 + 2i$.

Q19 2025 Sydney Grammar Trial Q12(a)(i) · **2 marks**

Find the two square roots of $5 - 12i$.

Criteria	Marks
obtains $x^2 - y^2 = 5$ and $2xy = -12$ (or equivalent merit)	1
both square roots correct	1

Solution.

Let $\sqrt{5-12i} = x + iy$. Then $x^2 - y^2 = 5$ and $2xy = -12$. From $y = -6/x$: $x^4 - 5x^2 - 36 = 0$, i.e. $(x^2 - 9)(x^2 + 4) = 0$, so $x = \pm 3$, $y = \mp 2$. The two square roots are $3 - 2i$ and $-3 + 2i$.

Q20 2021 HSC Q11(d) · **4 marks**

(i) Find the two square roots of $-i$, giving the answers in the form $x + iy$, where x and y are real numbers.

(ii) Hence, or otherwise, solve $z^2 + 2z + 1 + i = 0$, giving your solutions in the form $a + ib$ where a and b are real numbers.

Criteria	Marks
(i) obtains $x^2 - y^2 = 0$ and $2xy = -1$ (or equivalent merit)	1
(i) both square roots correct	1
(ii) recognises $(z + 1)^2 = -i$ and uses part (i) (or equivalent merit)	1
(ii) both solutions correct in Cartesian form	1

Solution.

(i) Let $\sqrt{-i} = x + iy$. Then $x^2 - y^2 = 0$ and $2xy = -1$. With $x^2 = y^2$ and $xy < 0$, $y = -x$ and $x^2 = \frac{1}{2}$, so $x = \pm\dfrac{\sqrt{2}}{2}$. The two square roots are $\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$ and $-\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$.

(ii) Completing the square, $z^2 + 2z + 1 + i = (z + 1)^2 + i = 0$, so $(z + 1)^2 = -i$. Hence $z + 1 = \pm\sqrt{-i}$, using part (i). Therefore

$$z = -1 + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \qquad \text{or} \qquad z = -1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

Modulus, argument & polar form

4. Modulus & argument

Q21 2025 Fort St Trial Q11(a)(i) · **2 marks**

Let $z = 3 - i$ and $w = 2 + 5i$. Find $|z + w|$.

Criteria	Marks
correct value of $z + w$, or finds a modulus	1
correct value $\|z + w\| = \sqrt{41}$	1

Solution.

$z + w = (3 + 2) + (-1 + 5)i = 5 + 4i$, so

$$|z + w| = \sqrt{5^2 + 4^2} = \sqrt{41}.$$

Q22 2025 Cheltenham Girls Trial Q11(a)(i) · **2 marks**

Consider the complex numbers $z = 1 - i$ and $w = \sqrt{3} + i$. Evaluate $|z| + |w|$, leaving your answer in exact form.

Criteria	Marks
correct value of one modulus, $\|z\| = \sqrt{2}$ or $\|w\| = 2$	1
correct exact value $\|z\| + \|w\| = 2 + \sqrt{2}$	1

Solution.

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \qquad |w| = \sqrt{(\sqrt{3})^2 + 1^2} = 2.$$

So $|z| + |w| = 2 + \sqrt{2}$.

Q23 2025 Barker Trial Q1 · **1 mark**

The argument of a complex number z is $\frac{\pi}{7}$. What is $\text{Arg}(z^{10})$?

(A) $\frac{3\pi}{7}$

(B) $-\frac{3\pi}{7}$

(C) $\frac{4\pi}{7}$

(D) $-\frac{4\pi}{7}$

Criteria	Marks
correct option (D)	1

Solution.

By De Moivre's theorem, $\arg(z^{10}) = 10 \times \frac{\pi}{7} = \frac{10\pi}{7}$. Reducing to the principal value in $(-\pi, \pi]$:

$$\frac{10\pi}{7} - 2\pi = -\frac{4\pi}{7}.$$

The answer is (D).

Q24 2025 Trinity Grammar Trial Q11(a) · **3 marks**

Let $z_1 = -3 - 2i$, $z_2 = 2 - i$, and $z_3 = i(z_1 - \overline{z_2})$. Calculate $|z_3|$.

Criteria	Marks
correct conjugate $\overline{z_2} = 2 + i$, or correct $z_1 - \overline{z_2} = -5 - 3i$	1
correct simplified form $z_3 = 3 - 5i$	1
correct value $\lvert z_3\rvert = \sqrt{34}$	1

Solution.

$\overline{z_2} = 2 + i$, so $z_1 - \overline{z_2} = (-3 - 2i) - (2 + i) = -5 - 3i$. Then

$$z_3 = i(-5 - 3i) = -5i - 3i^2 = 3 - 5i,$$

$$|z_3| = \sqrt{3^2 + (-5)^2} = \sqrt{34}.$$

Q25 2025 Sydney Girls Trial Q11(f) · **2 marks**

Evaluate $\text{Arg}(i^{2025})$, where $\text{Arg}(z)$ denotes the principal argument of z.

Criteria	Marks
simplifies i^{2025} to i using $2025 = 4 \times 506 + 1$	1
correct value $\text{Arg}(i^{2025}) = \dfrac{\pi}{2}$	1

Solution.

Since $i^4 = 1$ and $2025 = 4 \times 506 + 1$, $i^{2025} = i$. Therefore

$$\text{Arg}(i^{2025}) = \text{Arg}(i) = \frac{\pi}{2}.$$

Q26 2025 Baulkham Hills Trial Q3 · **1 mark**

What is the principal argument of $\dfrac{-3\sqrt{2} - i\sqrt{6}}{2 + 2i}$?

(A) $-\dfrac{11\pi}{12}$

(B) $\dfrac{7\pi}{12}$

(C) $\dfrac{11\pi}{12}$

(D) $\dfrac{13\pi}{12}$

Criteria	Marks
correct option (C)	1

Solution.

Use $\arg\left(\dfrac{u}{v}\right) = \arg(u) - \arg(v)$. The numerator $-3\sqrt{2} - i\sqrt{6}$ lies in the third quadrant, with argument $-\pi + \arctan\left(\dfrac{\sqrt{6}}{3\sqrt{2}}\right) = -\pi + \dfrac{\pi}{6} = -\dfrac{5\pi}{6}$. The denominator $2 + 2i$ has argument $\dfrac{\pi}{4}$. Then

$$-\frac{5\pi}{6} - \frac{\pi}{4} = -\frac{13\pi}{12},$$

which is outside $(-\pi, \pi]$. Adding 2π gives the principal argument $\dfrac{11\pi}{12}$. The answer is (C).

Q27 2025 Riverview Trial Q9 · **1 mark**

The complex number $z = a + ib$, where $0 < a < b$. Which of the following best describes z^4?

(A) $\text{Re}(z^4) < 0$

(B) $\text{Re}(z^4) \leq 0$

(C) $\text{Im}(z^4) < 0$

(D) $\text{Im}(z^4) \leq 0$

Criteria	Marks
correct option (C)	1

Solution.

Expanding, $\text{Im}(z^4) = 4a^3b - 4ab^3 = 4ab(a-b)(a+b)$. With $0 < a < b$, we have $ab > 0$, $(a+b) > 0$ and $(a-b) < 0$, so $\text{Im}(z^4) < 0$ strictly. The real part $\text{Re}(z^4) = a^4 - 6a^2b^2 + b^4$ can be positive or negative (for example $a = 1, b = 10$ gives 9401; $a = 1, b = 2$ gives -7), so options A and B fail. The answer is (C).

5. Modulus-argument (polar) form & converting

Q28 2025 Trinity Grammar Trial Q12(a)(i) · **1 mark**

Consider the complex number $z = 1 + \sqrt{3}\,i$. Write z in modulus-argument form.

Criteria	Marks
correct modulus-argument form	1

Solution.

$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$. Since z lies in the first quadrant, $\arg z = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$. Hence

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

Q29 2024 HSC Q11(e)(i) · **2 marks**

Write the number $\sqrt{3} + i$ in modulus-argument form.

Criteria	Marks
correct modulus	1
correct argument and final form	1

Solution.

$|\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$. The point lies in the first quadrant, so $\arg(\sqrt{3} + i) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$. Hence

$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

Q30 2025 Hornsby Girls Trial Q11(b)(i) · **2 marks**

Express $-1 + \sqrt{3}\,i$ in modulus-argument form.

Criteria	Marks
correct modulus	1
correct argument and final form	1

Solution.

$|-1+\sqrt{3}\,i| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$. The point lies in the second quadrant, so $\arg(-1+\sqrt{3}\,i) = \pi - \tan^{-1}(\sqrt{3}) = \dfrac{2\pi}{3}$. Hence

$$-1 + \sqrt{3}\,i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Q31 2025 Sydney Grammar Trial Q11(a)(i) · **2 marks**

Consider the complex numbers $z = 1 + \sqrt{3}\,i$ and $w = 1 - i$. Express both z and w in modulus-argument form.

Criteria	Marks
correct modulus-argument form for z	1
correct modulus-argument form for w	1

Solution.

For z: $|z| = 2$ and $\arg z = \dfrac{\pi}{3}$, so $z = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

For w: $|w| = \sqrt{2}$ and, since w lies in the fourth quadrant, $\arg w = -\dfrac{\pi}{4}$, so

$$w = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right).$$

Q32 2025 Killara Trial Q11(a) · **4 marks**

Let $z = 1 + 3i$ and $w = 2 - 6i$.

(i) Find $\overline{z} + w$ in modulus-argument form.

(ii) Write $\dfrac{z}{w}$ in the form $x + iy$, where x and y are real numbers.

Criteria	Marks
(i) correct value of $\overline{z} + w$	1
(i) correct modulus-argument form	1
(ii) rationalises using the conjugate of the denominator	1
(ii) correct answer in the form $x + iy$	1

Solution.

(i) $\overline{z} = 1 - 3i$, so $\overline{z} + w = (1 - 3i) + (2 - 6i) = 3 - 9i$. Then $|3 - 9i| = \sqrt{9 + 81} = 3\sqrt{10}$, and since the point lies in the fourth quadrant, $\arg(3 - 9i) = -\tan^{-1}(3)$. Hence

$$\overline{z} + w = 3\sqrt{10}\left(\cos\left(-\tan^{-1}3\right) + i\sin\left(-\tan^{-1}3\right)\right).$$

(ii) Multiply by the conjugate of the denominator:

$$\frac{z}{w} = \frac{(1 + 3i)(2 + 6i)}{(2 - 6i)(2 + 6i)} = \frac{2 + 6i + 6i + 18i^2}{4 + 36} = \frac{-16 + 12i}{40} = -\frac{2}{5} + \frac{3}{10}i.$$

Q33 2025 Normanhurst Boys Trial Q11(b) · **2 marks**

Given that $w = 2 - 2\sqrt{3}\,i$ and $\text{Arg}(zw) = \dfrac{\pi}{3}$, find the exact value of $\text{Arg}(z)$.

Criteria	Marks
states $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$ and finds $\text{Arg}(w)$	1
correct value of $\text{Arg}(z)$	1

Solution.

$w = 2 - 2\sqrt{3}\,i$ lies in the fourth quadrant, so $\text{Arg}(w) = -\tan^{-1}\left(\dfrac{2\sqrt{3}}{2}\right) = -\tan^{-1}(\sqrt{3}) = -\dfrac{\pi}{3}$. Using $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$:

$$\text{Arg}(z) = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

Since $\dfrac{2\pi}{3} \in (-\pi, \pi]$, this is the principal argument.

6. Modulus, argument & conjugate identities

Q34 2025 Sydney Grammar Trial Q14(b)(i) · **1 mark**

Consider the equation $z^2 = \dfrac{az + w}{\overline{z}}$, where a is real, and both w and z are complex. Show that $\text{Im}\left(\dfrac{z}{w}\right) = 0$.

Criteria	Marks
uses $z\overline{z} = \lvert z\rvert^2$ to obtain $w = z\left(\lvert z\rvert^2 - a\right)$	1

Solution.

Multiply both sides by $\overline{z}$: $z^2\overline{z} = az + w$. Since $z^2\overline{z} = z\,(z\overline{z}) = z\,\lvert z\rvert^2$, this gives

$$w = z\left(\lvert z\rvert^2 - a\right).$$

Both $\lvert z\rvert^2$ and a are real, so $\lvert z\rvert^2 - a$ is real. Hence

$$\frac{z}{w} = \frac{z}{z\left(\lvert z\rvert^2 - a\right)} = \frac{1}{\lvert z\rvert^2 - a},$$

which is real, so $\text{Im}\left(\dfrac{z}{w}\right) = 0$.

Q35 2025 Manly Trial Q12(c) · 2 marks

Show that $\text{Re}\left(\dfrac{1-z}{1+z}\right) = 0$ for any complex number z with $|z| = 1$.

Criteria	Marks
realises the denominator (or uses $z\overline{z} = 1$) to isolate the real part	1
shows the real part equals 0	1

Solution.

Take $z \neq -1$, so $1 + z \neq 0$. Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{1-z}{1+z} = \frac{(1-z)(1+\overline{z})}{(1+z)(1+\overline{z})} = \frac{1+\overline{z}-z-z\overline{z}}{|1+z|^2}.$$

Since $|z| = 1$, $z\overline{z} = 1$, so the numerator is $\overline{z} - z = -2i\ \text{Im}(z)$, which is purely imaginary. As $|1+z|^2$ is real and positive, the whole expression is purely imaginary, so

$$\text{Re}\left(\frac{1-z}{1+z}\right) = 0.$$

Q36 2025 St George Girls Trial Q13(c)(i) · 3 marks

For any complex numbers $w_1 = a + bi$ and $w_2 = c + di$, where a, b, c, d are real, prove that

$$|w_1 + w_2|^2 + |w_1 - w_2|^2 = 2\left(|w_1|^2 + |w_2|^2\right).$$

Criteria	Marks
expands $\lvert w_1 + w_2\rvert^2$ and $\lvert w_1 - w_2\rvert^2$ in terms of a, b, c, d	1
adds and cancels the cross terms	1
reaches $2\left(\lvert w_1\rvert^2 + \lvert w_2\rvert^2\right)$	1

Solution.

With $w_1 + w_2 = (a+c) + (b+d)i$ and $w_1 - w_2 = (a-c) + (b-d)i$,

$$|w_1 + w_2|^2 = (a+c)^2 + (b+d)^2, \qquad |w_1 - w_2|^2 = (a-c)^2 + (b-d)^2.$$

Adding, the cross terms $\pm 2ac$ and $\pm 2bd$ cancel:

$$|w_1 + w_2|^2 + |w_1 - w_2|^2 = 2a^2 + 2b^2 + 2c^2 + 2d^2 = 2\left(a^2 + b^2\right) + 2\left(c^2 + d^2\right).$$

Since $|w_1|^2 = a^2 + b^2$ and $|w_2|^2 = c^2 + d^2$, this equals $2\left(|w_1|^2 + |w_2|^2\right)$.

Q37 2024 HSC Q14(c) · **2 marks**

For the complex numbers z and w, it is known that $\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$. Find $\left|\frac{z+w}{z-w}\right|$.

Criteria	**Marks**
recognises that $\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$ makes $\frac{z}{w}$ purely imaginary	1
concludes $\left\|\frac{z+w}{z-w}\right\| = 1$	1

Solution.

Let $\frac{z}{w} = t$. Since $\arg t = -\frac{\pi}{2}$, t is purely imaginary, say $t = -ki$ with $k > 0$. Dividing numerator and denominator by w,

$$\left|\frac{z+w}{z-w}\right| = \left|\frac{t+1}{t-1}\right| = \frac{|1-ki|}{|-1-ki|} = \frac{\sqrt{1+k^2}}{\sqrt{1+k^2}} = 1.$$

Q38 2025 Ascham Trial Q16(c) · **4 marks**

Let z_1 and z_2 be two non-zero complex numbers. It is known that $z_1 + z_2 + 1 = 0$.

(i) Show that if $|z_1| = |z_2| = 1$, then $\dfrac{1}{z_1} + \dfrac{1}{z_2} + 1 = 0$.

(ii) Show that if $\dfrac{1}{z_1} + \dfrac{1}{z_2} + 1 = 0$, then $|z_1| = |z_2| = 1$.

(iii) Hence, or otherwise, show that if $|z_1| = |z_2| = 1$, then $z_1^2 + z_2^2 + 1 = 0$.

Criteria	**Marks**
(i) uses $\lvert z\rvert = 1 \Rightarrow \frac{1}{z} = \overline{z}$ and conjugates $z_1 + z_2 + 1 = 0$	1
(ii) deduces $z_1 z_2 = 1$ and identifies the moduli	1
(iii) shows $z_1^2 + z_2^2 + 1 = 2(1 - z_1 z_2)$	1
(iii) uses $z_1 z_2 = 1$ to conclude $z_1^2 + z_2^2 + 1 = 0$	1

Solution.

(i) If $|z_1| = |z_2| = 1$ then $\dfrac{1}{z_1} = \overline{z_1}$ and $\dfrac{1}{z_2} = \overline{z_2}$. Taking the conjugate of $z_1 + z_2 + 1 = 0$,

$$\overline{z_1} + \overline{z_2} + 1 = 0, \qquad \text{that is} \qquad \frac{1}{z_1} + \frac{1}{z_2} + 1 = 0.$$

(ii) From $\dfrac{1}{z_1} + \dfrac{1}{z_2} + 1 = 0$, $\dfrac{z_1 + z_2}{z_1 z_2} = -1$, so $z_1 + z_2 = -z_1 z_2$. Since $z_1 + z_2 = -1$, this gives $z_1 z_2 = 1$. Then z_1, z_2 are the roots of $t^2 + t + 1 = 0$, namely $t = \dfrac{-1 \pm \sqrt{3}\, i}{2}$, each of modulus 1. So $|z_1| = |z_2| = 1$.

(iii) Since $z_1 + z_2 = -1$,

$$z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1 z_2 = 1 - 2z_1 z_2,$$

so $z_1^2 + z_2^2 + 1 = 2(1 - z_1 z_2)$. If $|z_1| = |z_2| = 1$, then by part (i) $\dfrac{1}{z_1} + \dfrac{1}{z_2} + 1 = 0$, which gives $z_1 + z_2 = -z_1 z_2$; since $z_1 + z_2 = -1$, this forces $z_1 z_2 = 1$. Therefore

$$z_1^2 + z_2^2 + 1 = 2(1 - 1) = 0.$$

7. Real and imaginary conditions on z and $\overline{z}$

Q39 2025 St George Girls Trial Q5 · 1 mark

Consider the complex number $z = a + ib$ $(a, b \in \mathbb{R},\ a \neq 0,\ b \neq 0)$ and its conjugate $\overline{z}$. Which row correctly classifies $z + \overline{z}$, $z - \overline{z}$, $z\overline{z}$ and $z \div \overline{z}$?

	$z + \overline{z}$	$z - \overline{z}$	$z\overline{z}$	$z \div \overline{z}$
A.	Real	Imaginary	Real	Imaginary
B.	Complex	Real	Complex	Real
C.	Real	Imaginary	Real	Complex
D.	Real	Real	Imaginary	Complex

Criteria	Marks
correct option (C)	1

Solution.

$z + \overline{z} = 2a$ (real); $z - \overline{z} = 2ib$ (imaginary, $b \neq 0$); $z\overline{z} = a^2 + b^2$ (real); and

$$\frac{z}{\overline{z}} = \frac{(a+ib)^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}\,i,$$

which has both parts non-zero (since $a, b \neq 0$), so it is complex. The classification is Real, Imaginary, Real, Complex: row (C).

Q40 2025 Sydney Grammar Trial Q7 · 1 mark

Let z be a complex number. Which of the following will always be real?

(A) $(z + \overline{z})(z - \overline{z})$

(B) $(z + i\overline{z})(z - i\overline{z})$

(C) $(z + i\overline{z})(\overline{z} + iz)$

(D) $(z - i\overline{z})(\overline{z} - iz)$

Criteria	Marks
correct option (B)	1

Solution.

With $z = a + ib$: option A $= (2a)(2ib) = 4abi$ (imaginary). Option B $= z^2 - i^2\overline{z}^2 = z^2 + \overline{z}^2 = 2(a^2 - b^2)$, always real. Options C and D give $\pm 2i(a^2 - b^2)$ (imaginary). Only B is always real.

Q41 2022 HSC Q6 · 1 mark

A complex number z is NOT real. Which of the following could be true for z?

(A) $\overline{z} = iz$

(B) $\overline{z} = |z^2|$

(C) $\text{Re}(iz) = \text{Im}(z)$

(D) $\text{Arg}(z^3) = \text{Arg}(z)$

Criteria	Marks
correct option (A)	1

Solution.

With $z = a + ib$, $b \neq 0$: option A, $\overline{z} = iz$ gives $a - ib = -b + ai$, i.e. $a = -b$, which holds for a non-real z (e.g. $z = 1 - i$). Option B forces $\overline{z}$ real, so z real. Option C gives $-b = b$, so $b = 0$. Option D forces Arg real ($\theta = 0$ or π). Only A can hold. The answer is (A).

Q42 2025 Hurlstone Trial Q1 · 1 mark

Let $z = a + ib$ with a, b real and non-zero. If $z + \dfrac{1}{z}$ is real, which must be true?

(A) $\arg z = \dfrac{\pi}{4}$

(B) $a = -b$

(C) $a = b$

(D) $|z| = 1$

Criteria	Marks
correct option (D)	1

Solution.

$z + \dfrac{1}{z} = z + \dfrac{\overline{z}}{|z|^2}$, whose imaginary part is $b\left(1 - \dfrac{1}{a^2 + b^2}\right) = \dfrac{b(a^2 + b^2 - 1)}{a^2 + b^2}$. For this to be zero with $b \neq 0$, we need $a^2 + b^2 = 1$, i.e. $|z| = 1$. The answer is (D).

Q43 2025 Caringbah Trial Q3 · **1 mark**

Given $z = a + ib$ $(a, b \in \mathbb{R},\ a \neq 0)$, the expression $\dfrac{4\overline{z}z}{(z+\overline{z})^2}$ is equivalent to:

(A) $4[\text{Re}(z) \times \text{Im}(z)]$

(B) $1 + \left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

(C) $4\left([\text{Re}(z)]^2 + [\text{Im}(z)]^2\right)$

(D) $\dfrac{2 \times \text{Im}(z)}{[\text{Re}(z)]^2}$

Criteria	**Marks**
correct option (B)	1

Solution.

$\overline{z}z = a^2 + b^2$ and $z + \overline{z} = 2a$, so

$$\frac{4(a^2+b^2)}{(2a)^2} = \frac{a^2+b^2}{a^2} = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)^2.$$

The answer is (B).

Q44 2025 Normanhurst Boys Trial Q7 · 1 mark

Let z be a complex number with $\text{Im}(z) \neq 0$. Which is equivalent to $\dfrac{8z\overline{z}}{(\overline{z}-z)^2}$?

(A) $2+2\left(\dfrac{\text{Re}(z)}{\text{Im}(z)}\right)^2$

(B) $2+2\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

(C) $-2-2\left(\dfrac{\text{Re}(z)}{\text{Im}(z)}\right)^2$

(D) $-2-2\left(\dfrac{\text{Im}(z)}{\text{Re}(z)}\right)^2$

Criteria	Marks
correct option (C)	1

Solution.

$z\overline{z} = a^2+b^2$ and $\overline{z}-z = -2ib$, so $(\overline{z}-z)^2 = -4b^2$. Then

$$\frac{8(a^2+b^2)}{-4b^2} = -2-\frac{2a^2}{b^2} = -2-2\left(\frac{\text{Re}(z)}{\text{Im}(z)}\right)^2.$$

The answer is (C).

Q45 2025 Pymble Trial Q4 · 1 mark

The sum of the non-real complex numbers z and w is zero. Which statement is FALSE?

(A) $\dfrac{w}{z} = -1$

(B) $z-w = 2z$

(C) $w-z = 2w$

(D) $z+\overline{w} = 0$

Criteria	Marks
correct option (D)	1

Solution.

From $z+w=0$, $w=-z$; write $z=a+ib$, $b\neq 0$, so $\overline{w}=-a+ib$. Then A: $\frac{w}{z}=-1$ (true); B: $z-w=2z$ (true); C: $w-z=-2z=2w$ (true); D: $z+\overline{w}=2ib\neq 0$ (false). The false statement is (D).