Year 11 Mathematics Extension 1

Topic: Polynomials (4 of 4)

Review Questions (Worked Solutions): Mixed One-off Exam Questions

Term 3, 2026

Mixed one-off exam questions

Q1 2023 Blacktown Boys Trial Q10 · 1 mark

Given that the roots of $2x^2 - \sqrt{3}\,x + 1 = 0$ are $\tan\alpha$ and $\tan\beta$, what is the value of $\alpha + \beta$?

(A) $\dfrac{\pi}{6}$

(B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{3}$

(D) $\dfrac{2\pi}{3}$

Criteria	Marks
Correct option (C)	1

Solution.

For the quadratic $2x^2 - \sqrt{3}\,x + 1 = 0$ with roots $\tan\alpha$ and $\tan\beta$,

$$\tan\alpha + \tan\beta = \frac{\sqrt{3}}{2}, \qquad \tan\alpha\tan\beta = \frac{1}{2}.$$

Then

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\sqrt{3}/2}{1 - 1/2} = \sqrt{3},$$

so $\alpha + \beta = \dfrac{\pi}{3}$. The answer is (C).

Q2 2022 Normanhurst Boys Trial Q12(b)(i) · **1 mark**

Show that

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

Criteria	**Marks**
Correct derivation of the identity (division or common denominator)	1

Solution.

Dividing x^2 by $x+1$ gives quotient $x-1$ and remainder 1, since

$$x^2 = (x-1)(x+1) + 1.$$

Dividing both sides by $x+1$,

$$\frac{x^2}{x+1} = \frac{(x-1)(x+1)+1}{x+1} = x - 1 + \frac{1}{x+1},$$

as required.

Q3 2020 Killara Trial Q13(d)(i) · **1 mark**

Part of the graph of $x^3 + 8y = 64$ is shown. A tangent is drawn to the curve at the point $P(2, 7)$, intersecting the curve again at the point K. The equation of the line KP is

$$y = -\frac{3}{2}x + 10.$$

Find the coordinates of K.

Criteria	**Marks**
Correct coordinates $K(-4, 16)$	1

Solution.

From the curve, $y = \dfrac{64 - x^3}{8}$. Setting this equal to the line,

$$\frac{64 - x^3}{8} = -\frac{3}{2}x + 10.$$

Multiplying by 8: $64 - x^3 = -12x + 80$, so $x^3 - 12x + 16 = 0$. Since the line is tangent at $x = 2$, $(x - 2)$ is a repeated factor:

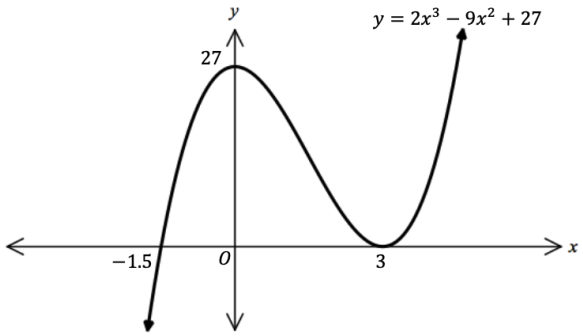
$$x^3 - 12x + 16 = (x - 2)^2(x + 4) = 0.$$

The tangent point gives $x = 2$; the other intersection is $x = -4$, and then $y = -\frac{3}{2}(-4) + 10 = 16$. Hence

$$K(-4, 16).$$

Q4 2021 Scots Trial Q10 · **1 mark**

The diagram shows the graph of $y = 2x^3 - 9x^2 + 27$ with x-intercepts at $x = 3$ and $x = -1.5$ and turning points at $(0, 27)$ and $(3, 0)$.



For what values of k does the function $y = 2x^3 - 9x^2 + k$ have one x-intercept?

(A) $0 < k < 27$

(B) $k < -1.5$ or $k > 27$

(C) $k < -1.5$ or $k > 3$

(D) $k < 0$ or $k > 27$

Criteria	**Marks**
Correct option (D)	1

Solution.

For $y = 2x^3 - 9x^2 + k$, $y' = 6x^2 - 18x = 6x(x - 3)$, so the turning points are at $x = 0$ and $x = 3$. Their y-values are the local maximum $y(0) = k$ and the local minimum $y(3) = 54 - 81 + k = k - 27$. The cubic has a single x-intercept when the two turning points lie on the *same* side of the x-axis, i.e. their product is positive:

$$k(k - 27) > 0 \quad \Longrightarrow \quad k < 0 \text{ or } k > 27.$$

This is option **(D)**.

Q5 2020 Sydney Boys Trial Q12(f)(i) · **2 marks**

Suppose $Q(x) = Ax^3 + Bx^2 + Cx + D$ with $A, B, C, D \in \mathbb{R}$, $A \neq 0$.

(i) What can be deduced about the zeros of $Q(x)$ if $B^2 - 3AC < 0$?

Criteria	**Marks**
Considers $Q'(x)$ and shows it has no real roots (discriminant < 0)	1
Concludes Q is monotonic, so exactly one real zero (two non-real conjugate zeros)	1

Solution.

$Q'(x) = 3Ax^2 + 2Bx + C$. Its discriminant is

$$(2B)^2 - 4(3A)(C) = 4B^2 - 12AC = 4\left(B^2 - 3AC\right).$$

If $B^2 - 3AC < 0$ then this discriminant is negative, so $Q'(x)$ has no real roots and never changes sign. Hence $Q(x)$ is strictly monotonic (always increasing if $A > 0$, always decreasing if $A < 0$) and has no stationary points. A strictly monotonic cubic crosses the x-axis exactly once, so $Q(x)$ has exactly *one* real zero (and its two remaining zeros are a non-real complex conjugate pair).

Q6 2025 Pymble Trial Q14(c)(ii) · 2 marks

(i) Solve $\tan 4\alpha = 1$ for $0 \le \alpha < \pi$.

(ii) Given

$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$

and by letting $x = \tan\alpha$, solve the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

Criteria	Marks
(i) All four solutions $\alpha = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$	1
(ii) States the four roots $x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$	1

Solution.

(i) $\tan 4\alpha = 1$ gives $4\alpha = \dfrac{\pi}{4} + n\pi$, so $\alpha = \dfrac{\pi}{16} + \dfrac{n\pi}{4}$. Taking $\alpha \in [0, \pi)$ $(n = 0, 1, 2, 3)$,

$$\alpha = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}.$$

(ii) With $x = \tan\alpha$, the identity gives

$$\tan 4\alpha = \frac{4x - 4x^3}{1 - 6x^2 + x^4}.$$

Setting $\tan 4\alpha = 1$ (the equation solved in part (i)) and cross-multiplying,

$$4x - 4x^3 = 1 - 6x^2 + x^4 \quad \Longrightarrow \quad x^4 + 4x^3 - 6x^2 - 4x + 1 = 0.$$

Thus the four solutions of the quartic are exactly $x = \tan\alpha$ for the four angles found in part (i):

$$x = \tan\frac{\pi}{16},\ \tan\frac{5\pi}{16},\ \tan\frac{9\pi}{16},\ \tan\frac{13\pi}{16}.$$

(Equivalently, factoring over $\mathbb{Q}(\sqrt{2})$, $x^4 + 4x^3 - 6x^2 - 4x + 1 = \left(x^2 + (2 - 2\sqrt{2})x - 1\right)\left(x^2 + (2 + 2\sqrt{2})x - 1\right)$, giving $x = (\sqrt{2} - 1) \pm \sqrt{4 - 2\sqrt{2}}$ and $x = -(\sqrt{2} + 1) \pm \sqrt{4 + 2\sqrt{2}}$.)

Q7 2023 Glenwood Trial Q13(b) · **3 marks**

(i) Consider the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. By letting $x = \cos\theta$ in the cubic equation $8x^3 - 6x - 1 = 0$, show that $\cos 3\theta = \dfrac{1}{2}$.

(ii) Hence prove that $\cos\dfrac{\pi}{9}\cos\dfrac{5\pi}{9}\cos\dfrac{7\pi}{9} = \dfrac{1}{8}$.

Criteria	**Marks**
(i) Substitutes $x = \cos\theta$ to get $2\cos 3\theta = 1$, hence $\cos 3\theta = \frac{1}{2}$	1
(ii) Identifies the three roots and applies product of roots to reach $\frac{1}{8}$	2

Solution.

(i) Let $x = \cos\theta$. Then

$$8x^3 - 6x = 2\left(4\cos^3\theta - 3\cos\theta\right) = 2\cos 3\theta.$$

The equation $8x^3 - 6x - 1 = 0$ becomes $2\cos 3\theta = 1$, so $\cos 3\theta = \dfrac{1}{2}$.

(ii) Solving $\cos 3\theta = \dfrac{1}{2}$ gives $3\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \ldots$, so

$$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9},$$

and the three roots $x = \cos\theta$ of $8x^3 - 6x - 1 = 0$ are $\cos\dfrac{\pi}{9}, \cos\dfrac{5\pi}{9}, \cos\dfrac{7\pi}{9}$. The product of the roots is $-\dfrac{\text{constant}}{\text{leading}} = -\dfrac{-1}{8} = \dfrac{1}{8}$, so

$$\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}.$$

Q8 2025 HSC Q14(e) · **3 marks**

It is given that $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are the three real roots of the polynomial equation

$$x^3 + bx^2 + cx - 1 + b + c = 0,$$

where b and c are real numbers and $c \neq 1$. Find the smallest positive value of $\alpha + \beta + \gamma$.

Criteria	**Marks**
Writes down the three symmetric functions of the roots from the coefficients	1
Applies the expansion of $\tan(\alpha+\beta+\gamma)$ and simplifies to -1	1
Deduces the smallest positive value $\dfrac{3\pi}{4}$	1

Solution.

With roots $\tan\alpha, \tan\beta, \tan\gamma$ of $x^3 + bx^2 + cx + (b + c - 1) = 0$,

$$\sum \tan\alpha = -b, \qquad \sum \tan\alpha\tan\beta = c, \qquad \tan\alpha\tan\beta\tan\gamma = -(b+c-1) = 1 - b - c.$$

Using the expansion

$$\tan(\alpha+\beta+\gamma) = \frac{\sum \tan\alpha - \tan\alpha\tan\beta\tan\gamma}{1 - \sum \tan\alpha\tan\beta},$$

we obtain

$$\tan(\alpha+\beta+\gamma) = \frac{-b - (1 - b - c)}{1 - c} = \frac{c - 1}{1 - c} = -1.$$

Hence $\alpha + \beta + \gamma = -\dfrac{\pi}{4} + n\pi$ for integer n. The smallest positive value (taking $n = 1$) is

$$\alpha + \beta + \gamma = \frac{3\pi}{4}.$$

Q9 ★ 2022 Moriah Trial Q13(d) · **4 marks**

★ *Uses a Year 12 topic (the second derivative): kept as genuine cross-topic exam practice.*

Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real numbers and $a \neq 0$. Let α, β and γ be the zeros of $f(x)$.

(i) Write down an expression for $\alpha + \beta + \gamma$.

(ii) All cubic polynomial functions have a single point of inflexion where the second derivative is equal to zero. Using part (i), or otherwise, show that the x-coordinate of the point of inflexion on the curve $y = f(x)$ is given by

$$x = \frac{\alpha + \beta + \gamma}{3}.$$

(iii) A certain cubic polynomial has x-intercepts at -1, 3 and 4. Find the x-coordinate of its point of inflexion.

Criteria	**Marks**
(i) States $\alpha + \beta + \gamma = -\dfrac{b}{a}$	1
(ii) Finds $f''(x) = 6ax + 2b$ and solves $f''(x) = 0$ to $x = -\dfrac{b}{3a}$	1
(ii) Identifies $-\dfrac{b}{3a} = \dfrac{\alpha + \beta + \gamma}{3}$	1
(iii) Evaluates $x = \dfrac{-1 + 3 + 4}{3} = 2$	1

Solution.

(i) By the sum of the roots of a cubic, $\alpha + \beta + \gamma = -\dfrac{b}{a}$.

(ii) Differentiating twice, $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. The point of inflexion occurs where $f''(x) = 0$:

$$6ax + 2b = 0 \implies x = -\frac{b}{3a} = \frac{1}{3}\left(-\frac{b}{a}\right) = \frac{\alpha + \beta + \gamma}{3},$$

using part (i).

(iii) With zeros -1, 3 and 4, the sum of the roots is $-1 + 3 + 4 = 6$, so

$$x = \frac{\alpha + \beta + \gamma}{3} = \frac{6}{3} = 2.$$

Q10 2025 Baulkham Hills Prelim Q12(a) · **4 marks**

The polynomial $P(x)$ is given by $P(x) = x^3 - (k-1)x^2 + (1-k)x + 1$ for some real number k.

(i) Show that $x = -1$ is a root of the equation $P(x) = 0$.

(ii) Given that $P(x) = (x+1)(x^2 - kx + 1)$, find the set of values of k such that the equation $P(x) = 0$ has three distinct real roots.

Criteria	Marks
(i) Shows $P(-1) = 0$	1
(ii) Quadratic factor needs distinct real roots: discriminant $k^2 - 4 > 0$	1
(ii) Excludes the values giving a repeated root ($k = \pm 2$, including $x = -1$ at $k = -2$)	1
(ii) Correct set $k < -2$ or $k > 2$	1

Solution.

(i)

$$P(-1) = (-1)^3 - (k-1)(-1)^2 + (1-k)(-1) + 1 = -1 - (k-1) - (1-k) + 1 = -1 - k + 1 - 1 + k + 1 = 0.$$

So $x = -1$ is a root.

(ii) Since $P(x) = (x+1)(x^2 - kx + 1)$, one root is $x = -1$; the other two are the roots of $x^2 - kx + 1 = 0$. For three *distinct* real roots we need the quadratic to have two distinct real roots, neither equal to -1.

Distinct real roots require the discriminant to be positive:

$$k^2 - 4 > 0 \quad \Longrightarrow \quad k < -2 \text{ or } k > 2.$$

Substituting $x = -1$ into the quadratic gives $1 + k + 1 = k + 2$, which is zero only when $k = -2$ (a boundary already excluded). At $k = 2$ the quadratic is $(x-1)^2$ (a repeated root), also excluded. Hence the quadratic gives two distinct roots, both different from -1, exactly when

$$k < -2 \quad \text{or} \quad k > 2.$$

Q11 2021 St George Girls Trial Q15(b) · 6 marks

(i) By using $\cos 4\theta = 2\cos^2 2\theta - 1$, show that

$$8\cos^4\theta - 8\cos^2\theta + 1 - \cos 4\theta = 0.$$

(ii) Using part (i) and by letting $x = \cos\theta$ into the equation $16x^4 - 16x^2 + 2 - \sqrt{2} = 0$, show that $\cos 4\theta = \dfrac{\sqrt{2}}{2}$.

(iii) Hence, prove that $\cos^2\dfrac{\pi}{16}\cos^2\dfrac{7\pi}{16} = \dfrac{2-\sqrt{2}}{16}$.

Criteria	Marks
(i) Substitutes $\cos 2\theta = 2\cos^2\theta - 1$ into $\cos 4\theta = 2\cos^2 2\theta - 1$ and rearranges	2
(ii) Substitutes $x = \cos\theta$ to reach $2\cos 4\theta = \sqrt{2}$, hence $\cos 4\theta = \frac{\sqrt{2}}{2}$	2
(iii) Identifies the four roots and uses product of roots to reach $\frac{2-\sqrt{2}}{16}$	2

Solution.

(i) Since $\cos 2\theta = 2\cos^2\theta - 1$,

$$\cos 4\theta = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1 = 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Rearranging, $8\cos^4\theta - 8\cos^2\theta + 1 - \cos 4\theta = 0$.

(ii) Let $x = \cos\theta$. From part (i), $8x^4 - 8x^2 + 1 = \cos 4\theta$. Multiplying by 2,

$$16x^4 - 16x^2 + 2 = 2\cos 4\theta.$$

The equation $16x^4 - 16x^2 + 2 - \sqrt{2} = 0$ gives $2\cos 4\theta = \sqrt{2}$, so $\cos 4\theta = \dfrac{\sqrt{2}}{2}$.

(iii) Solving $\cos 4\theta = \dfrac{\sqrt{2}}{2}$ for θ in $[0, \pi]$ gives $4\theta = \dfrac{\pi}{4}, \dfrac{7\pi}{4}, \dfrac{9\pi}{4}, \dfrac{15\pi}{4}$, i.e.

$$\theta = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16},$$

so the four roots $x = \cos\theta$ of $16x^4 - 16x^2 + 2 - \sqrt{2} = 0$ are $\cos\dfrac{\pi}{16}$, $\cos\dfrac{7\pi}{16}$, $\cos\dfrac{9\pi}{16}$, $\cos\dfrac{15\pi}{16}$. The product of the roots is $\dfrac{2-\sqrt{2}}{16}$. Since $\cos\dfrac{15\pi}{16} = -\cos\dfrac{\pi}{16}$ and $\cos\dfrac{9\pi}{16} = -\cos\dfrac{7\pi}{16}$,

$$\cos\frac{\pi}{16}\cos\frac{7\pi}{16}\left(-\cos\frac{7\pi}{16}\right)\left(-\cos\frac{\pi}{16}\right) = \cos^2\frac{\pi}{16}\cos^2\frac{7\pi}{16} = \frac{2-\sqrt{2}}{16}.$$

Q12 2023 Barker Prelim Q15(c) · 6 marks

The polynomial $P(x) = 2x^4 - 16x^3 + 46x^2 - 55x + 21$ is tangent to the quadratic function $Q(x) = 2x^2 - 7x + 3$ at two distinct points A and B. The point B lies on the x-axis.

(i) Find the coordinates of the points A and B.

(ii) Hence, show that the line passing through A and B is tangent to $P(x)$ at the midpoint of AB.

Criteria	Marks
(i) Forms $P(x) - Q(x)$ and factorises it as $2(x-1)^2(x-3)^2$	2
(i) States $A = (1, -2)$ and $B = (3, 0)$ (with B on the x-axis)	1
(ii) Finds the line through A, B: $y = x - 3$, and the midpoint $(2, -1)$	1
(ii) Shows $P(x) - (x-3) = 2(x-1)(x-2)^2(x-3)$, giving a double root at $x = 2$	2

Solution.

(i) The curves touch where $P(x) - Q(x) = 0$ has repeated roots:

$$P(x) - Q(x) = 2x^4 - 16x^3 + 44x^2 - 48x + 18 = 2(x-1)^2(x-3)^2.$$

The double roots at $x = 1$ and $x = 3$ are the two points of tangency. At $x = 1$, $Q(1) = 2 - 7 + 3 = -2$, so $A = (1, -2)$; at $x = 3$, $Q(3) = 18 - 21 + 3 = 0$, so $B = (3, 0)$, which lies on the x-axis as stated.

(ii) The line through $A(1, -2)$ and $B(3, 0)$ has gradient $\dfrac{0 - (-2)}{3 - 1} = 1$, so its equation is $y = x - 3$. The midpoint of AB is $\left(\dfrac{1+3}{2}, \dfrac{-2+0}{2}\right) = (2, -1)$. Now

$$P(x) - (x-3) = 2x^4 - 16x^3 + 46x^2 - 56x + 24 = 2(x-1)(x-2)^2(x-3).$$

The factor $(x-2)^2$ shows $x = 2$ is a double root, so the line $y = x - 3$ is tangent to $P(x)$ at $x = 2$, i.e. at the point $(2, -1)$, which is the midpoint of AB. Hence the line through A and B is tangent to $P(x)$ at the midpoint of AB.

Q13 2020 St George Girls Trial Q16(a) · 7 marks

(i) Prove the trigonometric identity $\sin 3A = 3\sin A - 4\sin^3 A$.

(ii) Hence, show that the equation $6x - 8x^3 = 1$ has the roots

$$\sin\frac{\pi}{18}, \quad \sin\frac{5\pi}{18} \quad \text{and} \quad \sin\frac{25\pi}{18}. \qquad \text{Hint: let } x = \sin A.$$

(iii) Hence show that $\sin\frac{\pi}{18} \times \sin\frac{5\pi}{18} \times \sin\frac{25\pi}{18} = -\frac{1}{8}$.

Criteria	Marks
(i) Expands $\sin 3A = \sin(2A + A)$ and reduces to $3\sin A - 4\sin^3 A$	3
(ii) Sets $x = \sin A$ to get $2\sin 3A = 1$, i.e. $\sin 3A = \frac{1}{2}$	2
(ii) Solves for the three stated roots	1
(iii) Uses product of roots of $8x^3 - 6x + 1 = 0$ to get $-\frac{1}{8}$	1

Solution.

(i)

$$\sin 3A = \sin(2A + A) = \sin 2A\cos A + \cos 2A \sin A.$$

Using $\sin 2A = 2\sin A\cos A$ and $\cos 2A = 1 - 2\sin^2 A$,

$$\sin 3A = 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A = 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A = 3\sin A - 4\sin^3 A.$$

(ii) Let $x = \sin A$. Then

$$6x - 8x^3 = 2\left(3\sin A - 4\sin^3 A\right) = 2\sin 3A.$$

The equation $6x - 8x^3 = 1$ becomes $2\sin 3A = 1$, i.e. $\sin 3A = \dfrac{1}{2}$, so

$$3A = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots \quad \Rightarrow \quad A = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \ldots$$

Since $x = \sin A$, the three distinct roots are $\sin\dfrac{\pi}{18}$, $\sin\dfrac{5\pi}{18}$ and $\sin\dfrac{25\pi}{18}$ (note $\sin\dfrac{13\pi}{18} = \sin\dfrac{5\pi}{18}$, so the third distinct value comes from $3A = \dfrac{25\pi}{6}$).

(iii) Writing $6x - 8x^3 = 1$ as $8x^3 - 6x + 1 = 0$, the product of the roots is $-\dfrac{1}{8}$ (that is, $-\dfrac{\text{constant}}{\text{leading}} = -\dfrac{1}{8}$). Hence

$$\sin\frac{\pi}{18} \times \sin\frac{5\pi}{18} \times \sin\frac{25\pi}{18} = -\frac{1}{8}.$$

Q14 ★ 2019 HSC Q14(b) · 7 marks

★ *Uses a Year 12 topic (Newton's method): kept as genuine cross-topic exam practice.*

The diagram shows the graph of $y = \dfrac{1}{x-k}$, where k is a positive real number.

(i) By considering the graphs of $y = x^2$ and $y = \dfrac{1}{x-k}$, explain why the function $f(x) = x^3 - kx^2 - 1$ has exactly one real zero.

Let this zero be α and let $x_1 = k$ be a first approximation to α.

(ii) Show that using one application of Newton's method gives a second approximation, $x_2 = k + \dfrac{1}{k^2}$.

(iii) Show that $x_1 < \alpha < x_2$.

Criteria	Marks
(i) Rearranges $f(x) = 0$ to $x^2 = \dfrac{1}{x-k}$ and argues the graphs meet exactly once	2
(ii) Applies Newton's method $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ with $x_1 = k$ to get $x_2 = k + \dfrac{1}{k^2}$	2
(iii) Shows $f(x_1) < 0$, so $\alpha > x_1$	1
(iii) Shows $f(x_2) > 0$, so $\alpha < x_2$, hence $x_1 < \alpha < x_2$	2

Solution.

(i) Setting $f(x) = 0$ gives $x^3 - kx^2 - 1 = 0$, i.e. $x^2(x-k) = 1$, so $x^2 = \dfrac{1}{x-k}$. Real zeros of f correspond to intersections of $y = x^2$ and $y = \dfrac{1}{x-k}$. For $x < k$ the hyperbola branch is negative while $x^2 \geq 0$, so there is no intersection there; for $x > k$ the branch $\dfrac{1}{x-k}$ decreases from $+\infty$ to 0 while x^2 increases from k^2 to $+\infty$, so the two curves meet exactly once. Hence f has exactly one real zero.

(ii) With $f(x) = x^3 - kx^2 - 1$ and $f'(x) = 3x^2 - 2kx$, Newton's method gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = k - \frac{f(k)}{f'(k)}.$$

Now $f(k) = k^3 - k^3 - 1 = -1$ and $f'(k) = 3k^2 - 2k^2 = k^2$, so

$$x_2 = k - \frac{-1}{k^2} = k + \frac{1}{k^2}.$$

(iii) Since $f(x_1) = f(k) = -1 < 0$ and the single real zero α is the only sign change, $\alpha > x_1$. Evaluating at x_2,

$$f(x_2) = f\left(k + \tfrac{1}{k^2}\right) = \frac{2k^3 + 1}{k^6} > 0 \quad (k > 0),$$

so $\alpha < x_2$. Combining, $x_1 < \alpha < x_2$.

Q15 2022 Normanhurst Boys Trial Q13(c) · **8 marks**

(i) You are given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Use this to show that

$$\sin^4\theta + \cos^4\theta = \frac{1}{4}\left(\cos 4\theta + 3\right).$$

(ii) By letting $x = \cos\theta$, show that the equation $8x^4 + 8\left(1 - x^2\right)^2 = 7$ has roots $\pm\cos\dfrac{\pi}{12}$ and $\pm\cos\dfrac{5\pi}{12}$.

(iii) Deduce that $\cos\dfrac{\pi}{12}\cos\dfrac{5\pi}{12} = \dfrac{1}{4}$.

Criteria	Marks
(i) Establishes $\sin^4\theta + \cos^4\theta = 1 - \frac{1}{2}\sin^2 2\theta$ (or equivalent) and converts to $\frac{1}{4}(\cos 4\theta + 3)$	3
(ii) Substitutes $x = \cos\theta$ so the equation becomes $\cos 4\theta = \frac{1}{2}$, and solves for the four roots	3
(iii) Uses product of roots to deduce $\frac{1}{4}$	2

Solution.

(i) Using $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = 1 - \dfrac{1}{2}\sin^2 2\theta$, and $\sin^2 2\theta = \dfrac{1 - \cos 4\theta}{2}$,

$$\sin^4\theta + \cos^4\theta = 1 - \frac{1}{2}\cdot\frac{1-\cos 4\theta}{2} = 1 - \frac{1-\cos 4\theta}{4} = \frac{4 - 1 + \cos 4\theta}{4} = \frac{1}{4}\left(\cos 4\theta + 3\right).$$

(ii) Let $x = \cos\theta$, so $\sin^2\theta = 1 - x^2$ and $\sin^4\theta + \cos^4\theta = (1-x^2)^2 + x^4$. Then

$$8x^4 + 8\left(1 - x^2\right)^2 = 8\left(\cos^4\theta + \sin^4\theta\right) = 8\cdot\frac{1}{4}\left(\cos 4\theta + 3\right) = 2\cos 4\theta + 6.$$

Setting this equal to 7 gives $2\cos 4\theta + 6 = 7$, i.e. $\cos 4\theta = \dfrac{1}{2}$. Solving in $[0, \pi]$,

$$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}.$$

Since $\cos\dfrac{7\pi}{12} = -\cos\dfrac{5\pi}{12}$ and $\cos\dfrac{11\pi}{12} = -\cos\dfrac{\pi}{12}$, the four roots $x = \cos\theta$ are $\pm\cos\dfrac{\pi}{12}$ and $\pm\cos\dfrac{5\pi}{12}$.

(iii) The quartic $8x^4 + 8(1-x^2)^2 = 7$ expands to $16x^4 - 16x^2 + (8-7) = 0$, i.e. $16x^4 - 16x^2 + 1 = 0$, whose product of roots is $\dfrac{1}{16}$. Hence

$$\left(\cos\frac{\pi}{12}\right)\left(-\cos\frac{\pi}{12}\right)\left(\cos\frac{5\pi}{12}\right)\left(-\cos\frac{5\pi}{12}\right) = \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}.$$

Taking positive square roots, $\cos\dfrac{\pi}{12}\cos\dfrac{5\pi}{12} = \dfrac{1}{4}$.

Q16 ★ 2022 Sydney Boys Trial Q14(b) · **11 marks**

★ *Uses Year 12 topics (parametric equations and angle of inclination): kept as genuine cross-topic exam practice.*

Let L be the straight line passing through the point $P(x_0, y_0)$ with an angle of inclination θ to the x-axis. The line L is defined parametrically by

$$x = x_0 + t\cos\theta, \qquad y = y_0 + t\sin\theta,$$

where $t \in \mathbb{R}$ and $\theta \in [0, \pi]$.

(i) Show that for any point T on the line L, $PT = |t|$.

(ii) The line L cuts the parabola $y = 3x^2 + 2$ at points A and B, where P is the point $\left(-1, -\frac{1}{3}\right)$. Let $PA = t_1$ and $PB = t_2$.

(α) Show that t_1 and t_2 are the roots of the equation

$$9t^2\cos^2\theta - 3t(\sin\theta + 6\cos\theta) + 16 = 0.$$

(β) Hence show that

$$AB^2 = \frac{\sin^2\theta + 12\sin\theta\cos\theta - 28\cos^2\theta}{9\cos^4\theta}.$$

(γ) If L is a tangent, show that the two possible gradients of L are 2 and -14.

(δ) Hence, if one of the tangents has a gradient of 2, show that $PZ = \frac{4\sqrt{5}}{3}$, where Z is the point of tangency of the line L to the parabola.

Criteria	**Marks**
(i) Computes $PT = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2} = \lvert t\rvert$	2
(ii)(α) Substitutes the parametric x, y into $y = 3x^2 + 2$ and reduces to the required quadratic	2
(ii)(β) Uses $AB = \lvert t_1 - t_2\rvert$ with sum/product of roots to obtain AB^2	3
(ii)(γ) Sets the discriminant to 0 and solves $\tan\theta = 2$ or $\tan\theta = -14$	2
(ii)(δ) Uses the repeated root $t = \frac{\sin\theta + 6\cos\theta}{6\cos^2\theta}$ with $\tan\theta = 2$ to get $PZ = \frac{4\sqrt{5}}{3}$	2

Solution.

(i) If T is on L then $T = (x_0 + t\cos\theta,\ y_0 + t\sin\theta)$, so

$$PT = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2} = \sqrt{t^2(\cos^2\theta + \sin^2\theta)} = \sqrt{t^2} = |t|.$$

(ii)(α) With $P = \left(-1, -\frac{1}{3}\right)$, a point on L is $\left(-1 + t\cos\theta,\ -\frac{1}{3} + t\sin\theta\right)$. Substituting into $y = 3x^2 + 2$:

$$-\tfrac{1}{3} + t\sin\theta = 3(-1 + t\cos\theta)^2 + 2 = 3t^2\cos^2\theta - 6t\cos\theta + 5.$$

Rearranging, $3t^2\cos^2\theta - t(\sin\theta + 6\cos\theta) + \dfrac{16}{3} = 0$, and multiplying through by 3,

$$9t^2\cos^2\theta - 3t(\sin\theta + 6\cos\theta) + 16 = 0,$$

whose roots are $t_1 = PA$ and $t_2 = PB$.

(β) For this quadratic, $t_1 + t_2 = \dfrac{3(\sin\theta + 6\cos\theta)}{9\cos^2\theta} = \dfrac{\sin\theta + 6\cos\theta}{3\cos^2\theta}$ and $t_1t_2 = \dfrac{16}{9\cos^2\theta}$. Since A, B lie on L, $AB = |t_1 - t_2|$, so

$$AB^2 = (t_1 + t_2)^2 - 4t_1t_2 = \frac{(\sin\theta + 6\cos\theta)^2}{9\cos^4\theta} - \frac{64}{9\cos^2\theta}.$$

Writing the second term over $9\cos^4\theta$ as $\dfrac{64\cos^2\theta}{9\cos^4\theta}$,

$$AB^2 = \frac{\sin^2\theta + 12\sin\theta\cos\theta + 36\cos^2\theta - 64\cos^2\theta}{9\cos^4\theta} = \frac{\sin^2\theta + 12\sin\theta\cos\theta - 28\cos^2\theta}{9\cos^4\theta}.$$

(γ) L is a tangent when $A = B$, i.e. the quadratic in t has a repeated root, so its discriminant is zero:

$$9(\sin\theta + 6\cos\theta)^2 - 4(9\cos^2\theta)(16) = 0 \implies (\sin\theta + 6\cos\theta)^2 - 64\cos^2\theta = 0.$$

Dividing by $\cos^2\theta$ and writing $m = \tan\theta$, $(m + 6)^2 - 64 = 0$, so $m + 6 = \pm 8$, giving $m = 2$ or $m = -14$. The two possible gradients are 2 and -14.

(δ) At tangency the repeated root is $t = \dfrac{3(\sin\theta + 6\cos\theta)}{2(9\cos^2\theta)} = \dfrac{\sin\theta + 6\cos\theta}{6\cos^2\theta}$, and $PZ = |t|$. With gradient 2, $\tan\theta = 2$ (with θ acute), so $\sin\theta = \dfrac{2}{\sqrt{5}}$ and $\cos\theta = \dfrac{1}{\sqrt{5}}$. Then

$$PZ = \frac{\frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}}}{6\cdot\frac{1}{5}} = \frac{\frac{8}{\sqrt{5}}}{\frac{6}{5}} = \frac{8}{\sqrt{5}}\cdot\frac{5}{6} = \frac{40}{6\sqrt{5}} = \frac{20}{3\sqrt{5}} = \frac{4\sqrt{5}}{3}.$$