Year 11 Mathematics Extension 1

Topic: Polynomials (3 of 4)

Review Questions (Worked Solutions): Sums and Products of Zeroes

Term 3, 2026

Sums and products of the roots

1. Sum, product and pairwise sums of the roots

Q1 2023 St George Girls Trial Q2 · 1 mark

Consider the equation $3x^3 + 2x^2 - x + 5 = 0$. What is the product of the roots of this polynomial equation?

(A) $\frac{5}{3}$

(B) $\frac{1}{3}$

(C) $-\frac{1}{3}$

(D) $-\frac{5}{3}$

Criteria	Marks
Correct option (D)	1

Solution.

For a cubic $ax^3 + bx^2 + cx + d = 0$ the product of the roots is $-\frac{d}{a}$. Here $a = 3$, $d = 5$, so

$$\alpha\beta\gamma = -\frac{5}{3}.$$

This is option **(D)**.

Q2 2024 HSC Q1 · **1 mark**

The polynomial $x^3 + 2x^2 - 5x - 6$ has zeros -1, -3 and a. What is the value of a?

(A) -2

(B) 2

(C) 3

(D) 6

Criteria	Marks
Correct option (B)	1

Solution.

For $x^3 + 2x^2 - 5x - 6$ the sum of the zeros is $-\dfrac{2}{1} = -2$. Hence

$$-1 + (-3) + a = -2 \implies a = 2.$$

(Check the product: $(-1)(-3)(2) = 6 = -\dfrac{-6}{1}$.) This is option **(B)**.

Q3 2022 Sydney Boys Trial Q1 · **1 mark**

What is the sum of the zeroes of $Q(x) = 4x^5 + 8x^3 - 2x^2 + 5x - 2$?

(A) -8

(B) -2

(C) 0

(D) 2

Criteria	Marks
Correct option (C)	1

Solution.

The sum of the zeros of a degree-5 polynomial equals $-\dfrac{(\text{coeff of } x^4)}{(\text{coeff of } x^5)}$. Here the x^4 coefficient is 0, so

$$\sum \text{zeros} = -\frac{0}{4} = 0.$$

This is option **(C)**.

Q4 2022 Glenwood Trial Q6 · **1 mark**

The equation $x^3 - 8x^2 - 4x + 2 = 0$ has roots α, β and γ. What is the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$?

(A) -4

(B) -2

(C) 2

(D) 4

Criteria	**Marks**
Correct option (A)	1

Solution.

Combining over a common denominator,

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}.$$

For $x^3 - 8x^2 - 4x + 2 = 0$: $\alpha + \beta + \gamma = -\dfrac{-8}{1} = 8$ and $\alpha\beta\gamma = -\dfrac{2}{1} = -2$. Hence

$$\frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = \frac{8}{-2} = -4.$$

This is option **(A)**.

Q5 2023 Penrith Prelim Q7 · **1 mark**

The roots of $4x^4 - 2x^3 + 9x - 10$ are α, β, γ and δ. What is the value of $\alpha\beta\gamma\delta(\alpha + \beta + \gamma + \delta)$?

(A) $\dfrac{5}{4}$

(B) $-\dfrac{5}{4}$

(C) $\dfrac{9}{4}$

(D) $-\dfrac{9}{4}$

Criteria	**Marks**
Correct option (B)	1

Solution.

For a quartic $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ the product of the roots is $\dfrac{a_0}{a_4}$ and their sum is $-\dfrac{a_3}{a_4}$. Here $a_4 = 4$, $a_3 = -2$, $a_0 = -10$, so

$$\alpha\beta\gamma\delta = \frac{-10}{4} = -\frac{5}{2}, \qquad \alpha + \beta + \gamma + \delta = -\frac{-2}{4} = \frac{1}{2}.$$

Therefore

$$\alpha\beta\gamma\delta(\alpha + \beta + \gamma + \delta) = \left(-\frac{5}{2}\right)\left(\frac{1}{2}\right) = -\frac{5}{4}.$$

This is option **(B)**.

Q6 2020 Ascham Trial Q11(b) · **3 marks**

The polynomial $P(x) = x^3 - 4x^2 + tx + 10$ has roots α, $-\alpha$ and β. Find the three roots and hence find the value of t.

Criteria	**Marks**
Uses sum and product of roots to find $\beta = 4$ and $\alpha^2 = \frac{5}{2}$	1
States the three roots $x = \pm\sqrt{\frac{5}{2}},\ 4$	1
Correct value $t = -\frac{5}{2}$	1

Solution.

The sum of the roots is $-\dfrac{-4}{1} = 4$:

$$\alpha + (-\alpha) + \beta = 4 \implies \beta = 4.$$

The product of the roots is $-\dfrac{10}{1} = -10$:

$$\alpha(-\alpha)\beta = -10 \implies -\alpha^2(4) = -10 \implies \alpha^2 = \frac{5}{2}.$$

So the three roots are

$$x = \sqrt{\tfrac{5}{2}}, \quad x = -\sqrt{\tfrac{5}{2}}, \quad x = 4.$$

Finally, t equals the sum of products of roots taken two at a time:

$$t = \alpha(-\alpha) + \alpha\beta + (-\alpha)\beta = -\alpha^2 = -\frac{5}{2}.$$

Q7 2020 Normanhurst Boys Prelim Q11(a) · **3 marks**

Solve $3x^3 - 17x^2 - 8x + 12 = 0$, given that the product of two of the roots is 4.

Criteria	**Marks**
States the product of all three roots $\alpha\beta\gamma = -4$	1
Uses $\alpha\beta = 4$ to deduce the third root $\gamma = -1$	1
Factorises and gives all roots $x = -1,\ \frac{2}{3},\ 6$	1

Solution.

The product of all three roots is $-\dfrac{12}{3} = -4$. If two of the roots have product $\alpha\beta = 4$, then the third root γ satisfies

$$\alpha\beta\gamma = -4 \implies 4\gamma = -4 \implies \gamma = -1.$$

Since $x = -1$ is a root, $(x + 1)$ is a factor. Dividing,

$$3x^3 - 17x^2 - 8x + 12 = (x + 1)(3x^2 - 20x + 12) = (x + 1)(3x - 2)(x - 6).$$

Hence the solutions are

$$x = -1, \qquad x = \frac{2}{3}, \qquad x = 6.$$

(Check: the product of two of these, $\dfrac{2}{3} \times 6 = 4$, as required.)

Q8 2021 Sydney Grammar Prelim Q4(b) · **4 marks**

The polynomial $P(x) = 2x^3 + 6x^2 + cx + d$ has a triple zero α.

(i) Find the value of α.

(ii) Find the values of c and d.

Criteria	**Marks**
(i) Uses sum of roots $3\alpha = -3$ to find $\alpha = -1$	2
(ii) Correct value $c = 6$	1
(ii) Correct value $d = 2$	1

Solution.

(i) A triple zero means all three roots equal α, so the sum of the roots is 3α. But the sum of the roots is $-\dfrac{6}{2} = -3$, hence

$$3\alpha = -3 \implies \alpha = -1.$$

(ii) With a triple zero at $\alpha = -1$, $P(x) = 2(x+1)^3$. Expanding,

$$2(x+1)^3 = 2x^3 + 6x^2 + 6x + 2.$$

Comparing with $2x^3 + 6x^2 + cx + d$ gives

$$c = 6, \qquad d = 2.$$

Q9 2020 Fort St Prelim Q9(c) · 5 marks

The cubic equation $2x^3 - 3x + 2 = 0$ has three real roots α, β and γ. Evaluate:

(i) $\alpha^3\beta^3\gamma^2 + \alpha^3\beta^2\gamma^3 + \alpha^2\beta^3\gamma^3$

(ii) $\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\alpha\gamma} + \dfrac{\gamma}{\alpha\beta}$

Criteria	Marks
States the elementary symmetric values: $\alpha+\beta+\gamma = 0$, $\alpha\beta+\beta\gamma+\gamma\alpha = -\frac{3}{2}$, $\alpha\beta\gamma = -1$	1
(i) Factorises as $(\alpha\beta\gamma)^2(\alpha\beta+\beta\gamma+\gamma\alpha)$ and evaluates to $-\frac{3}{2}$	2
(ii) Writes the sum as $\dfrac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$ with $\alpha^2+\beta^2+\gamma^2 = 3$	1
(ii) Correct value -3	1

Solution.

For $2x^3 + 0x^2 - 3x + 2 = 0$ the elementary symmetric functions are

$$\alpha+\beta+\gamma = -\frac{0}{2} = 0, \qquad \alpha\beta+\beta\gamma+\gamma\alpha = \frac{-3}{2} = -\frac{3}{2}, \qquad \alpha\beta\gamma = -\frac{2}{2} = -1.$$

(i) Take out the common factor $(\alpha\beta\gamma)^2$:

$$\alpha^3\beta^3\gamma^2 + \alpha^3\beta^2\gamma^3 + \alpha^2\beta^3\gamma^3 = (\alpha\beta\gamma)^2(\alpha\beta+\alpha\gamma+\beta\gamma) = (-1)^2\left(-\frac{3}{2}\right) = -\frac{3}{2}.$$

(ii) Over the common denominator $\alpha\beta\gamma$,

$$\frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}.$$

Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = 0^2 - 2\left(-\frac{3}{2}\right) = 3$, so

$$\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma} = \frac{3}{-1} = -3.$$

2. Symmetric functions of the roots

Q10 2021 Caringbah Trial Q3 · **1 mark**

The equation $x^3 - 2x^2 - 4x + 8 = 0$ has roots α, β and γ. What is the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$?

(A) $-\dfrac{1}{2}$

(B) $-\dfrac{1}{4}$

(C) $\dfrac{1}{4}$

(D) $\dfrac{1}{2}$

Criteria	Marks
Correct option (D)	1

Solution.

By Vieta, $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{-4}{1} = -4$ and $\alpha\beta\gamma = \dfrac{-8}{1} = -8$.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \frac{-4}{-8} = \frac{1}{2}.$$

The answer is (D).

Q11 2025 Normanhurst Boys Trial Q3 · **1 mark**

The equation $x^3 - 3x^2 - 18x + 40 = 0$ has roots α, β, γ. What is the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma}$?

(A) $-\dfrac{1}{18}$

(B) $\dfrac{1}{18}$

(C) $-\dfrac{3}{40}$

(D) $\dfrac{3}{40}$

Criteria	**Marks**
Correct option (C)	1

Solution.

Vieta gives $\alpha + \beta + \gamma = 3$ and $\alpha\beta\gamma = \dfrac{-40}{1} = -40$.

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \frac{3}{-40} = -\frac{3}{40}.$$

The answer is (C).

Q12 2022 Presbyterian Prelim Q5 · **1 mark**

Let α and β be the roots of the polynomial $P(x) = 3x^2 - x - 2$. What is the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$?

(A) $-\dfrac{2}{3}$

(B) $-\dfrac{1}{2}$

(C) $\dfrac{1}{2}$

(D) $\dfrac{2}{3}$

Criteria	**Marks**
Correct option (B)	1

Solution.

For $3x^2 - x - 2$, $\alpha + \beta = \dfrac{1}{3}$ and $\alpha\beta = \dfrac{-2}{3}$.

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{1/3}{-2/3} = -\frac{1}{2}.$$

The answer is (B).

Q13 2019 HSC Q7 · **1 mark**

Let $P(x) = qx^3 + rx^2 + rx + q$ where q and r are constants, $q \neq 0$. One of the zeros of $P(x)$ is -1. Given that a is a zero of $P(x)$, $a \neq -1$, which of the following is also a zero?

(A) $-\dfrac{1}{a}$

(B) $-\dfrac{q}{a}$

(C) $\dfrac{1}{a}$

(D) $\dfrac{q}{a}$

Criteria	Marks
Correct option (C)	1

Solution.

The coefficients q, r, r, q read the same forwards and backwards, so P is palindromic: $x^3 P\left(\frac{1}{x}\right) = P(x)$. Hence if a is a zero then $\dfrac{1}{a}$ is a zero. (Equivalently, the product of the three zeros is $-\dfrac{q}{q} = -1$; with zeros $-1, a, t$ this gives $-at = -1$, so $t = \dfrac{1}{a}$.) The answer is (C).

Q14 2025 Hills Grammar Trial Q11(f) · **2 marks**

The cubic polynomial $P(x) = x^3 + x^2 - 8x - 12$ has roots α, β and γ. Find $\alpha^2 + \beta^2 + \gamma^2$.

Criteria	Marks
States $\alpha + \beta + \gamma = -1$ and $\alpha\beta + \beta\gamma + \gamma\alpha = -8$ from the coefficients	1
Applies the identity to obtain $\alpha^2 + \beta^2 + \gamma^2 = 17$	1

Solution.

By Vieta, $\alpha + \beta + \gamma = -1$ and $\alpha\beta + \beta\gamma + \gamma\alpha = -8$. Using the identity

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = (-1)^2 - 2(-8) = 1 + 16 = 17.$$

Q15 2020 St George Girls Trial Q11(a) · **2 marks**

The polynomial $2x^3 - 4x^2 + 3x - 6 = 0$ has roots α, β and γ. Calculate the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$.

Criteria	Marks
Finds $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{3}{2}$ and $\alpha\beta\gamma = 3$	1
Correct value $\frac{1}{2}$	1

Solution.

By Vieta, $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{3}{2}$ and $\alpha\beta\gamma = \dfrac{-(-6)}{2} = 3$.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \frac{3/2}{3} = \frac{1}{2}.$$

Q16 2024 Hurlstone Prelim Q15(c) · **2 marks**

If α, β, γ are the roots of the equation $2x^3 - 5x^2 - 3x + 1 = 0$, find the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma}$.

Criteria	Marks
Finds $\alpha + \beta + \gamma = \frac{5}{2}$ and $\alpha\beta\gamma = -\frac{1}{2}$	1
Correct value -5	1

Solution.

By Vieta, $\alpha + \beta + \gamma = \dfrac{5}{2}$ and $\alpha\beta\gamma = \dfrac{-1}{2}$.

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \frac{5/2}{-1/2} = -5.$$

Q17 2023 Girraween Prelim Q9(b) · 3 marks

The polynomial $P(x) = x^4 + 4x^3 - 3x^2 + 5x - 7$ has roots α, β, γ and δ. Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$.

Criteria	Marks
Writes the sum of reciprocals as (sum of triple products)/(product of roots)	1
Finds sum of triple products $= -5$ and product $= -7$	1
Correct value $\frac{5}{7}$	1

Solution.

For a monic quartic $x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$ with roots $\alpha, \beta, \gamma, \delta$: the sum of triple products $= -c_1 = -5$ and the product $\alpha\beta\gamma\delta = c_0 = -7$. Then

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta} = \frac{-5}{-7} = \frac{5}{7}.$$

Q18 2022 SCEGGS Prelim Q8(f) · 3 marks

Given that α, β, γ are the roots of the equation $2x^3 - 6x^2 + 10x - 5 = 0$, find:

(i) $\alpha + \beta + \gamma$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

Criteria	Marks
(i) $\alpha + \beta + \gamma = 3$	1
(ii) Finds $\alpha\beta + \beta\gamma + \gamma\alpha = 5$ and applies the identity	1
(ii) Correct value -1	1

Solution.

(i) $\alpha + \beta + \gamma = \dfrac{-(-6)}{2} = 3.$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{10}{2} = 5$, so

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 3^2 - 2(5) = 9 - 10 = -1.$$

Q19 2024 Glenwood Prelim Q9(e) · 3 marks

If α, β, γ are roots of $3x^3 + 8x^2 - 1 = 0$, find the value of $\left(\beta + \frac{1}{\gamma}\right)\left(\gamma + \frac{1}{\alpha}\right)\left(\alpha + \frac{1}{\beta}\right)$. (Do not find the values of α, β and γ.)

Criteria	Marks
States $\alpha + \beta + \gamma = -\frac{8}{3}$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$, $\alpha\beta\gamma = \frac{1}{3}$	1
Expands the product to $\alpha\beta\gamma + (\alpha + \beta + \gamma) + \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) + \frac{1}{\alpha\beta\gamma}$	1
Correct value $\frac{2}{3}$	1

Solution.

By Vieta, $\alpha + \beta + \gamma = -\dfrac{8}{3}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{0}{3} = 0$ and $\alpha\beta\gamma = \dfrac{-(-1)}{3} = \dfrac{1}{3}$. Expanding,

$$\left(\beta + \tfrac{1}{\gamma}\right)\left(\gamma + \tfrac{1}{\alpha}\right)\left(\alpha + \tfrac{1}{\beta}\right) = \alpha\beta\gamma + (\alpha + \beta + \gamma) + \left(\tfrac{1}{\alpha} + \tfrac{1}{\beta} + \tfrac{1}{\gamma}\right) + \frac{1}{\alpha\beta\gamma}.$$

Now $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \dfrac{0}{1/3} = 0$, so

$$= \frac{1}{3} + \left(-\frac{8}{3}\right) + 0 + \frac{1}{1/3} = \frac{1}{3} - \frac{8}{3} + 3 = -\frac{7}{3} + 3 = \frac{2}{3}.$$

Q20 2023 Moriah Trial Q9(a) · 4 marks

The quadratic equation $x^2 + 3x + 1 = 0$ has roots α and β.

(i) Find the value of $\alpha + \beta$.

(ii) Find the value of $\alpha\beta$.

(iii) Hence, or otherwise, find the value of $\alpha^2 + \beta^2$.

Criteria	Marks
(i) $\alpha + \beta = -3$	1
(ii) $\alpha\beta = 1$	1
(iii) Applies $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$	1
(iii) Correct value 7	1

Solution.

(i) $\alpha + \beta = -\dfrac{3}{1} = -3$.

(ii) $\alpha\beta = \dfrac{1}{1} = 1$.

(iii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-3)^2 - 2(1) = 9 - 2 = 7$.

Q21 2021 Sydney Grammar Prelim Q2(b) · **4 marks**

The polynomial equation $x^3 + 9x^2 - 2x - 24 = 0$ has roots α, β and γ. Find the value of:

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

Criteria	**Marks**
(i) Uses $\dfrac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$ to obtain $-\frac{1}{12}$	2
(ii) Applies the identity to obtain 85	2

Solution.

By Vieta, $\alpha + \beta + \gamma = -9$, $\alpha\beta + \beta\gamma + \gamma\alpha = -2$ and $\alpha\beta\gamma = 24$.

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = \dfrac{-2}{24} = -\dfrac{1}{12}$.

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = (-9)^2 - 2(-2) = 81 + 4 = 85$.

Q22 2022 James Ruse Trial Q12(a) · 6 marks

The quartic polynomial $P(x) = x^4 + px^3 + qx^2 + px + 1$ has $x = \alpha$ as a double zero.

(i) Show that α^{-1} is also a zero of $P(x)$.

(ii) Find the fourth root in terms of α.

(iii) Hence show that $p^2 = 4(q-2)$.

Criteria	Marks
(i) Evaluates $\alpha^4 P\left(\frac{1}{\alpha}\right) = P(\alpha) = 0$, so $\frac{1}{\alpha}$ is a zero	2
(ii) Uses the product of roots = 1 to find the fourth root = $\frac{1}{\alpha}$	1
(iii) Sum of roots gives $p = -2\left(\alpha + \frac{1}{\alpha}\right)$	1
(iii) Sum of pairwise products gives $q = \alpha^2 + \frac{1}{\alpha^2} + 4$	1
(iii) Combines to show $p^2 = 4(q-2)$	1

Solution.

(i) Since $P(\alpha) = 0$,

$$\alpha^4 P\left(\tfrac{1}{\alpha}\right) = \alpha^4\left(\tfrac{1}{\alpha^4} + \tfrac{p}{\alpha^3} + \tfrac{q}{\alpha^2} + \tfrac{p}{\alpha} + 1\right) = 1 + p\alpha + q\alpha^2 + p\alpha^3 + \alpha^4 = P(\alpha) = 0.$$

As $\alpha^4 \neq 0$, $P\left(\frac{1}{\alpha}\right) = 0$, so $\frac{1}{\alpha}$ is a zero.

(ii) α is a double zero and $\frac{1}{\alpha}$ is a zero, so the four zeros are $\alpha, \alpha, \frac{1}{\alpha}, t$. The product of the roots is $\dfrac{1}{1} = 1$, so $\alpha \cdot \alpha \cdot \frac{1}{\alpha} \cdot t = \alpha t = 1$, giving the fourth root $t = \dfrac{1}{\alpha}$. Thus the zeros are $\alpha, \alpha, \frac{1}{\alpha}, \frac{1}{\alpha}$.

(iii) Sum of roots: $2\alpha + \dfrac{2}{\alpha} = -p$, so $p = -2\left(\alpha + \frac{1}{\alpha}\right)$ and

$$p^2 = 4\left(\alpha + \tfrac{1}{\alpha}\right)^2 = 4\left(\alpha^2 + 2 + \tfrac{1}{\alpha^2}\right).$$

Sum of products two at a time: $q = \alpha^2 + 1 + 1 + \dfrac{1}{\alpha^2} + 1 + 1 = \alpha^2 + \dfrac{1}{\alpha^2} + 4$, so $q - 2 = \alpha^2 + \dfrac{1}{\alpha^2} + 2$. Therefore

$$4(q-2) = 4\left(\alpha^2 + \tfrac{1}{\alpha^2} + 2\right) = p^2,$$

as required.

Q23 2020 St George Girls Prelim Q9(b) · 7 marks

Let α, β and γ be the roots of $2x^3 - 4x + 5 = 0$. Find, in simplest form:

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$

(iii) $\alpha\beta\gamma$

(iv) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha}$

(v) $\alpha^3 + \beta^3 + \gamma^3$

Criteria	Marks
(i) $\alpha + \beta + \gamma = 0$	1
(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = -2$	1
(iii) $\alpha\beta\gamma = -\frac{5}{2}$	1
(iv) Correct value 0	2
(v) Applies the sum-of-cubes identity to obtain $-\frac{15}{2}$	2

Solution.

The equation is $2x^3 + 0x^2 - 4x + 5 = 0$. By Vieta:

(i) $\alpha + \beta + \gamma = \dfrac{-0}{2} = 0.$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{-4}{2} = -2.$

(iii) $\alpha\beta\gamma = \dfrac{-5}{2}.$

(iv) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\gamma\alpha} = \dfrac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \dfrac{0}{-5/2} = 0.$

(v) Using $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$,

$$\alpha^3 + \beta^3 + \gamma^3 = 0^3 - 3(0)(-2) + 3\left(-\tfrac{5}{2}\right) = -\tfrac{15}{2}.$$

Relationships between roots

3. Finding a coefficient from a root condition

Q24 2020 Glenwood Prelim Q4 · 1 mark

It is known that two of the roots of the equation $3x^3 + x^2 - kx + 6 = 0$ are reciprocals of each other. What is the value of k?

(A) -2

(B) 6

(C) 7

(D) 17

Criteria	Marks
Correct option (C)	1

Solution.

Let the roots be α, $\frac{1}{\alpha}$ and γ. The product of the roots is $\alpha \cdot \frac{1}{\alpha} \cdot \gamma = \gamma$, and for this cubic the product equals $-\frac{6}{3} = -2$, so $\gamma = -2$. Since γ is a root,

$$3(-2)^3 + (-2)^2 - k(-2) + 6 = 0 \;\Rightarrow\; -24 + 4 + 2k + 6 = 0 \;\Rightarrow\; k = 7.$$

The answer is (C).

Q25 2020 Normanhurst Boys Prelim Q3 · **1 mark**

Two of the roots of $4x^3 + 8x^2 + kx - 18 = 0$ are equal in magnitude but have opposite sign. What is the value of k?

(A) -9

(B) 9

(C) 2

(D) -2

Criteria	Marks
Correct option (A)	1

Solution.

Let the roots be α, $-\alpha$ and γ. The sum of the roots is $\gamma = -\dfrac{8}{4} = -2$. The product is $(\alpha)(-\alpha)(\gamma) = -\alpha^2\gamma = -\dfrac{-18}{4} = \dfrac{9}{2}$, so $-\alpha^2(-2) = \dfrac{9}{2}$ gives $\alpha^2 = \dfrac{9}{4}$. Then k is 4 times the sum of the products of roots in pairs:

$$k = 4\big(-\alpha^2 + \alpha\gamma - \alpha\gamma\big) = 4(-\alpha^2) = 4\left(-\tfrac{9}{4}\right) = -9.$$

The answer is (A).

Q26 2024 Penrith Prelim Q9 · **1 mark**

Two of the roots of the equation $x^3 - 3x^2 + kx + 15 = 0$ are reciprocals of each other. What is the value of k?

(A) -69

(B) -169

(C) -269

(D) -369

Criteria	Marks
Correct option (C)	1

Solution.

Let the roots be α, $\frac{1}{\alpha}$ and γ. The product of the roots is $\alpha \cdot \frac{1}{\alpha} \cdot \gamma = \gamma = -\dfrac{15}{1} = -15$, so $\gamma = -15$. Since $\gamma = -15$ is a root,

$$(-15)^3 - 3(-15)^2 + k(-15) + 15 = 0 \;\Rightarrow\; -3375 - 675 - 15k + 15 = 0 \;\Rightarrow\; -15k = 4035 \;\Rightarrow\; k = -269.$$

The answer is (C).

Q27 2025 Pymble Trial Q11(e) · **2 marks**

The sum of two of the roots of the equation $2x^3 + 6x^2 + kx - 150 = 0$ is zero. Find the value of k.

Criteria	Marks
Identifies the third root $\gamma = -3$ (from the sum of all roots)	1
Correct value $k = -50$	1

Solution.

Let the roots be α, $-\alpha$ (their sum is zero) and γ. The sum of all three roots is $\gamma = -\dfrac{6}{2} = -3$. The product is $(\alpha)(-\alpha)(\gamma) = -\alpha^2\gamma = -\dfrac{-150}{2} = 75$, so $-\alpha^2(-3) = 75$ gives $\alpha^2 = 25$. Then

$$\frac{k}{2} = (\alpha)(-\alpha) + (\alpha)\gamma + (-\alpha)\gamma = -\alpha^2 = -25 \;\Rightarrow\; k = -50.$$

Q28 2023 Baulkham Hills Prelim Q13(d) · **3 marks**

Two of the roots of the equation $4x^3 + px^2 - 20x + 12 = 0$ are reciprocals. Find the value of p.

Criteria	Marks
Uses the product of roots to find the third root $\gamma = -3$	1
Substitutes $\gamma = -3$ into the equation	1
Correct value $p = 4$	1

Solution.

Let the roots be α, $\frac{1}{\alpha}$ and γ. The product of the roots is $\alpha \cdot \frac{1}{\alpha} \cdot \gamma = \gamma = -\dfrac{12}{4} = -3$, so $\gamma = -3$. Since $\gamma = -3$ is a root,

$$4(-3)^3 + p(-3)^2 - 20(-3) + 12 = 0 \;\Rightarrow\; -108 + 9p + 60 + 12 = 0 \;\Rightarrow\; 9p = 36 \;\Rightarrow\; p = 4.$$

Q29 2025 Baulkham Hills Prelim Q11(c) · **3 marks**

The polynomial $P(x) = x^3 - 4x^2 + kx + 10$ has two zeros that are the same in magnitude but opposite in sign. Find the value of k and all zeros of the polynomial.

Criteria	**Marks**
Finds the third zero $\gamma = 4$ (from the sum of zeros)	1
Correct value $k = -\frac{5}{2}$	1
States all three zeros 4, $\frac{\sqrt{10}}{2}$, $-\frac{\sqrt{10}}{2}$	1

Solution.

Let the zeros be α, $-\alpha$ and γ. The sum of the zeros is $\gamma = 4$ (since $-\dfrac{-4}{1} = 4$). The product of the zeros is $(\alpha)(-\alpha)(\gamma) = -\alpha^2\gamma = -10$ (since $-\dfrac{10}{1} = -10$), so $-4\alpha^2 = -10$ gives $\alpha^2 = \dfrac{5}{2}$, i.e. $\alpha = \dfrac{\sqrt{10}}{2}$. Then

$$k = (\alpha)(-\alpha) + (\alpha)\gamma + (-\alpha)\gamma = -\alpha^2 = -\frac{5}{2}.$$

The three zeros are $x = 4,\ \dfrac{\sqrt{10}}{2},\ -\dfrac{\sqrt{10}}{2}$.

Q30 2021 Newington Trial Q6(b) · 5 marks

Consider the polynomial $P(x) = x^3 - 4x^2 + kx + 12$, where k is a constant. It is given that α, β and γ are the roots of the equation $P(x) = 0$.

(i) Find the values of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

(ii) If two of the roots are equal in magnitude but opposite in sign, find the value of the roots and the value of k.

Criteria	Marks
(i) Correct sum $\alpha + \beta + \gamma = 4$	1
(i) Correct product $\alpha\beta\gamma = -12$	1
(ii) Finds the third root = 4	1
(ii) Finds the equal-and-opposite roots $\pm\sqrt{3}$	1
(ii) Correct value $k = -3$	1

Solution.

(i) For $P(x) = x^3 - 4x^2 + kx + 12$, the sum of the roots is $\alpha + \beta + \gamma = -\dfrac{-4}{1} = 4$ and the product is $\alpha\beta\gamma = -\dfrac{12}{1} = -12$.

(ii) Let the roots be ρ, $-\rho$ and γ. From part (i) the sum is $\gamma = 4$. The product is $(\rho)(-\rho)(\gamma) = -\rho^2\gamma = -12$, so $-4\rho^2 = -12$ gives $\rho^2 = 3$, i.e. $\rho = \sqrt{3}$. The roots are 4, $\sqrt{3}$, $-\sqrt{3}$. Then

$$k = (\rho)(-\rho) + (\rho)\gamma + (-\rho)\gamma = -\rho^2 = -3.$$

4. Root relationships, proofs and forming polynomials

Q31 2022 Barker Trial Q1 · 1 mark

The equation $x^3 + bx^2 + cx + d = 0$ has roots $1 + \sqrt{2}$, $1 - \sqrt{2}$ and 4. What is the value of b?

(A) -6

(B) -4

(C) 4

(D) 6

Criteria	Marks
Correct option (A)	1

Solution.

For a monic cubic, the sum of the roots equals $-b$. Here

$$\text{sum} = (1 + \sqrt{2}) + (1 - \sqrt{2}) + 4 = 6,$$

so $-b = 6$, giving $b = -6$. The answer is (A).

Q32 2023 SCEGGS Prelim Q1 · 1 mark

The sum of two roots of $4x^4 - 56x^3 + 239x^2 - 316x + 105 = 0$ is 12. What is the sum of the other two roots?

(A) $-\dfrac{14}{3}$

(B) -2

(C) 2

(D) $\dfrac{14}{3}$

Criteria	Marks
Correct option (C)	1

Solution.

The sum of all four roots is $-\dfrac{-56}{4} = 14$. If two of them sum to 12, the remaining two sum to $14 - 12 = 2$. The answer is (C).

Q33 2020 Knox Prelim Q3 · **1 mark**

A polynomial equation has roots α, β and γ where

$$\alpha + \beta + \gamma = -2, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = 3 \qquad \text{and} \qquad \alpha\beta\gamma = 1.$$

Which polynomial equation has the roots α, β and γ?

(A) $x^3 + 2x^2 + 3x + 1 = 0$

(B) $x^3 + 2x^2 + 3x - 1 = 0$

(C) $x^3 - 2x^2 + 3x + 1 = 0$

(D) $x^3 - 2x^2 + 3x - 1 = 0$

Criteria	**Marks**
Correct option (B)	1

Solution.

A monic cubic with these roots is

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0.$$

Substituting gives $x^3 - (-2)x^2 + 3x - 1 = x^3 + 2x^2 + 3x - 1 = 0$. The answer is (B).

Q34 2025 Hornsby Girls Trial Q13(b) · **2 marks**

The three solutions of the equation $x^3 - 2x^2 + 3x + 1 = 0$ are denoted by α, β and γ. Consider the equation $x^3 - x^2 + 7x + C = 0$ whose solutions are

$$2\alpha - 1, \quad 2\beta - 1 \quad \text{and} \quad 2\gamma - 1.$$

Find the value of C, where C is an integer.

Criteria	**Marks**
Sets up the substitution $y = 2x - 1$, i.e. $x = \dfrac{y+1}{2}$	1
Substitutes and reads off $C = 17$	1

Solution.

If $y = 2x - 1$ is a root of the new cubic whenever x is a root of the old one, then $x = \dfrac{y+1}{2}$. Substituting into $x^3 - 2x^2 + 3x + 1 = 0$:

$$\left(\frac{y+1}{2}\right)^3 - 2\left(\frac{y+1}{2}\right)^2 + 3\left(\frac{y+1}{2}\right) + 1 = 0.$$

Multiplying through by 8:

$$(y+1)^3 - 4(y+1)^2 + 12(y+1) + 8 = 0.$$

Expanding gives $y^3 - y^2 + 7y + 17 = 0$. Comparing with $x^3 - x^2 + 7x + C = 0$, we read off $C = 17$.

Q35 2020 Gosford Trial Q12(a) · **3 marks**

The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β and γ.

(i) Show that $\beta + \gamma = 1 - \alpha$.

(ii) Using part (i), and similar results, evaluate $\dfrac{\beta+\gamma}{\alpha} + \dfrac{\gamma+\alpha}{\beta} + \dfrac{\alpha+\beta}{\gamma}$.

Criteria	**Marks**
(i) Uses $\alpha + \beta + \gamma = 1$ to obtain $\beta + \gamma = 1 - \alpha$	1
(ii) Rewrites each term as $\dfrac{1}{\alpha} - 1$ (and similar) using part (i)	1
(ii) Evaluates $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{5}{2}$ and reaches $-\dfrac{1}{2}$	1

Solution.

(i) The sum of the roots is $\alpha + \beta + \gamma = -\dfrac{-1}{1} = 1$. Rearranging, $\beta + \gamma = 1 - \alpha$.

(ii) By part (i) and the symmetric results $\gamma + \alpha = 1 - \beta$ and $\alpha + \beta = 1 - \gamma$,

$$\frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} + \frac{\alpha+\beta}{\gamma} = \frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} + \frac{1-\gamma}{\gamma} = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3.$$

Now $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \dfrac{-5}{-2} = \dfrac{5}{2}$, so the value is $\dfrac{5}{2} - 3 = -\dfrac{1}{2}$.

Q36 2023 Sydney Grammar Prelim Q11(a) · 3 marks

The polynomial $P(x) = 2x^3 - x^2 + 16x - 8$ has zeroes α, β and γ. Find the value of:

(i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

(ii) $(\alpha + 1)(\beta + 1)(\gamma + 1)$

Criteria	Marks
States the elementary symmetric sums: $\alpha + \beta + \gamma = \frac{1}{2}$, $\alpha\beta + \alpha\gamma + \beta\gamma = 8$, $\alpha\beta\gamma = 4$	1
(i) Factorises as $\alpha\beta\gamma(\alpha + \beta + \gamma) = 2$	1
(ii) Expands and evaluates $(\alpha + 1)(\beta + 1)(\gamma + 1) = \frac{27}{2}$	1

Solution.

By Vieta's formulas for $2x^3 - x^2 + 16x - 8$:

$$\alpha + \beta + \gamma = \frac{1}{2}, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{16}{2} = 8, \qquad \alpha\beta\gamma = \frac{8}{2} = 4.$$

(i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma) = 4 \cdot \frac{1}{2} = 2.$

(ii) Expanding,

$$(\alpha + 1)(\beta + 1)(\gamma + 1) = \alpha\beta\gamma + (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) + 1 = 4 + 8 + \frac{1}{2} + 1 = \frac{27}{2}.$$

Q37 2022 Fort St Prelim Q10(b) · 3 marks

The three roots of $x^3 + px^2 + qx + r = 0$ are integers with a common difference of β. That is, if the roots are x_1, x_2 and x_3 then $x_2 - x_1 = x_3 - x_2 = \beta$. Show that $9pq = 27r + 2p^3$.

Criteria	Marks
Writes the roots as $\alpha - \beta,\ \alpha,\ \alpha + \beta$ and uses sum of roots to get $\alpha = -\frac{p}{3}$	1
Forms the pair-sum $q = 3\alpha^2 - \beta^2$ and product $-r = \alpha(\alpha^2 - \beta^2)$	1
Eliminates α, β to obtain $9pq = 27r + 2p^3$	1

Solution.

Since the roots are in arithmetic progression, write them as $\alpha - \beta,\ \alpha,\ \alpha + \beta$. Then:

$$\text{sum} = (\alpha - \beta) + \alpha + (\alpha + \beta) = 3\alpha = -p \implies \alpha = -\frac{p}{3}.$$

$$\text{pair-sum} = q = \alpha(\alpha - \beta) + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = 3\alpha^2 - \beta^2.$$

$$\text{product} = (\alpha - \beta)\alpha(\alpha + \beta) = \alpha(\alpha^2 - \beta^2) = -r.$$

From the pair-sum, $\alpha^2 - \beta^2 = q - 2\alpha^2$. Substituting into the product,

$$\alpha(q - 2\alpha^2) = -r \implies \alpha q - 2\alpha^3 = -r.$$

Now put $\alpha = -\frac{p}{3}$:

$$-\frac{p}{3}q - 2\left(-\frac{p}{3}\right)^3 = -r \implies -\frac{pq}{3} + \frac{2p^3}{27} = -r.$$

Multiplying through by -27:

$$9pq - 2p^3 = 27r \implies 9pq = 27r + 2p^3.$$

Q38 2022 Sydney Girls Prelim Q4(d) · **3 marks**

Suppose that the roots of $Q(x) = Ax^3 + Bx^2 + Cx + D$ are α, α and γ, where A, B, C and $D \in \mathbb{R}$, and $A \neq 0$. Show that $B\alpha^2 + 2C\alpha + 3D = 0$.

Criteria	**Marks**
Recognises α is a double root, so $Q(\alpha) = 0$ and $Q'(\alpha) = 0$	1
Writes $Q(\alpha) = A\alpha^3 + B\alpha^2 + C\alpha + D = 0$ and $Q'(\alpha) = 3A\alpha^2 + 2B\alpha + C = 0$	1
Eliminates A (e.g. $3Q(\alpha) - \alpha Q'(\alpha)$) to obtain $B\alpha^2 + 2C\alpha + 3D = 0$	1

Solution.

Since α is a repeated (double) root of $Q(x)$, both $Q(\alpha) = 0$ and $Q'(\alpha) = 0$:

$$Q(\alpha) = A\alpha^3 + B\alpha^2 + C\alpha + D = 0, \qquad Q'(\alpha) = 3A\alpha^2 + 2B\alpha + C = 0.$$

Eliminate the $A\alpha^3$ term by forming $3Q(\alpha) - \alpha\, Q'(\alpha)$:

$$3(A\alpha^3 + B\alpha^2 + C\alpha + D) - \alpha(3A\alpha^2 + 2B\alpha + C) = 0.$$

The $3A\alpha^3$ terms cancel, leaving

$$3B\alpha^2 + 3C\alpha + 3D - 2B\alpha^2 - C\alpha = B\alpha^2 + 2C\alpha + 3D = 0,$$

as required.

Q39 2021 Caringbah Trial Q13(b) · **4 marks**

One of the roots of the equation $x^3 + ax^2 + 1 = 0$ is equal to the sum of the other two roots.

(i) Using your knowledge of the sum and products of the roots, show that $x = -\dfrac{a}{2}$ is a root of the equation.

(ii) Find the value of a.

Criteria	Marks
(i) Uses sum of roots $= -a$ with one root equal to the sum of the other two to get the special root $= -\dfrac{a}{2}$	1
(ii) Substitutes $x = -\dfrac{a}{2}$ into the equation	1
(ii) Simplifies to $\dfrac{a^3}{8} = 1$ (equivalently $a^3 = 8$)	1
(ii) Concludes $a = -2$	1

Solution.

(i) Let the roots be r_1, r_2, r_3 with $r_3 = r_1 + r_2$. The sum of the roots is

$$r_1 + r_2 + r_3 = -a \implies 2r_3 = -a \implies r_3 = -\frac{a}{2}.$$

So $x = -\dfrac{a}{2}$ is a root.

(ii) Substituting $x = -\dfrac{a}{2}$ into $x^3 + ax^2 + 1 = 0$:

$$-\frac{a^3}{8} + a \cdot \frac{a^2}{4} + 1 = 0 \implies -\frac{a^3}{8} + \frac{a^3}{4} + 1 = 0 \implies \frac{a^3}{8} = -1.$$

Hence $a^3 = -8$, so $a = -2$.

Q40 2023 Sydney Grammar Prelim Q13(c) · 4 marks

Given that the polynomial $x^3 + px^2 + qx + r$ has zeroes α, β and $\alpha + \beta$, show that

$$4pq = p^3 + 8r.$$

Criteria	Marks
Sum of zeroes: $2(\alpha + \beta) = -p$, so $\alpha + \beta = -\frac{p}{2}$	1
Pair-sum: $\alpha\beta + (\alpha + \beta)^2 = q$, giving $\alpha\beta = q - \frac{p^2}{4}$	1
Product: $\alpha\beta(\alpha + \beta) = -r$	1
Substitutes and rearranges to $4pq = p^3 + 8r$	1

Solution.

The zeroes are α, β and $\alpha + \beta$.
Sum of zeroes: $\alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta) = -p$, so $\alpha + \beta = -\frac{p}{2}$.
Sum of products in pairs: $\alpha\beta + \alpha(\alpha + \beta) + \beta(\alpha + \beta) = \alpha\beta + (\alpha + \beta)^2 = q$, so

$$\alpha\beta = q - (\alpha + \beta)^2 = q - \frac{p^2}{4}.$$

Product of zeroes: $\alpha\beta(\alpha + \beta) = -r$. Substituting the two results,

$$\left(q - \frac{p^2}{4}\right)\left(-\frac{p}{2}\right) = -r \implies -\frac{pq}{2} + \frac{p^3}{8} = -r.$$

Multiplying through by -8:

$$4pq - p^3 = 8r \implies 4pq = p^3 + 8r,$$

as required.

Q41 2023 St George Girls Trial Q11(c) · 5 marks

The polynomial $P(x) = 2x^3 + mx^2 + nx - 20$ has a double root and $P(2) = P'(2) = 0$.

(i) Find the values of m and n.

(ii) Hence, find the other root of $P(x)$.

Criteria	Marks
(i) Forms $P(2) = 0$: $8m + 2n = -4$ (i.e. $4m + n = -2$)	1
(i) Forms $P'(2) = 0$: $24 + 4m + n = 0$	1
(i) Solves to $m = -13$, $n = 28$	1
(ii) Uses the product of roots $= \frac{20}{2} = 10$ with the double root $x = 2$	1
(ii) Concludes the other root is $x = \frac{5}{2}$	1

Solution.

(i) With $P(x) = 2x^3 + mx^2 + nx - 20$ and $P'(x) = 6x^2 + 2mx + n$:

$$P(2) = 16 + 4m + 2n - 20 = 0 \implies 4m + 2n = 4 \implies 2m + n = 2,$$

$$P'(2) = 24 + 4m + n = 0 \implies 4m + n = -24.$$

Subtracting, $2m = -26$, so $m = -13$ and then $n = 2 - 2(-13) = 28$.

(ii) Now $P(x) = 2x^3 - 13x^2 + 28x - 20$. The product of the roots is $-\frac{-20}{2} = 10$. The double root is $x = 2$, so if the third root is t then $2 \cdot 2 \cdot t = 10$, giving $t = \frac{5}{2}$. Thus the other root is $x = \frac{5}{2}$.