Year 11 Mathematics Extension 1

Topic: Permutations & Combinations (1 of 2)

Review Questions (Worked Solutions): Permutations & Combinations

Term 3, 2026

Counting arrangements

1. Multiplication principle & basic counting

Q1 2020 Cherrybrook Tech Trial Q1 · **1 mark**

A PIN is to be chosen using four of the digits 0 through 9. Digits cannot be repeated. How many possible PINs are there?

(A) 256

(B) 210

(C) 10 000

(D) 5040

Criteria	Marks
correct option (D)	1

Solution.

Four of the ten digits 0 to 9 are chosen and placed in order, with no repeats, so this is an arrangement of 4 objects taken from 10:

$$^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040.$$

The answer is **(D)**.

Q2 2025 Hornsby Girls Trial Q3 · **1 mark**

A four-digit number is formed by choosing four counters from the set $\{9, 8, 7, 6, 5, 4\}$ and placing them in order. How many numbers greater than 6 000 can be formed?

(A) 240

(B) 360

(C) 840

(D) 3024

Criteria	Marks
correct option (A)	1

Solution.

Four counters are drawn from $\{9, 8, 7, 6, 5, 4\}$ and placed in order, so no digit repeats. For the number to exceed 6000 the leading (thousands) digit must be 6, 7, 8 or 9: that is 4 choices. Since every digit in the set is at least 4, any of these leading digits already forces a number of at least 6000, and because the smallest available remaining digits still give a value strictly above 6000, all four leading choices are valid.

Fill the remaining three places from the 5 digits left, in order:

$$4 \times (5 \times 4 \times 3) = 4 \times 60 = 240.$$

The answer is **(A)**.

Q3 2021 Sydney Girls Prelim Q1(a) · **1 mark**

The number plates for scooters appear as 3 letters followed by 2 numbers such as MAX21. How many different number plates are possible?

Criteria	Marks
correct count 1 757 600	1

Solution.

By the multiplication principle, each of the three letter positions can be any of the 26 letters and each of the two number positions can be any of the 10 digits, with repetition allowed:

$$26 \times 26 \times 26 \times 10 \times 10 = 26^3 \times 10^2 = 1\,757\,600.$$

Q4 2024 North Sydney Boys Trial Q11(d) · **2 marks**

How many numbers greater than 8000 can be formed from the digits 1, 2, 4, 6, 9 if no digit is repeated?

Criteria	**Marks**
counts the four-digit numbers greater than 8000 (leading digit 9): 24	1
adds the five-digit numbers (5! = 120) for a total of 144	1

Solution.

The five available digits are $1, 2, 4, 6, 9$ with no repeats. A number formed from them is greater than 8000 in two separate cases.

Case 1: four-digit numbers greater than 8000**.** The leading digit must be at least 8. The only digit available that is 8 or more is 9, so the thousands digit is fixed as 9 (1 way), and the remaining three places are filled from the other four digits:

$$1 \times 4 \times 3 \times 2 = 24.$$

Case 2: five-digit numbers. Every five-digit number made from these digits is at least 12 000, hence automatically greater than 8000. All five distinct digits are arranged:

$$5! = 120.$$

Adding the two disjoint cases:

$$24 + 120 = 144.$$

Q5 2023 Trinity Grammar Trial Q11(e) · **2 marks**

A 6-character password is to be chosen from the following characters.

Digits	4	5	7
Letters	T	G	S
Symbols	#	&	%

No character may be used more than once in any password. Find the number of different passwords that may be chosen if the password starts with two letters and ends with two digits.

Criteria	**Marks**
arranges the two letters and the two digits in their fixed blocks: ${}^3P_2 \times {}^3P_2$	1
fills the two middle places from the remaining characters and multiplies to 720	1

Solution.

The password has six positions in the fixed pattern

$$\underbrace{L\,L}_{\text{letters}}\ \underbrace{\square\,\square}_{\text{any}}\ \underbrace{D\,D}_{\text{digits}},$$

and no character is used more than once. There are 3 letters (T, G, S), 3 digits $(4, 5, 7)$ and 3 symbols $(\#, \&, \%)$.

First two places (letters). Choose and order 2 of the 3 letters: ${}^3P_2 = 3 \times 2 = 6$.

Last two places (digits). Choose and order 2 of the 3 digits: ${}^3P_2 = 3 \times 2 = 6$.

Middle two places. After using 2 letters and 2 digits, $9 - 4 = 5$ characters remain (the 1 leftover letter, 1 leftover digit, and all 3 symbols). Order 2 of these 5: ${}^5P_2 = 5 \times 4 = 20$.

By the multiplication principle:

$${}^3P_2 \times {}^5P_2 \times {}^3P_2 = 6 \times 20 \times 6 = 720.$$

2. Basic arrangements (nPr, distinct)

Q6 2023 Penrith Trial Q14(a) · 1 mark

In how many ways can an Academic Award, Citizenship Award and Integrity Award be awarded in a class of 24 students?

Criteria	Marks
correct answer ${}^{24}P_3 = 12\,144$	1

Solution.

Three different awards are given to three different students from a class of 24, and no student receives two of these awards. The awards are distinct, so order matters: this is an arrangement of 3 students taken from 24:

$$ {}^{24}P_3 = 24 \times 23 \times 22 = 12\,144. $$

Q7 2023 Gosford Trial Q5 · 1 mark

In how many ways can 5 people be selected from a group of 6 and then arranged in a line so that the two oldest people in the selected group are at either end of the line? (Note, no two people are the same age.)

(A) 720

(B) 144

(C) 72

(D) 36

Criteria	Marks
correct option (C)	1

Solution.

Five people are selected from 6 and arranged in a line so the two oldest of the selected five occupy the two ends.

Choose the five. ${}^{6}C_5 = 6$ ways.

Place the two oldest at the ends. Within any chosen group of five, the two oldest are fixed people; they go on the two ends, which can be done in $2! = 2$ ways (left and right).

Arrange the middle three. The remaining 3 people fill the 3 inner seats: $3! = 6$ ways.

$$ {}^{6}C_5 \times 2! \times 3! = 6 \times 2 \times 6 = 72. $$

The answer is **(C)**.

Q8 2020 Hunters Hill Trial Q10 · **1 mark**

The letters A, B, C and D are used to form a four letter word. How many words can be written such that D comes before A?

(A) 3

(B) 6

(C) 12

(D) 24

Criteria	Marks
correct option (C)	1

Solution.

The four distinct letters A, B, C, D can be arranged in $4! = 24$ ways with no restriction. By symmetry, in exactly half of these arrangements D comes before A and in the other half A comes before D (swapping the positions of A and D pairs the arrangements one-to-one). Hence

$$\frac{4!}{2} = \frac{24}{2} = 12.$$

The answer is **(C)**.

Q9 2020 Baulkham Hills Prelim Q11(d)(iii) · **1 mark**

A group of 12 students is to be divided into discussion groups. In how many ways can a president and vice president be chosen for the group of 12?

Criteria	Marks
correct count 132	1

Solution.

Two distinct roles are filled from 12 people, so order matters. There are 12 choices for president and then 11 for vice president:

$$^{12}P_2 = 12 \times 11 = 132.$$

Q10 2022 Killara Trial Q12(f) · **2 marks**

A set of encyclopedias numbered 1 to 13 is placed randomly on a bookshelf. In how many ways can they be placed if encyclopedias 1 and 2 are in the correct order, but not necessarily next to each other?

Criteria	**Marks**
recognises there are 13! total arrangements	1
halves for the fixed relative order of encyclopedias 1 and 2: $\frac{13!}{2} = 3\,113\,510\,400$	1

Solution.

The 13 distinct encyclopedias can be placed on the shelf in 13! ways with no restriction. The condition is only that encyclopedia 1 appears before encyclopedia 2 (in the correct order), not that they are adjacent.

For every arrangement, either 1 precedes 2 or 2 precedes 1, and these two cases are in one-to-one correspondence by swapping the two books. So exactly half of all arrangements have 1 before 2:

$$\frac{13!}{2} = \frac{6\,227\,020\,800}{2} = 3\,113\,510\,400.$$

3. Arrangements with identical objects

Q11 2022 Abbotsleigh Trial Q1 · 1 mark

How many distinct arrangements of the letters of the word ARRANGEMENT are possible?

(A) 2 494 800

(B) 4 989 600

(C) 9 979 200

(D) 39 916 800

Criteria	Marks
correct option (A)	1

Solution.

ARRANGEMENT has 11 letters, in which A, R, N and E each appear twice (G, M, T appear once). Dividing the 11! orderings by the repeats of each doubled letter,

$$\frac{11!}{2!\,2!\,2!\,2!} = \frac{39\,916\,800}{16} = 2\,494\,800,$$

which is option (A).

Q12 2023 HSC Q11(b) · 2 marks

In how many different ways can all the letters of the word CONDOBOLIN be arranged in a line?

Criteria	Marks
correct total number of letters (10) and identifies repeats (O appears 3 times, N appears 2 times)	1
evaluates $\frac{10!}{3!\,2!} = 302\,400$	1

Solution.

The word CONDOBOLIN has 10 letters. Counting repeats: O appears 3 times, N appears 2 times, and C, D, B, L, I each appear once.

Arrangements of 10 objects with these repeated letters:

$$\frac{10!}{3!\,2!} = \frac{3\,628\,800}{6 \times 2} = \frac{3\,628\,800}{12} = 302\,400.$$

Q13 2025 Abbotsleigh Trial Q11(c) · **2 marks**

In how many different ways can all the letters of the word TENNANT be arranged in a line?

Criteria	**Marks**
identifies the repeats (N appears 3 times, T appears 2 times) among the 7 letters	1
evaluates $\frac{7!}{3!\,2!} = 420$	1

Solution.

The word TENNANT has 7 letters. Counting repeats: N appears 3 times, T appears 2 times, and E, A each appear once.

Arrangements of 7 objects with these repeats:

$$\frac{7!}{3!\,2!} = \frac{5040}{6 \times 2} = \frac{5040}{12} = 420.$$

Q14 2023 Sydney Grammar Trial Q11(c) · **2 marks**

Which boy's name has more unique arrangements of letters: MILO or SAVVAS? Show your working.

Criteria	**Marks**
computes the arrangements of MILO: $4! = 24$	1
computes the arrangements of SAVVAS ($\frac{6!}{2!\,2!\,2!} = 90$) and concludes SAVVAS has more	1

Solution.

Count the distinct arrangements of each name and compare.

MILO. All 4 letters are different, so the number of arrangements is

$$4! = 24.$$

SAVVAS. There are 6 letters, with S appearing 2 times, A appearing 2 times and V appearing 2 times. So the number of distinct arrangements is

$$\frac{6!}{2!\,2!\,2!} = \frac{720}{2 \times 2 \times 2} = \frac{720}{8} = 90.$$

Since $90 > 24$, **SAVVAS** has more unique arrangements of its letters.

Q15 2024 Baulkham Hills Prelim Q14(b)(i) · **2 marks**

Consider the word PHOTOGRAPHS. How many arrangements of the letters of the word are possible?

Criteria	Marks
divides 11! by the arrangements of the repeated letters	1
correct count 4 989 600	1

Solution.

PHOTOGRAPHS has 11 letters, of which P, H and O each appear twice (T, G, R, A, S appear once). Dividing the 11! orderings by the arrangements of each doubled letter,

$$\frac{11!}{2!\,2!\,2!} = \frac{39\,916\,800}{8} = 4\,989\,600.$$

4. Basic circular arrangements

Q16 2022 Girraween Trial Q13(c)(i) · **1 mark**

A class of fifteen Mathematics students are seated around a circular table to discuss Mathematical problems. In how many different ways can the students be arranged?

Criteria	Marks
Correct answer 14!	1

Solution.

Fifteen distinct students are seated around a circular table. For a circular arrangement of n distinct objects we fix one person to remove the rotational symmetry, then arrange the remaining $n-1$ people:

$$(15-1)! = 14! = 87\,178\,291\,200.$$

Q17 2025 North Sydney Boys Trial Q11(d)(i) · **1 mark**

The letters from the word VOLUME are placed at random on the circumference of a circle. Each letter is used only once. Find the number of different arrangements that can be formed.

Criteria	Marks
Correct answer $5! = 120$	1

Solution.

The six distinct letters of VOLUME are placed around a circle. A circular arrangement of n distinct objects has $(n-1)!$ arrangements, since rotations of the same order are not counted separately. With $n = 6$:

$$(6-1)! = 5! = 120.$$

Q18 2021 Knox Trial Q8 · **1 mark**

A team of 7 is to be selected from 10 people and then sat around a circular table. In how many ways can this be done?

(A) $\dfrac{10!}{7!\,3!} \times 7!$

(B) $\dfrac{10!}{7!\,3!} \times \dfrac{1}{5!}$

(C) $\dfrac{10!}{7 \times 3!}$

(D) $\dfrac{10!}{7!\,3!}$

Criteria	Marks
Correct answer (C)	1

Solution.

This is a two-step job: first choose the team, then seat it. Choose 7 people from 10 in ${}^{10}C_7 = \dfrac{10!}{7!\,3!}$ ways. Seat those 7 around a circular table in $(7-1)! = 6!$ ways. Multiply:

$$\frac{10!}{7!\,3!} \times 6! = \frac{10! \cdot 6!}{7! \cdot 3!} = \frac{10!}{7 \times 6! \times 3!} \times 6! = \frac{10!}{7 \times 3!}.$$

This equals 86 400, matching option (C).

Q19 2025 Blacktown Boys Trial Q8 · **1 mark**

In how many ways can 8 people from a group of 21 people be chosen and then arranged in a circle?

(A) $\dfrac{20!}{12!}$

(B) $\dfrac{20!}{12! \times 8}$

(C) $\dfrac{21!}{13!}$

(D) $\dfrac{21!}{13! \times 8}$

Criteria	Marks
Correct answer (D)	1

Solution.

Choose 8 people from 21, then seat them around a circle. The number of choices is ${}^{21}C_8 = \dfrac{21!}{8!\,13!}$, and each chosen group can be seated in a circle in $(8-1)! = 7!$ ways:

$$\frac{21!}{8!\,13!} \times 7! = \frac{21! \cdot 7!}{8! \cdot 13!} = \frac{21!}{8 \times 7! \times 13!} \times 7! = \frac{21!}{13! \times 8}.$$

This is option (D).

Q20 2020 International Grammar Prelim Q11(e)(i) · **1 mark**

Four Year 12 students, three Year 11 students, and two Year 10 students are seated around a circular table randomly. How many seating arrangements are possible?

Criteria	Marks
correct count 40 320	1

Solution.

There are $4 + 3 + 2 = 9$ students, all distinct, seated around a circular table. Circular arrangements of n distinct objects number $(n-1)!$, so

$$(9-1)! = 8! = 40\,320.$$

Arrangements with a restriction

5. Keeping a group together (block method)

Q21 2019 HSC Q8 · 1 mark

In how many ways can all the letters of the word PARALLEL be placed in a line with the three Ls together?

(A) $\dfrac{6!}{2!}$

(B) $\dfrac{6!}{2!\,3!}$

(C) $\dfrac{8!}{2!}$

(D) $\dfrac{8!}{2!\,3!}$

Criteria	Marks
Correct answer (A)	1

Solution.

PARALLEL has the letters P, A, A, R, L, L, L, E: eight letters with the A repeated twice and the L repeated three times. Treat the three Ls as one block, so there are 6 items to arrange in a line: P, A, A, R, [LLL], E. These have a repeated A, and the three Ls inside the block are identical (so no internal factor). The count is

$$\frac{6!}{2!}.$$

This is option (A), equal to 360.

Q22 2020 Abbotsleigh Trial Q1 · **1 mark**

In how many different ways can 5 boys and 3 girls stand in a queue if the girls all stand next to each other?

(A) 40 320

(B) 720

(C) 4 320

(D) 3 136

Criteria	Marks
Correct answer (C)	1

Solution.

Treat the 3 girls as a single block. That block plus the 5 boys gives 6 items to arrange in a queue: 6! ways. Inside the block the 3 girls can be ordered in 3! ways. Multiply:

$$6! \times 3! = 720 \times 6 = 4\,320.$$

This is option (C).

Q23 2020 James Ruse Trial Q3 · **1 mark**

How many arrangements of the letters of the word OLYMPIC are possible if the C and the L are to be together?

(A) 120

(B) 720

(C) 240

(D) 1440

Criteria	Marks
Correct answer (D)	1

Solution.

OLYMPIC has 7 distinct letters. Treat C and L as a single block, giving 6 items to arrange: 6! ways. Inside the block, C and L can be ordered in 2! ways (CL or LC). Multiply:

$$6! \times 2! = 720 \times 2 = 1\,440.$$

This is option (D).

Q24 2023 Cranbrook Trial Q11(e) · **2 marks**

In how many ways can the letters of the word POSSIBILITY be arranged so that the two Ss are together?

Criteria	**Marks**
Blocks the two Ss and sets up $\frac{10!}{r!}$ form (accounts for the three Is)	1
Correct answer $\frac{10!}{3!} = 604\,800$	1

Solution.

POSSIBILITY has 11 letters: P, O, S, S, I, I, I, B, L, T, Y, with the S repeated twice and the I repeated three times. To keep the two Ss together, treat SS as a single block. This leaves 10 items to arrange in a line: [SS], P, O, I, I, I, B, L, T, Y. The two Ss inside the block are identical, so there is no internal factor; the only remaining repeat is the three identical Is. The count is

$$\frac{10!}{3!} = 604\,800.$$

Q25 2024 Fort St Trial Q12(a) · **2 marks**

There are four teams consisting of two tennis players in each team. They are seated around a circular table. In how many ways can the teams be arranged around the table if the players in each team are to stay together?

Criteria	**Marks**
Blocks each team and arranges the 4 blocks around the table: $(4-1)! = 3!$	1
Multiplies by the internal orders $(2!)^4$ to get 96	1

Solution.

Each of the four teams of two must stay together, so treat each team as a single block. Seating the 4 blocks around a circular table gives $(4-1)! = 3!$ arrangements. Within each team the two players can swap in 2! ways, and there are four teams, contributing $(2!)^4$. Multiply:

$$3! \times (2!)^4 = 6 \times 16 = 96.$$

Q26 2024 Killara Trial Q11(c) · **3 marks**

Three men and three women are to sit at a round table. How many seating arrangements are possible:

(i) If there are no restrictions?

(ii) If two particular women must sit together?

Criteria	**Marks**
(i) Correct answer $5! = 120$	1
(ii) Blocks the two women and arranges the resulting 5 units around the table: $(5-1)! = 4!$	1
(ii) Multiplies by the $2!$ internal orders to get 48	1

Solution.

(i) Six people are seated around a round table with no restrictions. A circular arrangement of n people gives $(n-1)!$ orders, so

$$(6-1)! = 5! = 120.$$

(ii) Treat the two particular women as a single block. This block plus the other 4 people gives 5 units to seat around the table, in $(5-1)! = 4!$ ways. Inside the block the two women can swap in $2!$ ways. Multiply:

$$4! \times 2! = 24 \times 2 = 48.$$

Q27 2024 Baulkham Hills Trial Q12(c) · **4 marks**

In how many ways can the letters of the word DIFFERENCE be arranged if:

(i) there are no restrictions?

(ii) all three Es are next to each other with no letters separating them, and this string of three Es are somewhere between the two Fs?

Criteria	**Marks**
(i) Correct answer $\dfrac{10!}{2!\,3!} = 302\,400$	2
(ii) Blocks the three Es and arranges the 8 items as $\dfrac{8!}{2!} = 20\,160$	1
(ii) Applies the $\frac{1}{3}$ symmetry (E-block between the two Fs) for 6 720	1

Solution.

DIFFERENCE has 10 letters: D, I, F, F, E, E, E, R, N, C, with the F repeated twice and the E repeated three times.

(i) With no restrictions the number of arrangements is

$$\frac{10!}{2!\,3!} = 302\,400.$$

(ii) First keep the three Es together by treating EEE as a single block. That leaves 8 items to arrange: D, I, F, F, R, N, C, [EEE], where the two Fs are identical (the three Es inside the block are identical, so no internal factor). The number of such arrangements is

$$\frac{8!}{2!} = 20\,160.$$

Now impose that the E-block sits between the two Fs. Look only at the relative order of the three tokens F, F and [EEE]. Their three distinct orderings, FF[EEE], F[EEE]F and [EEE]FF, are equally likely, and exactly one of them, F[EEE]F, places the Es between the two Fs. So we take one third:

$$\frac{1}{3} \times \frac{8!}{2!} = \frac{20\,160}{3} = 6\,720.$$

6. Keeping objects apart (gap method)

Q28 2021 Newington Trial Q4 · **1 mark**

In how many ways can the letters of the word OLYMPICS be arranged if the 2 vowels are not next to each other?

(A) 40320

(B) 30240

(C) 10080

(D) 2520

Criteria	Marks
Correct answer (D)	1

Solution.

OLYMPICS has 8 distinct letters, with vowels O and I. Use the gap method: keep the two vowels apart.

Arrange the 6 consonants (L, Y, M, P, C, S) in a line: $6! = 720$ ways. This creates 7 gaps (including the two ends). Slot the two different vowels O and I into 2 of these 7 gaps, order mattering: ${}^{7}P_{2} = 42$ ways.

$$6! \times {}^{7}P_{2} = 720 \times 42 = 30\,240.$$

Answer: (D) 30240. (Check: total $8! = 40320$ minus vowels-together $2 \times 7! = 10080$ gives 30240.)

Q29 2020 North Sydney Girls Trial Q11 · **2 marks**

Donald, Melania and five other people get on a bus one at a time. In how many ways can the seven people board the bus if Melania refuses to get on immediately following Donald?

Criteria	Marks
Counts the arrangements with Melania immediately after Donald (6!)	1
Subtracts from total: 7! – 6! = 4320	1

Solution.

Seven distinct people board in order, so there are 7! = 5040 orders in total. The forbidden situation is Melania boarding immediately after Donald, that is, the block DM appearing with M directly following D.

Treat this DM block as a single unit. Then there are 6 units (the DM block plus the other 5 people) to arrange in 6! = 720 ways. (The block is fixed as D then M, so there is no internal factor of 2.)

$$7! - 6! = 5040 - 720 = 4320.$$

Q30 2024 James Ruse Trial Q12(a) · **2 marks**

Three adults and five children go to the cinema and are seated next to each other in a row of eight seats. How many ways can these eight people sit so that at least two of the adults sit next to each other?

Criteria	Marks
Uses complement: total arrangements minus the no-two-adults-adjacent count (gap method)	1
Correct total 25 920	1

Solution.

Eight distinct people in a row; ”at least two adults adjacent” is the complement of ”no two adults adjacent.” Total arrangements = 8! = 40 320. **No two adults adjacent (gap method):** seat the 5 children first in 5! = 120 ways. This creates 6 gaps (including the ends); place the 3 distinct adults into 3 separate gaps in ${}^6P_3 = 6 \times 5 \times 4 = 120$ ways. So $120 \times 120 = 14\,400$. Answer = 40 320 – 14 400 = 25 920.

Q31 2025 Barker Trial Q12 · 3 marks

The letters of the word SUCCESS are to be arranged.

(i) In how many ways can the letters of the word SUCCESS be arranged in a line if there are no restrictions?

(ii) In how many ways can the letters of the word SUCCESS be arranged in a line if the two Cs are NOT adjacent?

Criteria	Marks
(i) Total arrangements $\frac{7!}{3!\,2!} = 420$	1
(ii) Arranges the 5 non-C letters ($\frac{5!}{3!}$) and slots the Cs into gaps	1
(ii) Correct answer 300	1

Solution.

SUCCESS has 7 letters: S×3, C×2, U, E.

(i) With three identical Ss and two identical Cs,

$$\frac{7!}{3!\,2!} = \frac{5040}{12} = 420.$$

(ii) Gap method to keep the two Cs apart. Arrange the other 5 letters first (S, S, S, U, E, which has three identical Ss):

$$\frac{5!}{3!} = 20 \text{ ways.}$$

This gives 6 gaps. The two Cs are identical, so choose 2 gaps (order does not matter): ${}^6C_2 = 15$.

$$20 \times 15 = 300.$$

(Check: total 420 minus Cs-together $\frac{6!}{3!} = 120$ gives 300.)

Q32 2022 Cherrybrook Tech Trial Q11(b) · 3 marks

The letters of the word EXAMINATION are to be arranged in a row.

(i) How many eleven letter words can be made?

(ii) How many eleven letter words will NOT contain adjacent N's?

Criteria	Marks
(i) Total arrangements $\frac{11!}{2!\,2!\,2!} = 4\,989\,600$	1
(ii) Arranges the 9 non-N letters ($\frac{9!}{2!\,2!}$) and slots the Ns into gaps	1
(ii) Correct answer 4 082 400	1

Solution.

EXAMINATION has 11 letters: A×2, I×2, N×2, and E, X, M, T, O single.

(i)

$$\frac{11!}{2!\,2!\,2!} = \frac{39\,916\,800}{8} = 4\,989\,600.$$

(ii) Gap method to keep the two Ns apart. Arrange the other 9 letters first (E, X, A, M, I, A, T, I, O, which has A×2 and I×2):

$$\frac{9!}{2!\,2!} = \frac{362\,880}{4} = 90\,720 \text{ ways.}$$

This gives 10 gaps. The two Ns are identical, so choose 2 gaps: ${}^{10}C_2 = 45$.

$$90\,720 \times 45 = 4\,082\,400.$$

(Check: total 4 989 600 minus Ns-together $\frac{10!}{2!\,2!} = 907\,200$ gives 4 082 400.)

Q33 2023 Baulkham Hills Trial Q11 · **3 marks**

At a dinner party, the host, hostess and their 6 guests sit at a round table. In how many ways can they be arranged if:

(i) the host and hostess are separated?

(ii) the host and hostess are sitting directly opposite each other?

Criteria	**Marks**
(i) Host and hostess separated: $7! - 2 \times 6! = 3600$	1
(ii) Host and hostess opposite: fixes host, hostess opposite, remaining 6!	1
(ii) Correct answer 720	1

Solution.

8 people (host, hostess, 6 guests) sit at a round table. Total circular arrangements = $(8-1)! = 7! = 5040$.

(i) Separated (not next to each other) = total minus together. Glue host and hostess as a block: $(7-1)! = 6!$ circular arrangements of the 7 units, times 2! for the internal order of the pair.

$$7! - 2 \times 6! = 5040 - 1440 = 3600.$$

(ii) Directly opposite. Fix the host in any seat (this removes the rotational symmetry). At an 8-seat table there is exactly one seat directly opposite, which the hostess must take. The remaining 6 guests fill the remaining 6 seats in $6! = 720$ ways.

$$6! = 720.$$

Q34 2025 Roseville Trial Q13 · **3 marks**

There are seven distinct people who are to be seated around a round table. Of these seven people, Alice and Bob must sit next to each other, while Charlie and Dean must not sit next to each other. In how many ways could these seven people be seated with the above conditions?

Criteria	Marks
Counts A,B together at round table: $2 \times 5! = 240$	1
Subtracts arrangements where C,D are also together: $2 \times 2 \times 4! = 96$	1
Correct answer 144	1

Solution.

7 distinct people at a round table. Handle the ”together” condition first, then subtract to enforce the ”apart” condition.

Step 1: A and B together. Glue A and B into a block. There are then 6 units around the table: $(6-1)! = 5! = 120$ circular arrangements, times 2! for A and B internally.

$$2 \times 5! = 240.$$

Step 2: subtract A,B together AND C,D together. Glue both pairs into blocks. There are then 5 units: $(5-1)! = 4! = 24$ circular arrangements, times 2! for A,B and 2! for C,D.

$$2 \times 2 \times 4! = 96.$$

Result: (A,B together) minus (both pairs together)

$$240 - 96 = 144.$$

7. Alternating arrangements

Q35 2020 Riverview Trial Q5 · **1 mark**

Four males and four females are to sit around a table. In how many ways can this be done if the males and females alternate?

(A) 144

(B) 2880

(C) 5040

(D) 40 320

Criteria	Marks
Correct answer (A)	1

Solution.

4 males and 4 females around a table, alternating. Fix an alternating pattern: seat the 4 males first around the circular table. This uses $(4-1)! = 3! = 6$ arrangements (circular, so one position is fixed to remove rotations).

Seating the males creates 4 gaps between them, one for each female. Place the 4 distinct females into these 4 gaps: $4! = 24$ ways.

$$3! \times 4! = 6 \times 24 = 144.$$

Answer: (A) 144.

Q36 2020 Newcastle Grammar Trial Q7 · **1 mark**

Four female and four male students are to be seated around a circular table. In how many ways can this be done if the males and females must alternate?

(A) $4! \times 4!$

(B) $3! \times 4!$

(C) $3! \times 3!$

(D) $2 \times 3! \times 3!$

Criteria	Marks
Correct answer (B)	1

Solution.

4 females and 4 males around a circular table, alternating. Fix an alternating pattern: seat the 4 males around the table first, $(4-1)! = 3!$ ways (circular). This creates 4 gaps; place the 4 distinct females into them, $4!$ ways.

$$3! \times 4!.$$

Answer: (B) $3! \times 4!$. (Numerically $6 \times 24 = 144$.)

Q37 2022 Blacktown Boys Trial Q1 · 1 mark

Three women and three men are to be seated around a circular table. In how many ways can this be done if the men and women must alternate?

(A) $3! \times 2!$

(B) $2! \times 2!$

(C) $5!$

(D) $6!$

Criteria	Marks
Correct answer (A)	1

Solution.

3 women and 3 men around a circular table, alternating. Fix an alternating pattern: seat the 3 men around the table first, $(3-1)! = 2!$ ways (circular). This creates 3 gaps; place the 3 distinct women into them, $3!$ ways.

$$2! \times 3! = 2 \times 6 = 12.$$

Answer: (A) $3! \times 2!$.

Q38 2021 Hurlstone Trial Q11 · 1 mark

Four politicians and four activists are seated alternately at a round table (i.e. alternating so that no two politicians or two activists sit next to each other). In how many ways can this be done?

Criteria	Marks
Correct answer 144	1

Solution.

4 politicians and 4 activists at a round table, alternating (no two of the same type adjacent). Fix an alternating pattern: seat the 4 politicians around the table first, $(4-1)! = 3! = 6$ ways (circular). This creates 4 gaps between them; place the 4 distinct activists into these gaps, $4! = 24$ ways.

$$3! \times 4! = 6 \times 24 = 144.$$

Q39 2025 Normanhurst Boys Trial Q11(d) · **5 marks**

A group of six males and five females, including Aden and Ina, are to be placed into two separate lines. Line A consists of six people and line B consists of five people. How many ways can they be arranged across the two lines if:

(i) Aden and Ina are to be placed in separate lines?

(ii) The members in line A are selected first and must alternate between males and females within A?

Criteria	**Marks**
(i) Sets up separate-line count (Aden in one line, Ina in the other) with remaining 9 placed	1
(i) Correct answer $(6\times5+5\times6)\times9! = 21\,772\,800$	1
(ii) Identifies the two alternating patterns for line A and counts each with ${}^6P_3\times{}^5P_3$	1
(ii) Line A total $2\times{}^6P_3\times{}^5P_3 = 14\,400$	1
(ii) Arranges the remaining 5 in line B and gives $14\,400\times5! = 1\,728\,000$	1

Solution.

11 people (6 males including Aden, 5 females including Ina) fill two ordered lines: Line A (6 positions) and Line B (5 positions). With all positions distinct, the unrestricted total is 11!.

(i) Aden and Ina in separate lines. Place Aden and Ina first, one in each line, then fill the rest.

- Aden in Line A (6 position choices) and Ina in Line B (5 position choices): 6×5 ways; or
- Aden in Line B (5 choices) and Ina in Line A (6 choices): 5×6 ways.

The remaining 9 people fill the remaining 9 positions in 9! ways.

$$(6\times5+5\times6)\times9! = 60\times362\,880 = 21\,772\,800.$$

(ii) Line A selected first and must alternate male/female. Line A has 6 ordered positions. An alternating line of even length uses one of two patterns:

- **MFMFMF:** 3 males in the 3 male-positions and 3 females in the 3 female-positions. Choose and order: ${}^6P_3\times{}^5P_3 = 120\times60 = 7200$.
- **FMFMFM:** same count by symmetry, ${}^5P_3\times{}^6P_3 = 7200$.

So Line A can be filled in $2\times{}^6P_3\times{}^5P_3 = 14\,400$ ways. The remaining $11-6=5$ people fill Line B (5 ordered positions) with no restriction: $5! = 120$.

$$14\,400\times5! = 14\,400\times120 = 1\,728\,000.$$

8. Fixed positions / relative order / numbers

Q40 2022 Sydney Tech Trial Q1 · 1 mark

The number of different arrangements of the letters of the word NORTHERN which begin and end with the letter N is:

(A) $\dfrac{8!}{2!}$

(B) $\dfrac{6!}{2!}$

(C) $\dfrac{8!}{2!\,2!}$

(D) $\dfrac{6!}{2!\,2!}$

Criteria	Marks
Correct answer (B)	1

Solution.

NORTHERN has 8 letters with N:2, R:2 and O,T,H,E each once. Fixing an N at the front and an N at the back uses both Ns, leaving the 6 letters O, R, T, H, E, R to fill the middle six positions. Those six contain a repeated R, so the count is

$$\frac{6!}{2!} = 360.$$

The answer is (B).

Q41 2022 Glenwood Trial Q13(b)(ii) · 1 mark

Eight students, including Ari, Bobbi, and Cali, form a single file line and walk into their classroom. How many ways can the students walk into the classroom if Ari enters before Bobbi?

Criteria	Marks
Correct value $\dfrac{8!}{2} = 20160$	1

Solution.

Eight distinct students can enter in 8! orders. For any arrangement of the other six students, the pair Ari and Bobbi is equally likely to appear in the order (Ari, Bobbi) or (Bobbi, Ari), so exactly half of all orderings have Ari before Bobbi:

$$\frac{8!}{2} = \frac{40320}{2} = 20160.$$

Q42 2024 Trinity Grammar Trial Q5 · **1 mark**

Using the digits 0, 1, 2, 3, 4 and 5, how many odd numbers between 2000 and 5000 can be formed if no digits are repeated?

(A) 48

(B) 96

(C) 120

(D) 360

Criteria	Marks
Correct answer (B)	1

Solution.

We form 4-digit numbers from 0, 1, 2, 3, 4, 5 (no repeats) that are odd and satisfy $2000 < n < 5000$. **Thousands digit** must be 2, 3 or 4 (to land strictly between 2000 and 5000).

Case: thousands digit is 3. The number is odd, so the units digit must be odd: 1 or 5 (2 choices; 3 is used). The two middle digits come from the 4 remaining digits: $4 \times 3 = 12$. Total $2 \times 12 = 24$.

Case: thousands digit is 2 or 4 (even). The units digit must be odd, chosen from $\{1, 3, 5\}$: 3 choices. The two middle digits come from the 4 remaining digits: $4 \times 3 = 12$. Each thousands choice gives $3 \times 12 = 36$, and there are 2 such thousands digits: $2 \times 36 = 72$.

Total = 24 + 72 = 96. The answer is (B).

Q43 2022 St George Girls Trial Q11(c) · **2 marks**

In how many ways can the word GEOMETRY be arranged in a straight line if the vowels must occupy the 2nd, 4th and 6th position?

Criteria	**Marks**
Arranges the 3 vowels (E, E, O) in positions 2, 4, 6 as $\frac{3!}{2!}$	1
Arranges the 5 consonants in the remaining positions (5!) and multiplies for 360	1

Solution.

GEOMETRY has 8 letters: vowels E, O, E and consonants G, M, T, R, Y.

Vowels. The three vowels occupy the 2nd, 4th and 6th positions. Since two of them are the same (E, E), the number of distinct arrangements is

$$\frac{3!}{2!} = 3.$$

Consonants. The five distinct consonants fill the remaining five positions (1st, 3rd, 5th, 7th, 8th) in $5! = 120$ ways.

By the multiplication principle,

$$\frac{3!}{2!} \times 5! = 3 \times 120 = 360.$$

Q44 2021 Girraween Prelim Q2(b) · **2 marks**

How many numbers greater than 3 000 can be formed from the digits 1, 3, 5, 7, 9 if no repetition is permitted?

Criteria	**Marks**
counts the four-digit numbers greater than 3 000: $4 \times 4 \times 3 \times 2 = 96$	1
adds the five-digit numbers ($5! = 120$) for a total of 216	1

Solution.

The five available digits are $1, 3, 5, 7, 9$ with no repeats. Any three-digit (or shorter) number made from them is at most $975 < 3\,000$, so only four-digit and five-digit numbers can exceed 3 000.

Four-digit numbers greater than 3 000**.** The leading digit must be at least 3, so it is one of $3, 5, 7, 9$ (4 choices). The remaining three places are filled in order from the four unused digits:

$$4 \times (4 \times 3 \times 2) = 4 \times 24 = 96.$$

Five-digit numbers. Every five-digit number made from these digits is at least 13 000, so all arrangements of the five distinct digits qualify:

$$5! = 120.$$

Adding the two cases,

$$96 + 120 = 216.$$

Choosing (selections)

9. Basic selections (nCr)

Q45 2022 Moriah Trial Q11(c)(i) · 1 mark

A school has ten prefects. John and Alex are two of those prefects. A committee of six prefects is to be formed to organise a function. How many different committees can be formed?

Criteria	Marks
Correct value ${}^{10}C_6 = 210$	1

Solution.

John and Alex are simply two of the ten prefects, and there is no condition on them, so this is an unrestricted choice of 6 prefects from 10:

$$ {}^{10}C_6 = 210. $$

Q46 2025 Caringbah Trial Q13(a)(i) · 1 mark

A four-person team is to be chosen at random from nine women and seven men. In how many ways can the team be chosen?

Criteria	Marks
Correct value ${}^{16}C_4 = 1820$	1

Solution.

The nine women and seven men give 16 people in total, and any 4 of them may form the team with no restriction:

$$ {}^{16}C_4 = 1820. $$

Q47 2021 HSC Q11(d) · **1 mark**

A committee containing 5 men and 3 women is to be formed from a group of 10 men and 8 women. In how many different ways can the committee be formed?

Criteria	Marks
Correct value ${}^{10}C_5 \times {}^{8}C_3 = 14112$	1

Solution.

Choose the 5 men from the 10 men and, independently, the 3 women from the 8 women, then multiply:

$$ {}^{10}C_5 \times {}^{8}C_3 = 252 \times 56 = 14112. $$

Q48 2024 Penrith Trial Q11(b) · **2 marks**

A committee of 7 is to be chosen from 6 men and 8 women. How many different committees can be formed if the committee must have exactly 4 men?

Criteria	Marks
Sets up product ${}^{6}C_4 \times {}^{8}C_3$ (4 men, remaining 3 women)	1
Correct value 840	1

Solution.

The committee of 7 must have exactly 4 men, so it has $7 - 4 = 3$ women. Choose the 4 men from 6 and the 3 women from 8:

$$ {}^{6}C_4 \times {}^{8}C_3 = 15 \times 56 = 840. $$

Q49 2022 Penrith Trial Q12(c) · 3 marks

In how many ways can 5 people be selected from a group of 12 if:

(i) one particular person must be included.

(ii) two particular people cannot be selected.

(iii) one particular person must be included, and two particular people are excluded from the selection.

Criteria	Marks
(i) ${}^{11}C_4 = 330$	1
(ii) ${}^{10}C_5 = 252$	1
(iii) ${}^{9}C_4 = 126$	1

Solution.

Five people are selected from 12.

(i) One particular person must be included. Place that person on the committee, then choose the remaining 4 from the other 11:

$$ {}^{11}C_4 = 330. $$

(ii) Two particular people cannot be selected. Remove those two from the pool, leaving 10, and choose all 5 from them:

$$ {}^{10}C_5 = 252. $$

(iii) One must be included and two are excluded. Place the required person, remove the two excluded people, leaving $12 - 1 - 2 = 9$ candidates from whom to choose the remaining 4:

$$ {}^{9}C_4 = 126. $$

Q50 2024 Girraween Trial Q16(c) · 3 marks

A committee of 5 is to be chosen from six men and seven women.

(i) How many committees are possible if there are no restrictions?

(ii) How many committees are possible if there are more women than men?

Criteria	Marks
(i) ${}^{13}C_5 = 1287$	1
(ii) Identifies the valid cases (3W2M, 4W1M, 5W0M)	1
(ii) Correct total 756	1

Solution.

A committee of 5 is chosen from 6 men and 7 women (13 people).

(i) No restrictions.

$$ {}^{13}C_5 = 1287. $$

(ii) More women than men. With 5 members, "more women than men" means at least 3 women, giving the cases $(W, M) = (3, 2), (4, 1), (5, 0)$:

$$ {}^{7}C_3\,{}^{6}C_2 + {}^{7}C_4\,{}^{6}C_1 + {}^{7}C_5\,{}^{6}C_0 = 525 + 210 + 21 = 756. $$

10. Selections: at least / at most / cases

Q51 2020 Trinity Grammar Trial Q7 · 1 mark

From 8 green and 8 white cards, 4 are chosen at random. If each outcome is equally likely, how many combinations are possible that have at most 1 green card?

(A) 518

(B) 930

(C) 1820

(D) 3920

Criteria	Marks
Correct answer (A) 518	1

Solution.

"At most 1 green" from 4 chosen means 0 green or 1 green. **Case 0 green:** all 4 white, ${}^{8}C_{4} = 70$. **Case 1 green:** one green and three white, ${}^{8}C_{1} \times {}^{8}C_{3} = 8 \times 56 = 448$. Total $= 70 + 448 = 518$. Answer (A).

Q52 2022 Barker Trial Q11(d) · 2 marks

A state senate sub-committee is to be formed by choosing 3 senators from among 6 rural senators and 6 urban senators. How many different sub-committees can be formed if at least 1 rural and 1 urban senator must be chosen?

Criteria	Marks
Splits into the two valid cases (1R2U and 2R1U), or sets up the complement	1
Correct total 180	1

Solution.

Choosing 3 from 12 with at least one rural and one urban rules out the all-rural and all-urban selections. **Complement method:** total $= {}^{12}C_{3} = 220$; all rural $= {}^{6}C_{3} = 20$; all urban $= {}^{6}C_{3} = 20$. Answer $= 220 - 20 - 20 = 180$. **Direct (cases) check:** ${}^{6}C_{1}{}^{6}C_{2} + {}^{6}C_{2}{}^{6}C_{1} = 6(15) + 15(6) = 90 + 90 = 180$.

Q53 2022 Shore Trial Q12(d) · **2 marks**

A mixed volleyball team of eight players is selected from ten males and nine females. In how many ways can this be done if the team must have at least 2 female players?

Criteria	Marks
Sets up complement (or the correct case split) with the right terms	1
Correct total 74 457	1

Solution.

Team of 8 from 10 males and 9 females, with at least 2 females. Easiest by complement (subtract 0 and 1 female). Total $= {}^{19}C_8 = 75\,582$. **0 females:** all 8 from 10 males $= {}^{10}C_8 = 45$. **1 female:** ${}^{9}C_1 \times {}^{10}C_7 = 9 \times 120 = 1080$. Answer $= 75\,582 - 45 - 1080 = 74\,457$.

Q54 2023 Fort St Trial Q12(b) · **2 marks**

In a 16 member soccer squad 12 are right-handed while 4 are left-handed. If 11 members are to be selected as the starting line up, in how many ways will there be at least three left-handed players in the starting line up?

Criteria	Marks
Identifies the two valid cases (exactly 3 left, exactly 4 left) with correct terms	1
Correct total 2772	1

Solution.

Select 11 from 12 right-handed and 4 left-handed, with at least 3 left-handers. Since only 4 left-handers exist, the cases are exactly 3 or exactly 4 left-handed. **3 left:** ${}^{4}C_3 \times {}^{12}C_8 = 4 \times 495 = 1980$. **4 left:** ${}^{4}C_4 \times {}^{12}C_7 = 1 \times 792 = 792$. Answer $= 1980 + 792 = 2772$.

Q55 2025 Manly Trial Q14(a)(ii) · **3 marks**

A standard pack of 52 cards consists of 13 cards from each of the four suits: Spades, Hearts, Clubs and Diamonds. In how many ways can six cards be selected without replacement if at least five cards must be of the same suit?

Criteria	**Marks**
Correct "exactly 5 of one suit" count = $4 \times {}^{13}C_5 \times 39$	1
Correct "all 6 of one suit" count = $4 \times {}^{13}C_6$	1
Correct total 207 636	1

Solution.

Six cards from 52 with at least 5 of the same suit. Split by cases (this cannot happen for two suits at once, so no double counting). **Exactly 5 of one suit:** choose the suit (4 ways), choose 5 of its 13 cards $\left({}^{13}C_5 = 1287\right)$, and the 6th card from the other 39 cards. $4 \times 1287 \times 39 = 200\,772$. **All 6 of one suit:** choose the suit (4 ways), choose 6 of its 13 cards. $4 \times {}^{13}C_6 = 4 \times 1716 = 6864$. Answer = $200\,772 + 6864 = 207\,636$.

11. Grouping & splitting

Q56 2023 Penrith Trial Q9 · **1 mark**

In how many ways can twelve table tennis players be assigned into three groups each containing four players for the elimination round?

(A) 5 775

(B) 11 550

(C) 15 400

(D) 34 650

Criteria	Marks
Correct answer (A) 5775	1

Solution.

Assign 12 players into 3 groups of 4 each. Form the groups successively, then divide by 3! because the three equal-size groups are unlabelled.

$$\frac{{}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4}{3!} = \frac{495 \times 70 \times 1}{6} = \frac{34\,650}{6} = 5775.$$

Answer (A). (Option D 34 650 is the trap for forgetting to divide by 3!; option B 11 550 divides by 3 instead of 3!.)

Q57 2024 Girraween Trial Q12(d) · **2 marks**

In how many ways can 8 people be placed in groups of 2?

Criteria	Marks
Sets up the successive selections (or $8!/(2!)^4$) for four groups of two	1
Divides by 4! for the four identical-size groups; answer 105	1

Solution.

Divide 8 people into 4 unlabelled groups of 2. Form the groups one at a time, then divide out the ordering of the four equal-size groups.

$$\frac{{}^{8}C_2 \times {}^{6}C_2 \times {}^{4}C_2 \times {}^{2}C_2}{4!} = \frac{28 \times 15 \times 6 \times 1}{24} = \frac{2520}{24} = 105.$$

Equivalently $\dfrac{8!}{(2!)^4\,4!} = 105.$

Q58 2023 Fort St Trial Q13(d)(i) · **2 marks**

Ten friends are going to be divided into groups of 5, 3 and 2 members as part of a competition. In how many ways can the groups be formed?

Criteria	Marks
Correct product of selections ${}^{10}C_5 \times {}^{5}C_3 \times {}^{2}C_2$	1
Correct total 2520 (no symmetry division needed, all group sizes different)	1

Solution.

Divide 10 friends into groups of sizes 5, 3 and 2. The three groups are all different sizes, so they are distinguishable by size and there is no over-counting to divide out.

$$ {}^{10}C_5 \times {}^{5}C_3 \times {}^{2}C_2 = 252 \times 10 \times 1 = 2520. $$

Equivalently $\dfrac{10!}{5!\,3!\,2!} = 2520.$

Q59 2023 Normanhurst Boys Trial Q11(e) · **2 marks**

There are twelve people that need to be divided into groups.

(i) In how many ways can they be divided into two groups of six people?

(ii) In how many ways can they be divided into two groups of six people if two particular people must be in the same group?

Criteria	Marks
(i) Correct 462, showing division by 2! for the two equal groups	1
(ii) Correct 210 = ${}^{10}C_4$	1

Solution.

(i) Split 12 people into two unlabelled groups of 6. Choose one group, then halve for the symmetry of the two equal groups:

$$ \frac{{}^{12}C_6}{2!} = \frac{924}{2} = 462. $$

(ii) Two particular people must be in the same group. Placing that pair fixes which group is "theirs," so we just fill the remaining 4 spots of their group from the other 10 people; the second group of 6 is then determined. No further division is needed (the pair already breaks the symmetry).

$$ {}^{10}C_4 = 210. $$

Q60 2024 Ascham Trial Q14(e) · **2 marks**

Two teams of four players and an umpire are to be formed from nine people. If Amelia and Chloe cannot be on the same team, although they can be an umpire, find how many different ways the nine people can be grouped into two teams and an umpire.

Criteria	**Marks**
Handles the ”Amelia or Chloe is umpire” case correctly	1
Handles the ”neither is umpire” case with the not-same-team restriction	1

Solution.

Nine people form two teams of four plus one umpire. The two teams are of equal size and unlabelled, so any split of 8 people into two fours is counted as $\frac{{}^8C_4}{2!} = 35$. Amelia (A) and Chloe (C) may not be on the same team, though either may be the umpire. **Case 1: A or C is the umpire.** 2 choices of umpire. The other of the pair joins the remaining 7, and those 8 people split into two teams in $\frac{{}^8C_4}{2!} = 35$ ways (the not-same-team rule is automatically satisfied, since only one of A, C is on a team). This gives $2 \times 35 = 70$. **Case 2: umpire is one of the other 7.** 7 choices of umpire. The remaining 8 include both A and C. Total splits = 35; subtract the splits where A and C are together. A and C together: put them on a team and choose 2 more of the remaining 6 for that team, ${}^6C_2 = 15$ (the other team is then fixed). Valid splits = $35 - 15 = 20$. This gives $7 \times 20 = 140$. Total = $70 + 140 = 210$. (Confirmed by brute-force enumeration.)

Q61 2024 Sydney Tech Trial Q14(d) · 3 marks

(i) In how many ways can n students be placed in two distinct rooms so that neither room is empty? Give your answer in simplest form.

(ii) In how many ways can five students be placed in three distinct rooms so that no room is empty?

Criteria	Marks
(i) Correct $2^n - 2$ with reasoning	1
(ii) Sets up inclusion-exclusion (or the case split $C(5,1)\cdot$ arrangement) for surjections onto 3 rooms	1
(ii) Correct total 150	1

Solution.

(i) Each of the n students independently goes into one of two **distinct** rooms, giving 2^n assignments. Subtract the 2 forbidden cases where everyone is in the same room (all in room 1, or all in room 2), since neither room may be empty:

$$2^n - 2.$$

(ii) Five students into three **distinct** rooms, none empty (surjections). By inclusion-exclusion:

$$3^5 - {}^3C_1\,2^5 + {}^3C_2\,1^5 = 243 - 3(32) + 3(1) = 243 - 96 + 3 = 150.$$

(Independent check by the partition-then-label method: the only room-size patterns are 3+1+1 and 2+2+1. For 3+1+1: $\frac{{}^5C_3\,{}^2C_1\,{}^1C_1}{2!} \times 3! = 10 \times 3 = 30$ per... total 60; for 2+2+1: $\frac{{}^5C_2\,{}^3C_2\,{}^1C_1}{2!} \times 3! = 15 \times 6 = 90$; $60 + 90 = 150$.)

12. Inclusion-exclusion

Q62 2020 Sydney Grammar Trial Q6 · 1 mark

A school mathematics department consists of 5 male and 5 female teachers. How many different exam-writing committees comprising three teachers could be formed if there must be at least one teacher of each sex on the committee?

(A) 50

(B) 100

(C) 120

(D) 600

Criteria	Marks
Correct answer (B) 100	1

Solution.

Complement (subtract the disallowed all-one-sex committees from the total). Total committees of 3 from 10 teachers: ${}^{10}C_3 = 120$. Remove those with no valid mix: all-male ${}^{5}C_3 = 10$ and all-female ${}^{5}C_3 = 10$. So $120 - 10 - 10 = 100$. (Check by cases: $2M1F + 1M2F = {}^{5}C_2{}^{5}C_1 + {}^{5}C_1{}^{5}C_2 = 50 + 50 = 100$.) Answer **(B)**.

Q63 2021 Caringbah Trial Q7 · **1 mark**

A committee of 5 people is to be chosen from a group of 6 girls and 4 boys. How many different committees could be formed that have at least one boy?

(A) ${}^{10}C_5 - 1$

(B) ${}^{4}C_1 \times {}^{6}C_4$

(C) ${}^{4}C_1 + {}^{6}C_4$

(D) ${}^{10}C_5 - 6$

Criteria	**Marks**
Correct answer (D) ${}^{10}C_5 - 6$	1

Solution.

Complement (total minus the committees with no boys). Total committees of 5 from 10 people: ${}^{10}C_5$. Committees with at least one boy = total − (committees of all girls). All-girl committees choose 5 from the 6 girls: ${}^{6}C_5 = 6$. So the count is ${}^{10}C_5 - 6$. Answer **(D)**.

Q64 2023 Sydney Boys Trial Q11(d) · **2 marks**

A school principal calls a meeting with two student representatives from each year group from Year 7 to Year 12. The 12 students and the principal sit at a round table. In how many ways can the 13 participants sit around the table if at least one pair of students from a year group sits apart?

Criteria	**Marks**
Total $(13-1)! = 12!$ and complement all-together $6! \cdot 2^6$	1
Correct final expression $12! - 6! \cdot 2^6$ (or value)	1

Solution.

Complement (all seatings minus those where every pair sits together). Around a round table of 13 seats the total number of arrangements is $(13-1)! = 12!$.

"At least one pair apart" is the complement of "every pair together." Glue each of the 6 year-group pairs into a block: this gives 6 blocks plus the principal = 7 units around the table, arranged in $(7-1)! = 6!$ ways, and each pair can be internally ordered in 2 ways, giving 2^6. So all-pairs-together = $6! \cdot 2^6 = 46080$.

Therefore the number of seatings with at least one pair apart is

$$12! - 6! \cdot 2^6 = 479\,001\,600 - 46\,080 = 478\,955\,520.$$

Q65 2022 James Ruse Trial Q11(d) · **4 marks**

You are asked to provide an 8-letter password which may only be comprised of the letters A, B and C.

(i) How many 8-letter passwords are there to choose from?

(ii) How many of the 8-letter passwords contain only B's and C's?

(iii) If you decide that your 8-letter password must use each of the letters A, B and C at least once, how many possible passwords are there to choose from?

Criteria	**Marks**
(i) $3^8 = 6561$	1
(ii) $2^8 = 256$	1
(iii) sets up inclusion-exclusion (subtracts the one-letter-missing cases)	1
(iii) correct value 5796	1

Solution.

(i) Each of the 8 positions is filled independently by A, B or C, so there are $3^8 = 6561$ passwords.

(ii) Passwords using only B and C have 2 choices per position: $2^8 = 256$.

(iii) Require each of A, B, C at least once. Use inclusion-exclusion on the "missing letter" events. Passwords missing a chosen letter (only 2 letters used): 2^8 each, and there are 3 such letters. Passwords missing two chosen letters (only 1 letter used): 1^8 each, 3 such pairs. So

$$3^8 - {}^3C_1\, 2^8 + {}^3C_2\, 1^8 = 6561 - 3(256) + 3(1) = 6561 - 768 + 3 = 5796.$$

Q66 2020 Baulkham Hills Trial Q14(c) · 4 marks

Suppose there are n different books, and there are p identical copies of each of these n books. (You may assume n and p are non-zero integers by definition.)

(i) If all the books are to be placed in a line on one shelf, in how many ways can this be done?

(ii) Consider now a library with n different books, where there are p identical copies of each of these n books on one shelf. The library barcodes each book differently. Suppose you want to borrow at least one book from the shelf, but you do not want to borrow multiple copies of the same book. In how many ways can this be done?

Criteria	Marks
(i) $\dfrac{(np)!}{(p!)^n}$ (arrangement of identical copies)	2
(ii) each title contributes $(p+1)$ options and subtracts the empty selection	1
(ii) correct answer $(p+1)^n - 1$	1

Solution.

(i) There are np books in total placed in a line. They form n groups, each group being p identical copies. The number of distinct arrangements is the multinomial

$$\frac{(np)!}{\underbrace{p!\,p!\cdots p!}_{n \text{ times}}} = \frac{(np)!}{(p!)^n}.$$

(ii) Now the copies are individually barcoded (all np books distinct), and a selection may include at most one copy of each title, with at least one book borrowed. For each of the n titles there are $p+1$ independent choices: borrow none of its copies, or borrow exactly one of its p barcoded copies. That gives $(p+1)^n$ selections, which includes the empty selection. Subtracting the empty case:

$$(p+1)^n - 1.$$

Specialised techniques & proofs

13. Geometric counting

Q67 2024 North Sydney Girls Trial Q8 · 1 mark

A set of five distinct parallel lines intersects a different set of seven distinct parallel lines. How many possible parallelograms can be made from these lines?

(A) ${}^{12}C_4$

(B) ${}^{5}C_2 \times {}^{7}C_2$

(C) ${}^{5}C_2 + {}^{7}C_2$

(D) ${}^{5}P_2 \times {}^{7}P_2$

Criteria	Marks
Correct answer (B) ${}^{5}C_2 \times {}^{7}C_2$	1

Solution.

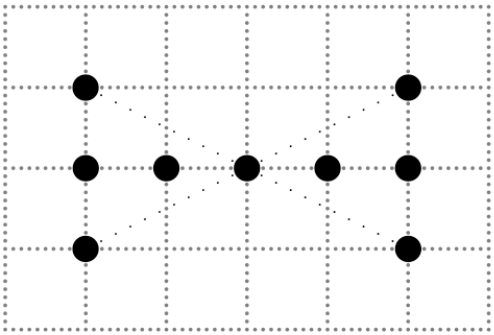
A parallelogram is fixed by choosing two lines from one parallel family (these give one pair of parallel sides) and two lines from the other family (the second pair). Choose 2 of the 5 lines and 2 of the 7 lines:

$$ {}^{5}C_2 \times {}^{7}C_2 = 10 \times 21 = 210. $$

Answer **(B)**.

Q68 2020 Normanhurst Boys Trial Q10 · **1 mark**

The diagram below shows a shape made by 9 points.



How many combinations of 3 points are collinear?

(A) 10

(B) 14

(C) 12

(D) 8

Criteria	**Marks**
Correct answer (D) 8	1

Solution.

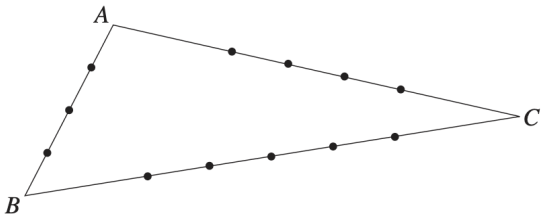
The 9 points form a 3×3 array. A set of 3 points is collinear when all three lie on one straight line of the array. These lines are: the 3 horizontal rows, the 3 vertical columns, and the 2 main diagonals, each containing exactly 3 points:

$$3 + 3 + 2 = 8.$$

So there are 8 collinear triples. Answer **(D)**. (Figure not shown in the .tex; solved from the standard 9-point 3×3 reading, which matches an offered option.)

Q69 2022 HSC Q7 · 1 mark

The diagram shows triangle ABC with points chosen on each of the sides. On side AB, 3 points are chosen. On side AC, 4 points are chosen. On side BC, 5 points are chosen.



How many triangles can be formed using the chosen points as vertices?

(A) 60

(B) 145

(C) 205

(D) 220

Criteria	Marks
Correct answer (C) 205	1

Solution.

There are $3 + 4 + 5 = 12$ chosen points. Any 3 non-collinear points form a triangle. Choose 3 of the 12 in ${}^{12}C_3 = 220$ ways, then subtract the collinear triples, which occur only when all 3 lie on the same side:

$$ {}^{12}C_3 - \left({}^{3}C_3 + {}^{4}C_3 + {}^{5}C_3\right) = 220 - (1 + 4 + 10) = 220 - 15 = 205. $$

Answer **(C)**.

Q70 2024 James Ruse Trial Q9 · **1 mark**

There are 11 points in a plane such that 4 of the points are collinear. Which of the following gives the number of lines that may be formed such that those lines pass through at least two of the 11 points?

(A) 55

(B) 49

(C) 50

(D) 52

Criteria	**Marks**
Correct answer (C) 50	1

Solution.

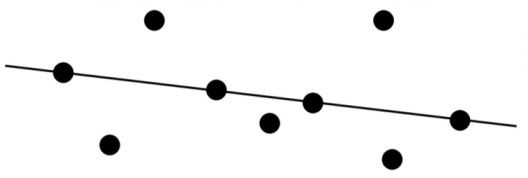
A line is determined by choosing 2 of the 11 points, giving ${}^{11}C_2 = 55$ if all points were in general position. But the 4 collinear points produce ${}^{4}C_2 = 6$ point-pairs that all lie on a single line rather than 6 distinct lines. Replace those 6 counted lines by the 1 actual line:

$$ {}^{11}C_2 - {}^{4}C_2 + 1 = 55 - 6 + 1 = 50. $$

Answer **(C)**.

Q71 2020 Abbotsleigh Trial Q13(a) · **2 marks**

Nine points are drawn on a page as shown, with four of them lying on one line. Other than these four points, no other set of three points is collinear.



How many different triangles can be drawn using the nine points as vertices?

Criteria	Marks
Choose 3 of 9 and identify the collinear correction: ${}^9C_3 - {}^4C_3$	1
Correct answer 80	1

Solution.

Any 3 non-collinear points give a triangle. Choose 3 of the 9 points in ${}^9C_3 = 84$ ways. The only collinear triples come from the 4 points on the single line, contributing ${}^4C_3 = 4$ degenerate (non-triangle) selections; no other 3 points are collinear. So

$$ {}^9C_3 - {}^4C_3 = 84 - 4 = 80. $$

14. Stars & bars / lattice paths

Q72 2022 Killara Trial Q8 · **1 mark**

How many solutions are there to the equation $x + y + z = 4$ given that x, y and z are non-negative integers?

(A) 4

(B) 15

(C) 17

(D) 24

Criteria	Marks
Correct answer (B)	1

Solution.

This is a stars and bars problem. We distribute 4 identical units among 3 non-negative integer variables x, y,

z. Represent the 4 units as stars and use 2 bars to split them into three groups:

$$\underbrace{\star\star\star\star}_{\text{4 stars}} \quad \text{with} \quad 2 \text{ bars.}$$

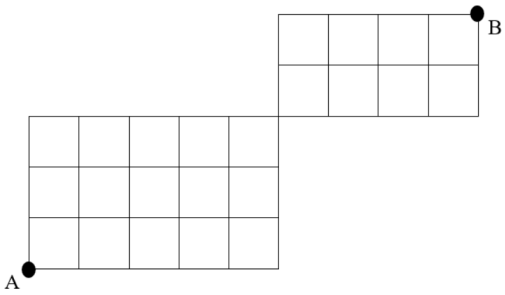
The number of arrangements of 4 stars and 2 bars in a row is

$$\binom{4+2}{2} = \binom{6}{2} = 15.$$

The answer is **(B)** 15.

Q73 2024 Manly Trial Q10 · **1 mark**

Moving only right or up, how many different paths exist to get from Point A to Point B?



(A) 45

(B) 840

(C) 480480

(D) 16!

Criteria	Marks
Correct answer (B) 840	1

Solution.

The region is L-shaped, so a single binomial coefficient does not apply directly. Every Right/Up path from A (bottom-left) to B (top-right) must pass through the inner corner node where the two rectangular arms meet, so we split the journey there and multiply.

Lower arm (5 wide, 3 tall) from A to the inner corner: 5 Rights and 3 Ups, giving

$$\binom{5+3}{3} = \binom{8}{3} = 56 \text{ paths.}$$

Upper arm (4 wide, 2 tall) from the inner corner to B: 4 Rights and 2 Ups, giving

$$\binom{4+2}{2} = \binom{6}{2} = 15 \text{ paths.}$$

By the multiplication principle,

$$56 \times 15 = 840.$$

The answer is **(B)** 840. (Note: $\binom{10}{2} = 45$ is the trap of treating the shape as a single rectangle; 16! is the trap of treating all moves as distinct.)

Q74 2022 James Ruse Prelim Q12(e)(i) · **1 mark**

A customer wishes to purchase a dozen donuts from a bakery that offers four different types of donut. The customer wishes to know how many ways there are of buying a dozen donuts. To do this, they represent each donut as a 'D' and place those twelve Ds in a line:

$$DDDDDDDDDDDD$$

They then insert three $*$s as place markers to indicate how many of each type of donut they buy. For example:

$$DD * DDD * DDDDDD * DD$$

means they buy 2 donuts of the first type, 3 donuts of the second type, 6 donuts of the third type and 2 donuts of the fourth type. In how many ways can the customer buy 12 donuts if there are 4 available types? Assume there are at least 12 donuts available for each type.

Criteria	**Marks**
Correct answer ${}^{15}C_3 = 455$	1

Solution.

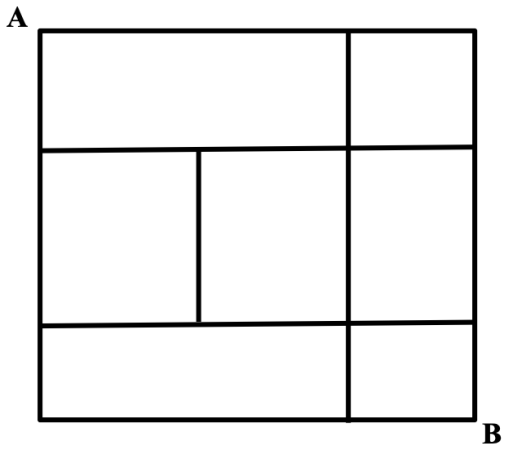
Each purchase corresponds to arranging the 12 identical Ds together with the 3 identical $*$ place markers in a line, a total of 15 symbols. The number of such arrangements is

$$\frac{15!}{12!\,3!} = \binom{15}{3} = 455.$$

Equivalently, this is the stars and bars count for the number of non-negative integer solutions of $a+b+c+d = 12$, namely $\binom{12+4-1}{4-1} = \binom{15}{3} = 455$.

Q75 2023 Baulkham Hills Prelim Q10 · 1 mark

Moving only to the right or down, how many different routes are there from A to B?



(A) 10

(B) 12

(C) 14

(D) 18

Criteria	Marks
Correct answer 14	1

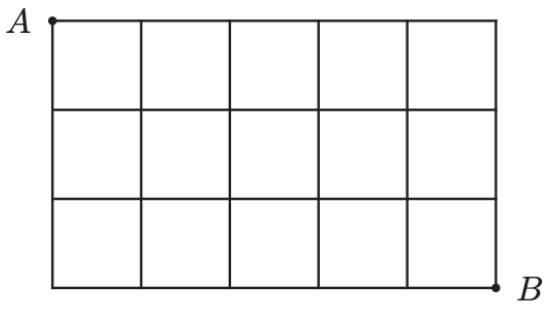
Solution.

The grid is not a full rectangle: some internal segments are missing, so the count is not a single binomial coefficient. Label each grid vertex with the number of monotone (right or down only) routes from A to that vertex. The count at a vertex is the sum of the labels of the vertex immediately above it and the vertex immediately to its left, taken only along grid segments that actually exist. Starting from A (label 1) and filling the grid in this way gives the label at B as the required total.

$$\text{Number of routes from } A \text{ to } B = 14.$$

Q76 2022 Normanhurst Boys Trial Q11(e) · **2 marks**

The diagram shows a grid consisting of unit squares. Chris needs to travel from point A to point B but can only do so by moving right or down along the grid lines.



How many paths are there for Chris?

Criteria	Marks
Identifies the grid requires 5 rights and 3 downs (or equivalent binomial setup)	1
Correct count 56	1

Solution.

The grid is 5 unit squares wide and 3 unit squares tall. Travelling from A (top-left) to B (bottom-right) using only Right and Down moves, every path consists of exactly 5 Right moves and 3 Down moves, in some order: a total of 8 moves.

The number of distinct paths is the number of ways to arrange these 8 moves, which is the number of ways to choose which 3 of the 8 moves are Down (or equivalently which 5 are Right):

$$\binom{8}{3} = \binom{8}{5} = 56.$$

There are 56 paths.

15. Solving nPr / nCr equations

Q77 2020 Sydney Boys Trial Q12(c) · **2 marks**

Solve the equation $(n+2)! = 72\,n!$.

Criteria	Marks
Simplifies $(n+2)!/n!$ to $(n+2)(n+1)$ and forms a quadratic	1
Solves to $n = 7$ (rejecting the negative root)	1

Solution.

Since $(n+2)! = (n+2)(n+1)\,n!$, dividing both sides of $(n+2)! = 72\,n!$ by $n!$ gives

$$(n+2)(n+1) = 72.$$

Expanding:

$$n^2 + 3n + 2 = 72 \quad \Longrightarrow \quad n^2 + 3n - 70 = 0.$$

Factorising:

$$(n+10)(n-7) = 0 \quad \Longrightarrow \quad n = -10 \text{ or } n = 7.$$

Since $n!$ requires n to be a non-negative integer, reject $n = -10$. Therefore

$$n = 7.$$

Q78 2025 Barker Trial Q12(d) · **2 marks**

Solve for n: ${}^nC_3 = 3\,({}^nP_2)$.

Criteria	**Marks**
Writes ${}^nC_3$ and ${}^nP_2$ in expanded form and cancels the common factor $n(n-1)$	1
Solves for $n = 20$ (rejecting the invalid root)	1

Solution.

Write both sides in expanded form:

$${}^nC_3 = \frac{n(n-1)(n-2)}{6}, \qquad {}^nP_2 = n(n-1).$$

The equation ${}^nC_3 = 3\,({}^nP_2)$ becomes

$$\frac{n(n-1)(n-2)}{6} = 3\,n(n-1).$$

Since ${}^nC_3$ requires $n \geq 3$, the factor $n(n-1)$ is non-zero and may be cancelled:

$$\frac{n-2}{6} = 3 \quad \Longrightarrow \quad n - 2 = 18 \quad \Longrightarrow \quad n = 20.$$

Check: ${}^{20}C_3 = 1140$ and $3\left({}^{20}P_2\right) = 3(380) = 1140$. Therefore $n = 20$.

Q79 2025 Blacktown Boys Trial Q12(a) · 3 marks

(i) Show that ${}^nC_2 = \dfrac{n(n-1)}{2}$.

(ii) Hence, or otherwise, solve for n where $2 \times {}^nC_4 = 51 \times {}^nC_2$.

Criteria	Marks
(i) Writes ${}^nC_2 = \dfrac{n!}{2!\,(n-2)!}$ and simplifies to $\dfrac{n(n-1)}{2}$	1
(ii) Expands ${}^nC_4$, ${}^nC_2$ and cancels $n(n-1)$ to reach $(n-2)(n-3) = 306$	1
(ii) Solves $(n-20)(n+15) = 0$ for $n = 20$ (rejecting $n = -15$)	1

Solution.

(i) By the definition of ${}^nC_r$,

$$ {}^nC_2 = \frac{n!}{2!\,(n-2)!}. $$

Write $n! = n(n-1)(n-2)!$ and cancel the common factor $(n-2)!$:

$$ {}^nC_2 = \frac{n(n-1)(n-2)!}{2!\,(n-2)!} = \frac{n(n-1)}{2}. $$

(ii) Using part (i) and the expansion of ${}^nC_4$,

$$ {}^nC_4 = \frac{n(n-1)(n-2)(n-3)}{24}, \qquad {}^nC_2 = \frac{n(n-1)}{2}. $$

The equation $2 \times {}^nC_4 = 51 \times {}^nC_2$ becomes

$$ 2 \cdot \frac{n(n-1)(n-2)(n-3)}{24} = 51 \cdot \frac{n(n-1)}{2}. $$

Since ${}^nC_4$ requires $n \geq 4$, the factor $n(n-1)$ is non-zero and may be cancelled:

$$ \frac{(n-2)(n-3)}{12} = \frac{51}{2} \quad \Longrightarrow \quad (n-2)(n-3) = 306. $$

Expanding,

$$ n^2 - 5n + 6 = 306 \quad \Longrightarrow \quad n^2 - 5n - 300 = 0 \quad \Longrightarrow \quad (n-20)(n+15) = 0. $$

So $n = 20$ or $n = -15$. Reject $n = -15$, therefore $n = 20$. **Check:** $2 \times {}^{20}C_4 = 2(4845) = 9690$ and $51 \times {}^{20}C_2 = 51(190) = 9690$.

Q80 2023 Knox Prelim Q13(b) · 3 marks

There are n swimmers in a swimming club. The trainer is to select a group of four or five of these swimmers to swim in a competition. The number of different groups of five swimmers he can select is twice the number of different groups of four swimmers. Find n, the number of swimmers in the swimming club.

Criteria	Marks
sets up ${}^nC_5 = 2\,{}^nC_4$ with combinations (order not important)	1
reduces the factorials to $\frac{1}{5} = \frac{2}{n-4}$	1
solves $n-4=10$ to obtain $n=14$	1

Solution.

Order does not matter when forming a group, so the counts are combinations. The condition "groups of five is twice groups of four" gives

$$ {}^nC_5 = 2\,{}^nC_4, \qquad \frac{n!}{5!\,(n-5)!} = 2\cdot\frac{n!}{4!\,(n-4)!}. $$

Dividing both sides by $\dfrac{n!}{4!\,(n-5)!}$,

$$ \frac{1}{5} = \frac{2}{n-4} \quad\Longrightarrow\quad n-4=10 \quad\Longrightarrow\quad n=14. $$

There are 14 swimmers in the club.

Q81 2024 Hurlstone Prelim Q11(c) · **3 marks**

Solve the equation $16(n-1)! = 5\,n! + (n+1)!$ where n is any positive integer.

Criteria	Marks
expresses $n!$ and $(n+1)!$ as multiples of $(n-1)!$ and factors it out	1
reduces to the quadratic $n^2 + 6n - 16 = 0$	1
solves and rejects $n = -8$, giving $n = 2$	1

Solution.

Write $n! = n(n-1)!$ and $(n+1)! = (n+1)n(n-1)!$, then take out the common factor $(n-1)!$:

$$16(n-1)! = 5n(n-1)! + (n+1)n(n-1)!,$$

$$(n-1)!\,\big(16 - 5n - n(n+1)\big) = 0.$$

Since $(n-1)! \neq 0$, the bracket must vanish:

$$16 - 5n - n^2 - n = 0 \quad\Longrightarrow\quad n^2 + 6n - 16 = 0 \quad\Longrightarrow\quad (n+8)(n-2) = 0.$$

So $n = -8$ or $n = 2$. As n must be a positive integer, $n = 2$.

16. Factorial manipulation: simplify & prove

Q82 2025 Roseville Trial Q14(c) · **1 mark**

Show that ${}^nC_r = {}^nC_{n-r}$.

Criteria	Marks
Writes both sides in factorial form and shows the denominators are identical	1

Solution.

By the definition of ${}^nC_r$,

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,\big(n-(n-r)\big)!} = \frac{n!}{(n-r)!\,r!}.$$

The factor order in the denominator does not matter, so

$${}^nC_{n-r} = \frac{n!}{r!\,(n-r)!} = {}^nC_r.$$

This is the required result. (Combinatorially: choosing r objects to include is the same as choosing the $n-r$ objects to leave out.)

Q83 2023 Abbotsleigh Trial Q12(c)(i) · 1 mark

Show that $\dfrac{1}{(n+1)!} - \dfrac{n+1}{(n+2)!} = \dfrac{1}{(n+2)!}$.

Criteria	Marks
Rewrites $(n+2)! = (n+2)(n+1)!$ to obtain a common denominator, then simplifies to the result	1

Solution.

Take the left-hand side and use $(n+2)! = (n+2)(n+1)!$ to give both terms the common denominator $(n+2)!$:

$$\frac{1}{(n+1)!} - \frac{n+1}{(n+2)!} = \frac{n+2}{(n+2)!} - \frac{n+1}{(n+2)!}.$$

Combining over the common denominator:

$$= \frac{(n+2)-(n+1)}{(n+2)!} = \frac{1}{(n+2)!}.$$

This is the required result.

Q84 2020 Girraween Prelim Q2 · 1 mark

Simplify $\dfrac{(n+1)!}{(n-1)!}$.

Criteria	Marks
correct simplification $n(n+1)$	1

Solution.

Expand the numerator down to $(n-1)!$ and cancel:

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1)\,n\,(n-1)!}{(n-1)!} = n(n+1) = n^2 + n.$$

Q85 2022 Girraween Prelim Q2 · 1 mark

Simplify $\dfrac{1}{n!} - \dfrac{1}{(n+1)!}$.

Criteria	Marks
correct single fraction $\dfrac{n}{(n+1)!}$	1

Solution.

Using the common denominator $(n+1)! = (n+1)\,n!$,

$$\frac{1}{n!} - \frac{1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{n}{(n+1)!}.$$

Q86 2024 Hurlstone Trial Q15(c) · **2 marks**

Show that ${}^{n-1}C_{k-1} + {}^{n-1}C_k = \dfrac{n!}{(n-k)!\,k!}$.

Criteria	Marks
Writes ${}^{n-1}C_{k-1}$ and ${}^{n-1}C_k$ in factorial form with a common denominator	1
Combines and simplifies the numerator to reach $\dfrac{n!}{(n-k)!\,k!}$	1

Solution.

Write each term using the definition of ${}^nC_r$:

$${}^{n-1}C_{k-1} = \frac{(n-1)!}{(k-1)!\,(n-k)!}, \qquad {}^{n-1}C_k = \frac{(n-1)!}{k!\,(n-1-k)!}.$$

Make the denominators identical by writing $k! = k(k-1)!$ in the first term and $(n-k)! = (n-k)(n-1-k)!$ in the second term, so both share the denominator $k!\,(n-k)!$:

$${}^{n-1}C_{k-1} = \frac{k\,(n-1)!}{k!\,(n-k)!}, \qquad {}^{n-1}C_k = \frac{(n-k)\,(n-1)!}{k!\,(n-k)!}.$$

Adding:

$${}^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{(n-1)!\big[k + (n-k)\big]}{k!\,(n-k)!} = \frac{(n-1)!\cdot n}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!}.$$

This is the required result. (This is Pascal's identity: the sum equals ${}^nC_k$.)

Q87 2024 Blacktown Boys Trial Q12(d) · 2 marks

It is known that ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ for all integers such that $1 \leq r \leq n-1$. (Do NOT prove this.) Find ONE possible set of values for p and q such that

$$ {}^{2026}C_{1924} - {}^{2024}C_{100} - {}^{2024}C_{101} = {}^{p}C_{q}. $$

Criteria	Marks
Applies Pascal's identity to combine two of the terms (e.g. rewrites ${}^{2026}C_{1924}$ as ${}^{2026}C_{102}$ and/or expands via Pascal)	1
Reaches a valid single ${}^{p}C_{q}$, e.g. $p = 2025,\ q = 102$ (or $q = 1923$)	1

Solution.

First use the symmetry ${}^{n}C_{r} = {}^{n}C_{n-r}$ on the leading term, since $2026 - 1924 = 102$:

$$ {}^{2026}C_{1924} = {}^{2026}C_{102}. $$

Now apply the given Pascal identity ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$ to peel the top index down from 2026 to 2024:

$$ {}^{2026}C_{102} = {}^{2025}C_{101} + {}^{2025}C_{102}, $$

and applying it again to each term,

$$ {}^{2025}C_{101} = {}^{2024}C_{100} + {}^{2024}C_{101}, \qquad {}^{2025}C_{102} = {}^{2024}C_{101} + {}^{2024}C_{102}. $$

Therefore

$$ {}^{2026}C_{102} = {}^{2024}C_{100} + {}^{2024}C_{101} + {}^{2024}C_{101} + {}^{2024}C_{102}. $$

Subtracting the two given terms:

$$ {}^{2026}C_{1924} - {}^{2024}C_{100} - {}^{2024}C_{101} = {}^{2024}C_{101} + {}^{2024}C_{102}. $$

One final application of Pascal's identity gives

$$ {}^{2024}C_{101} + {}^{2024}C_{102} = {}^{2025}C_{102}. $$

So one possible set of values is

$$ p = 2025, \qquad q = 102. $$

(Equivalently $p = 2025,\ q = 1923$, since ${}^{2025}C_{102} = {}^{2025}C_{1923}$.)

Q88 2024 Girraween Prelim Q8(a) · **2 marks**

Write $\dfrac{1}{(n+1)!} - \dfrac{1}{(n-1)!}$ as a single fraction.

Criteria	Marks
rewrites both terms over the common denominator $(n+1)!$	1
correct single fraction $\dfrac{1-n-n^2}{(n+1)!}$	1

Solution.

The common denominator is $(n+1)! = (n+1)\,n\,(n-1)!$, so $\dfrac{1}{(n-1)!} = \dfrac{(n+1)n}{(n+1)!}$. Then

$$\frac{1}{(n+1)!} - \frac{1}{(n-1)!} = \frac{1}{(n+1)!} - \frac{(n+1)n}{(n+1)!} = \frac{1-n(n+1)}{(n+1)!} = \frac{1-n-n^2}{(n+1)!}.$$

Q89 2020 Girraween Prelim Q9(a) · **2 marks**

Write $\dfrac{8}{(n+2)!} - \dfrac{1}{n!}$ as a single fraction.

Criteria	Marks
rewrites both terms over the common denominator $(n+2)!$	1
correct single fraction $\dfrac{6-3n-n^2}{(n+2)!}$	1

Solution.

The common denominator is $(n+2)! = (n+2)(n+1)\,n!$, so $\dfrac{1}{n!} = \dfrac{(n+2)(n+1)}{(n+2)!}$. Then

$$\frac{8}{(n+2)!} - \frac{1}{n!} = \frac{8}{(n+2)!} - \frac{(n+2)(n+1)}{(n+2)!} = \frac{8-(n^2+3n+2)}{(n+2)!} = \frac{6-3n-n^2}{(n+2)!}.$$

Q90 2024 Sydney Girls Trial Q13(c) · 4 marks

For positive integers n and k, where $k \leq n$:

(i) Show algebraically that $\dfrac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$, where $\binom{n}{k} = {}^nC_k$.

(ii) Prove the above identity by providing a combinatorial argument (i.e. by providing a counting argument).

Criteria	Marks
Part (i): writes $\dfrac{k}{n}\binom{n}{k}$ in factorial form	1
Part (i): simplifies using $k! = k(k-1)!$ and $n! = n(n-1)!$ to reach $\binom{n-1}{k-1}$	1
Part (ii): sets up the count of a committee of k with a chosen leader from n people, counted two ways	1
Part (ii): equates the two counts to establish the identity	1

Solution.

(i) Algebraic proof. Write the binomial coefficient in factorial form and simplify:

$$\frac{k}{n}\binom{n}{k} = \frac{k}{n} \cdot \frac{n!}{k!\,(n-k)!}.$$

Use $n! = n(n-1)!$ and $k! = k(k-1)!$:

$$= \frac{k}{n} \cdot \frac{n(n-1)!}{k(k-1)!\,(n-k)!} = \frac{(n-1)!}{(k-1)!\,(n-k)!}.$$

Now note that $(n-k)! = \big((n-1)-(k-1)\big)!$, so this is exactly the factorial form of $\binom{n-1}{k-1}$:

$$\frac{(n-1)!}{(k-1)!\,\big((n-1)-(k-1)\big)!} = \binom{n-1}{k-1}.$$

Hence $\dfrac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$.

(ii) Combinatorial proof. Count, in two ways, the number of ways to choose a committee of k people from n people and then appoint one of the k committee members as leader.

Method 1 (committee first, then leader). Choose the committee of k from the n people in $\binom{n}{k}$ ways, then choose the leader from within the committee in k ways. This gives

$$k\binom{n}{k} \text{ ways.}$$

Method 2 (leader first, then the rest). Choose the leader first: there are n choices. Then choose the

remaining $k-1$ committee members from the $n-1$ people not yet chosen, in $\binom{n-1}{k-1}$ ways. This gives

$$n\binom{n-1}{k-1} \text{ ways.}$$

Both methods count the same set of outcomes, so

$$k\binom{n}{k} = n\binom{n-1}{k-1} \quad \Longrightarrow \quad \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

This establishes the identity.

17. Combinatorial-argument and explain-why proofs

Q91 2021 Sydney Grammar Prelim Q5(b) · **2 marks**

Twelve pupils are taking a class that requires them to complete the same group project. They are to be split into three groups, with each group consisting of four pupils. In calculating the number of ways this can be done, one of the pupils suggests:

> "The first group can be selected in ${}^{12}C_4$ ways. The second group can be selected in ${}^{8}C_4$ ways. The third group can then be selected in ${}^{4}C_4$ ways. Therefore there are ${}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4 = 34\,650$ ways we can be split into three groups of four."

Explain why this reasoning is incorrect, and calculate the correct number of ways that the twelve pupils can be split into three groups of four.

Criteria	Marks
Explains that ordering the three equal groups as "first, second, third" overcounts each split by 3!, since the groups are indistinguishable	1
Divides by 3! to reach the correct answer 5775	1

Solution.

The pupil's product ${}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4 = 34\,650$ counts **ordered** triples of groups: it treats a first group, a second group and a third group as distinct roles. But the three groups of four are interchangeable, there is no first or last, so every genuine split has been counted once for each of the $3! = 6$ ways of ordering its three groups.

To correct the overcount, divide by 3!:

$$\frac{{}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4}{3!} = \frac{34\,650}{6} = 5775.$$

So there are 5775 ways to split the twelve pupils into three groups of four.

Q92 2023 Pymble Trial Q14(b) · **2 marks**

A Maths club has $2n$ members, half of which are male and half female. A Pi Day committee is made up of 3 members of this club. By considering all different combinations for this committee, show that

$$n\,{}^{n}C_2 + {}^{n}C_3 = \frac{1}{2}\,{}^{2n}C_3, \qquad \text{where } n \geq 3.$$

Criteria	**Marks**
Counts the total committees ${}^{2n}C_3$ and splits by gender composition, using symmetry to pair the majority-male cases with the majority-female cases	1
Identifies $n\,{}^{n}C_2 + {}^{n}C_3$ as the majority-female committees, equal to half the total	1

Solution.

The total number of committees of 3 chosen from all $2n$ members is ${}^{2n}C_3$. Split this total according to the number of females on the committee:

0 female, 3 male: ${}^{n}C_3$ committees;

1 female, 2 male: $n \cdot {}^{n}C_2$ committees;

2 female, 1 male: ${}^{n}C_2 \cdot n = n\,{}^{n}C_2$ committees;

3 female, 0 male: ${}^{n}C_3$ committees.

Because there are equally many males and females (n of each), swapping the roles of male and female matches the "0 female" case with the "3 female" case and the "1 female" case with the "2 female" case. Hence exactly half of all committees have a female majority (two or three females), giving

$$\tfrac{1}{2}\,{}^{2n}C_3 = (\text{committees with 2 females}) + (\text{committees with 3 females}) = n\,{}^{n}C_2 + {}^{n}C_3.$$

Therefore $n\,{}^{n}C_2 + {}^{n}C_3 = \frac{1}{2}\,{}^{2n}C_3$, as required.

Q93 2022 James Ruse Prelim Q14(e) · **2 marks**

Using a combinatorial argument, show that $\dfrac{(n^2)!}{(n!)^n} \in \mathbb{N}$ for all natural numbers n.

Criteria	**Marks**
Sets up a counting problem whose answer is $\dfrac{(n^2)!}{(n!)^n}$, e.g. the multinomial count of arranging n^2 distinct objects into n ordered groups of n	1
Concludes that, being a count of arrangements, the quantity must be a natural number	1

Solution.

Consider n^2 distinct objects to be placed into n labelled boxes (rows), with exactly n objects in each box. The number of ways to do this is the multinomial coefficient

$$\binom{n^2}{n, n, \ldots, n} = \frac{(n^2)!}{\underbrace{n!\,n!\cdots n!}_{n \text{ factors}}} = \frac{(n^2)!}{(n!)^n}.$$

Concretely: line up all n^2 objects in a row in $(n^2)!$ ways, then read them off n at a time to fill the n boxes. Within each box the order of its n objects does not matter, so we have over-counted by a factor of $n!$ for each of the n boxes, that is by $(n!)^n$ in total. Dividing gives

$$\frac{(n^2)!}{(n!)^n}.$$

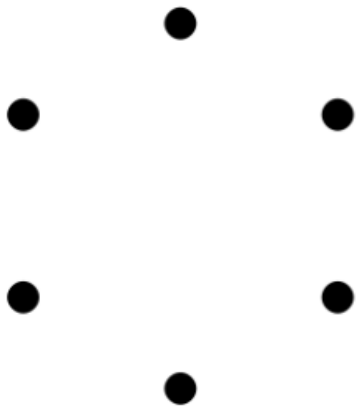
This quantity counts a set of distinct arrangements, so it is a non-negative whole number. It is clearly positive (at least one such arrangement exists for every n), hence

$$\frac{(n^2)!}{(n!)^n} \in \mathbb{N}$$

for all natural numbers n.

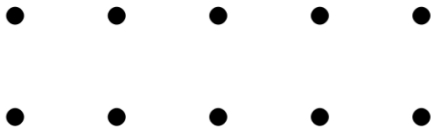
Q94 2023 Presbyterian (PLC Sydney) Prelim Q14(a) · 3 marks

Consider the dots below.



(i) Explain why ${}^6C_3$ distinct triangles can be formed using the dots as vertices.

(ii) How many distinct triangles can be formed using the dots below as vertices?



Criteria	Marks
(i) Explains that a triangle is a choice of 3 of the 6 dots, and no three of these dots are collinear, so every such choice gives a triangle: ${}^6C_3$	1
(ii) Counts the total selections ${}^{10}C_3 = 120$ and subtracts the collinear triples	1
(ii) Correct answer 100	1

Solution.

(i) A triangle is determined by choosing 3 of its vertices, and here the vertices must be dots. There are 6 dots, and no three of them lie on a straight line, so every selection of 3 distinct dots gives three non-collinear points, which form a genuine triangle. The number of ways to choose 3 dots from 6 without regard to order is therefore

$$ {}^6C_3 = 20. $$

(ii) The second figure has 10 dots arranged in two rows of 5. Choosing any 3 of them gives

$$ {}^{10}C_3 = 120 $$

selections, but a selection fails to form a triangle when the three chosen dots are collinear. The only lines containing three or more dots are the two rows of 5, and each row contributes ${}^5C_3 = 10$ collinear triples. Subtracting these degenerate cases,

$$ {}^{10}C_3 - 2\,{}^5C_3 = 120 - 2(10) = 120 - 20 = 100. $$

So 100 distinct triangles can be formed.

Q95 2021 Cherrybrook Tech Trial Q14(b) · 3 marks

Consider a set with n objects without regard to order. The set is to be split into r groups. The first group is of size n_1, the second is of size n_2, and so forth, with the last group of size n_r. Note $n_1 + n_2 + \cdots + n_r = n$.

(i) Explain why the number of ways to split the set into r groups is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}.$$

(ii) Show that the expression in part (i) is equal to

$$\frac{n!}{n_1!\,n_2!\,\ldots\,n_r!}.$$

Criteria	Marks
(i) Explains the sequential choice: choose n_1 from n, then n_2 from the $n-n_1$ remaining, and so on, multiplying by the counting (multiplication) principle	1
(ii) Writes each binomial coefficient as a factorial ratio and shows the internal terms cancel telescopically	1
(ii) Reaches $\dfrac{n!}{n_1!\,n_2!\,\ldots\,n_r!}$	1

Solution.

(i) Build the groups one at a time. The first group of size n_1 is any subset of the n objects, chosen in $\binom{n}{n_1}$ ways. Once those are removed, $n-n_1$ objects remain, and the second group of size n_2 is chosen from them in $\binom{n-n_1}{n_2}$ ways. Continuing, the kth group is chosen from whatever is left after the first $k-1$ groups have been taken out. Since each stage is an independent choice, the multiplication (counting) principle gives the number of ways to split the set as the product

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}.$$

(ii) Write each factor as a factorial quotient:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r} = \frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{r-1})!}{n_r!\;0!}.$$

The numerator of each factor after the first cancels with the trailing factorial in the previous denominator: $(n-n_1)!$ cancels, then $(n-n_1-n_2)!$ cancels, and so on down to $(n-n_1-\cdots-n_{r-1})! = n_r!$, and the final $0! = 1$. What survives is the leading $n!$ over the product of the group-size factorials:

$$= \frac{n!}{n_1!\,n_2!\,\ldots\,n_r!},$$

as required.

Q96 2021 Knox Prelim Q12(c) · 3 marks

12 people need to be split up into teams for a quiz.

(i) Explain why the number of ways of splitting them into two groups of the same size is

$$\frac{1}{2}\binom{12}{6}.$$

(ii) How many ways are there of splitting them into two groups of any size (but there must be at least one person in each group)?

Criteria	Marks
(i) Explains that $\binom{12}{6}$ chooses one group of six but counts each split twice (once as "this six", once as its complement), so divide by 2	1
(ii) Counts the ordered non-empty assignments $2^{12} - 2$	1
(ii) Halves for indistinguishable groups: $2^{11} - 1 = 2047$	1

Solution.

(i) A split into two groups of six is fixed as soon as we decide who is in one of the groups, since the other six are forced. Choosing six of the twelve people can be done in $\binom{12}{6}$ ways. However, the two groups of six are indistinguishable, so choosing a particular six and choosing the complementary six give the same split. Each split is therefore counted exactly twice, and the number of distinct splits is

$$\frac{1}{2}\binom{12}{6} = \frac{924}{2} = 462.$$

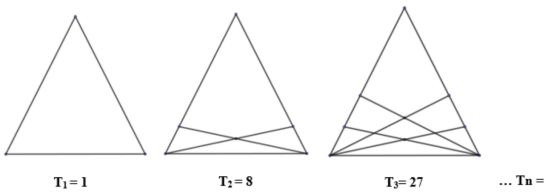
(ii) Now the two groups may be any (non-empty) sizes. Assign each of the 12 people to one of the two groups: there are 2^{12} such assignments. Two of them are illegal (everyone in the same group, leaving the other empty), so there are $2^{12} - 2$ assignments with both groups non-empty. As in part (i) the two groups are indistinguishable, so each split is counted twice; divide by 2:

$$\frac{2^{12} - 2}{2} = 2^{11} - 1 = 2047.$$

There are 2047 ways.

Q97 2020 James Ruse Prelim Q9(b) · 4 marks

Parminder is generating triangles from a given triangle, T_1. He takes the triangle and draws a ray from each corner of the base angles, as shown, and notes the number of triangles this creates in T_2. He then creates a third triangle, T_3, by drawing another ray from each base angle, and counts the number of triangles now. He continues doing this, creating a series: $T_1, T_2, T_3, \ldots$



(i) Explain why $T_n = \binom{2n+1}{3} - 2\binom{n+1}{3}, \quad n \geq 2.$

(ii) Hence, prove that $T_n = n^3$.

Criteria	Marks
(i) Identifies that $2n+1$ lines (or $2n+1$ points) are available and any 3 of them potentially make a triangle: $\binom{2n+1}{3}$	1
(i) Subtracts the two pencils of $n+1$ concurrent lines through each base vertex, $2\binom{n+1}{3}$, which do not form triangles	1
(ii) Expands both binomial coefficients as factorials / cubic polynomials	1
(ii) Simplifies to n^3	1

Solution.

(i) At stage n the figure is built from three families of lines: the base, and the n rays from each of the two base vertices (the two original sides of T_1 being the outermost of those rays). Counting the base together with the rays gives $1 + 2n = 2n + 1$ lines in total. A triangle is formed by choosing 3 of these lines, so there are $\binom{2n+1}{3}$ ways to pick three lines. However, three lines fail to bound a triangle when they are concurrent, and the only concurrences here are the $n + 1$ lines through the left base vertex and the $n + 1$ lines through the right base vertex. Any 3 lines chosen from one of these pencils meet at a single point and enclose no triangle, so we remove $\binom{n+1}{3}$ for each vertex:

$$T_n = \binom{2n+1}{3} - 2\binom{n+1}{3}, \qquad n \geq 2.$$

(ii) Expand each coefficient:

$$\binom{2n+1}{3} = \frac{(2n+1)(2n)(2n-1)}{6}, \qquad \binom{n+1}{3} = \frac{(n+1)(n)(n-1)}{6}.$$

Then

$$T_n = \frac{(2n+1)(2n)(2n-1) - 2\,(n+1)(n)(n-1)}{6} = \frac{2n\big[(2n+1)(2n-1) - (n+1)(n-1)\big]}{6}.$$

Now $(2n+1)(2n-1) = 4n^2 - 1$ and $(n+1)(n-1) = n^2 - 1$, so the bracket is $(4n^2 - 1) - (n^2 - 1) = 3n^2$. Hence

$$T_n = \frac{2n \cdot 3n^2}{6} = \frac{6n^3}{6} = n^3,$$

as required.

Probability

18. Arrangements, then a probability

Q98 2021 Hurlstone Trial Q11(b) · 1 mark

The letters of the word CONSIDER are rearranged at random. What is the probability that an arrangement begins with SI?

Criteria	Marks
Correct probability $\frac{1}{56}$ (MC / single mark)	1

Solution.

CONSIDER has 8 distinct letters, so the total number of arrangements is $8! = 40320$.

For an arrangement to begin with SI, fix S in the first position and I in the second. The remaining 6 distinct letters fill the last 6 positions in $6! = 720$ ways.

$$P(\text{begins with SI}) = \frac{6!}{8!} = \frac{720}{40320} = \frac{1}{56}.$$

Q99 2021 Ascham Trial Q6 · **1 mark**

Six girls and three boys are to be seated randomly around a circular table. What is the probability that the three boys are seated together?

(A) $\dfrac{1}{12}$

(B) $\dfrac{1}{24}$

(C) $\dfrac{3}{28}$

(D) $\dfrac{1}{56}$

Criteria	**Marks**
Correct probability $\dfrac{3}{28}$, option (C) (MC / single mark)	1

Solution.

Nine people are seated around a circular table, so the total number of arrangements is $(9-1)! = 8!$.

To seat the three boys together, treat them as a single block. This leaves 7 units (the block plus the 6 girls) around the table, arranged in $(7-1)! = 6!$ ways, and the 3 boys within the block can be ordered in $3!$ ways:

$$P(\text{boys together}) = \frac{6! \times 3!}{8!} = \frac{720 \times 6}{40320} = \frac{4320}{40320} = \frac{3}{28}.$$

The answer is **(C)**.

Q100 2020 Sydney Tech Trial Q11(a) · **2 marks**

Nine people are arranged around a circular table.

(i) How many ways can they be arranged?

(ii) If they are seated randomly, what is the probability that two individuals, Lois and Clark, are sitting next to each other?

Criteria	**Marks**
(i) Correct count $(9-1)! = 8! = 40320$	1
(ii) Correct probability $\frac{1}{4}$ (treating Lois and Clark as an adjacent pair)	1

Solution.

(i) For 9 people around a circular table, the number of arrangements is

$$(9-1)! = 8! = 40320.$$

(ii) For Lois and Clark to sit next to each other, treat the pair as a single block. This leaves 8 units around the table, arranged in $(8-1)! = 7!$ ways, and Lois and Clark can be ordered within the block in $2!$ ways:

$$P(\text{Lois and Clark adjacent}) = \frac{7! \times 2!}{8!} = \frac{2}{8} = \frac{1}{4}.$$

Q101 2025 Killara Trial Q11(c) · 3 marks

The letters of the name NARUTO are arranged in a row.

(i) In how many ways can this be done using all the letters?

(ii) What is the probability that the vowels are all together?

Criteria	Marks
(i) States 6! = 720	1
(ii) Treats the 3 vowels as a single block and counts favourable arrangements ($4! \times 3! = 144$)	1
(ii) Correct probability $\frac{1}{5}$	1

Solution.

NARUTO has 6 distinct letters.

(i) Using all the letters, the number of arrangements is $6! = 720$.

(ii) The vowels are A, U, O and the consonants are N, R, T. To keep the vowels together, treat the 3 vowels as a single block. This gives 4 units (the block plus N, R, T) arranged in 4! ways, and the 3 vowels within the block can be ordered in 3! ways:

$$\text{favourable} = 4! \times 3! = 24 \times 6 = 144.$$

$$P(\text{vowels together}) = \frac{144}{720} = \frac{1}{5}.$$

Q102 2025 Hurlstone Trial Q11(a) · 3 marks

The letters of the word CALCULUS are arranged in a row.

(i) How many different arrangements are there?

(ii) If one of these arrangements is selected at random, find the probability that it begins with "U" and ends with "U".

Criteria	Marks
(i) Correct arrangement count $\frac{8!}{2!\,2!\,2!} = 5040$	1
(ii) Counts favourable arrangements with a U fixed at each end ($\frac{6!}{2!\,2!} = 180$)	1
(ii) Correct probability $\frac{1}{28}$	1

Solution.

CALCULUS has 8 letters with repeats: C×2, L×2, U×2, and A, S each once.

(i) The number of distinct arrangements is

$$\frac{8!}{2!\,2!\,2!} = \frac{40320}{8} = 5040.$$

(ii) For an arrangement to begin and end with U, place a U in the first and last positions. Since the two U's are identical, this can be done in only one way. The remaining 6 letters (C×2, L×2, A, S) fill the middle positions in

$$\frac{6!}{2!\,2!} = \frac{720}{4} = 180 \text{ ways.}$$

$$P(\text{begins and ends with U}) = \frac{180}{5040} = \frac{1}{28}.$$

Q103 2025 Gosford Trial Q11(c) · **3 marks**

A group of six girls and two boys are to be seated around a round table.

(i) In how many different ways can the eight people be seated?

(ii) What is the probability the two boys won't be seated next to each other?

Criteria	**Marks**
(i) Correct count $(8-1)! = 7! = 5040$	1
(ii) Finds $P(\text{boys together}) = \frac{2}{7}$ via the block method	1
(ii) Uses the complement to get $P(\text{not together}) = \frac{5}{7}$	1

Solution.

(i) For 8 people around a round table, the number of arrangements is

$$(8-1)! = 7! = 5040.$$

(ii) First find the probability the two boys **are** together. Treat the two boys as a block: this leaves 7 units around the table in $(7-1)! = 6!$ ways, with the boys ordered in $2!$ ways:

$$P(\text{boys together}) = \frac{6! \times 2!}{7!} = \frac{2}{7}.$$

By the complement,

$$P(\text{boys not together}) = 1 - \frac{2}{7} = \frac{5}{7}.$$

Q104 2022 Manly Trial Q11(f) · 3 marks

A group of 10 people arrive to eat at a restaurant. The only seating available for them is at two circular tables, one that seats six people and another that seats four. Assuming that the seating arrangement is random, what is the probability that a particular couple will be seated at the same table? (The couple do not have to sit next to each other.)

Criteria	Marks
Sets up the split as $\binom{10}{6} = 210$ equally likely outcomes	1
Counts same-table cases: both at 6-seat ($\binom{8}{4} = 70$) and both at 4-seat ($\binom{8}{2} = 28$)	1
Correct probability $\frac{98}{210} = \frac{7}{15}$	1

Solution.

The seating only depends on **which** table each person is assigned to, so consider how the 10 people split into a group of 6 (first table) and a group of 4 (second table). The number of ways to choose the 6 people at the larger table is $\binom{10}{6} = 210$.

The couple sit at the same table in two ways.

Both at the 6-seat table: the couple take 2 of those 6 places, and the remaining 4 places are chosen from the other 8 people: $\binom{8}{4} = 70$ ways.

Both at the 4-seat table: the couple are then at the 4-seat table, so the 6-seat table is filled by 6 of the remaining 8 people: $\binom{8}{6} = \binom{8}{2} = 28$ ways.

$$P(\text{same table}) = \frac{70 + 28}{210} = \frac{98}{210} = \frac{7}{15}.$$

Alternative check: Fix one member of the couple. With probability $\frac{6}{10}$ they sit at the 6-seat table, leaving 5 of the remaining 9 seats there, so the partner joins with probability $\frac{5}{9}$; with probability $\frac{4}{10}$ they sit at the 4-seat table, giving $\frac{3}{9}$. Then $\frac{6}{10} \cdot \frac{5}{9} + \frac{4}{10} \cdot \frac{3}{9} = \frac{7}{15}$.

19. Selections, then a probability

Q105 2024 HSC Q9 · 1 mark

A bag contains n metal coins, $n \geq 3$, that are made from either silver or bronze. There are k silver coins in the bag and the rest are bronze. Two coins are to be drawn at random from the bag, with the first coin drawn not being replaced before the second coin is drawn. Which of the following expressions will give the probability that the two coins drawn are made of the same metal?

(A) $\dfrac{k(k-1)+(n-k)(n-k-1)}{n(n-1)}$

(B) $\binom{n}{2}\binom{n}{k}\left(1-\dfrac{k}{n}\right)^{n-2}$

(C) $\dfrac{\binom{k}{2}+\binom{n-k}{2}}{n(n-1)}$

(D) $\dfrac{k^2+(n-k)^2}{n^2}$

Criteria	Marks
Correct expression, option (A) (MC / single mark)	1

Solution.

Two coins are drawn without replacement from n coins (k silver, $n-k$ bronze). "Same metal" means both silver or both bronze. Counting ordered draws, the total number of outcomes is $n(n-1)$.

Both silver: $k(k-1)$ ordered outcomes. Both bronze: $(n-k)(n-k-1)$ ordered outcomes. Hence

$$P(\text{same metal}) = \frac{k(k-1)+(n-k)(n-k-1)}{n(n-1)},$$

which is option **(A)**.

Note on the trap: option (C), $\dfrac{\binom{k}{2}+\binom{n-k}{2}}{n(n-1)}$, mixes an unordered numerator with an ordered denominator. Its correct form would need denominator $\binom{n}{2}$; as written it is exactly half the true probability. Option (A) is correct.

Q106 2021 Ascham Trial Q11(d) · 2 marks

A committee of 5 is to be chosen from 5 men and 4 women. What is the probability that the committee contains three women?

Criteria	Marks
Correct favourable count $\binom{4}{3}\binom{5}{2}$ over $\binom{9}{5}$	1
Correct probability $\dfrac{20}{63}$	1

Solution.

A committee of 5 is chosen from $5+4=9$ people, so the total number of committees is $\binom{9}{5}=126$.

"Exactly three women" means 3 of the 4 women and 2 of the 5 men:

$$\text{favourable} = \binom{4}{3}\binom{5}{2} = 4\times 10 = 40.$$

$$P(\text{3 women}) = \frac{40}{126} = \frac{20}{63}.$$

Q107 2024 Newington Trial Q11(a) · **2 marks**

A student representative committee of 7 people is to be chosen from a group of 10 men and 11 women. Find the probability that the committee is made of 3 men and 4 women.

Criteria	**Marks**
Correct favourable count $\binom{10}{3}\binom{11}{4}$ over $\binom{21}{7}$	1
Correct probability $\frac{110}{323}$	1

Solution.

A committee of 7 is chosen from $10+11=21$ people, so the total number of committees is $\binom{21}{7}=116280$.

For 3 men and 4 women, choose 3 of the 10 men and 4 of the 11 women:

$$\text{favourable} = \binom{10}{3}\binom{11}{4} = 120\times 330 = 39600.$$

$$P(\text{3 men, 4 women}) = \frac{39600}{116280} = \frac{110}{323}.$$

Q108 2022 Sydney Boys Trial Q13(a) · 2 marks

A fireworks technician is testing fireworks for an upcoming display. He has 20 fireworks, of which 13 are red and 7 are yellow. He launches 8 of them in a random order. What is the probability that 5 of the launched fireworks are red?

Criteria	Marks
Correct favourable count $\binom{13}{5}\binom{7}{3}$ over $\binom{20}{8}$	1
Correct probability $\dfrac{231}{646}$	1

Solution.

Launching 8 of the 20 fireworks in a random order is equivalent to choosing which 8 are launched, so the total is $\binom{20}{8} = 125970$.

For exactly 5 red launched, choose 5 of the 13 red and 3 of the 7 yellow:

$$\text{favourable} = \binom{13}{5}\binom{7}{3} = 1287 \times 35 = 45045.$$

$$P(5 \text{ red}) = \frac{45045}{125970} = \frac{231}{646}.$$

Q109 2023 Manly Trial Q11(b) · **2 marks**

A Mathematics department consists of 12 teachers. Nine of the teachers wear glasses and three of the teachers do not wear glasses. Five of these teachers go out for dinner together. What is the probability there will be more teachers who wear glasses than teachers who do not wear glasses in the group who go out for dinner?

Criteria	**Marks**
Identifies the favourable cases (3, 4 or 5 glasses-wearers) and computes $\binom{9}{g}\binom{3}{5-g}$	1
Correct probability $\dfrac{21}{22}$ (direct or by complement)	1

Solution.

Five teachers are chosen from 12, so the total is $\binom{12}{5} = 792$.

Let g be the number of glasses-wearers chosen (out of 9), so $5-g$ do not wear glasses (out of only 3). "More glasses-wearers than non-glasses-wearers" needs $g > 5-g$, i.e. $g \geq 3$. Since there are only 3 non-wearers, g can be 3, 4 or 5:

$$\binom{9}{3}\binom{3}{2} + \binom{9}{4}\binom{3}{1} + \binom{9}{5}\binom{3}{0} = 252 + 378 + 126 = 756.$$

$$P = \frac{756}{792} = \frac{21}{22}.$$

Complement check: The only non-favourable case is $g = 2$ (3 non-wearers, which is all of them): $\binom{9}{2}\binom{3}{3} = 36$, giving $\frac{36}{792} = \frac{1}{22}$, so $P = 1 - \frac{1}{22} = \frac{21}{22}$.

Q110 2020 Newcastle Grammar Trial Q14(b) · **2 marks**

A bag contains n red marbles and one blue marble. Three marbles are drawn (without replacement). The probability that the three marbles are red is $\dfrac{5}{8}$. Find the value of n.

Criteria	**Marks**
Sets up $P(3 \text{ red}) = \dfrac{\binom{n}{3}}{\binom{n+1}{3}}$ (or the product form) equal to $\dfrac{5}{8}$	1
Solves correctly to give $n = 7$	1

Solution.

There are $n + 1$ marbles in total (n red and 1 blue). Drawing 3 without replacement, the probability all three are red is

$$P(3 \text{ red}) = \frac{\binom{n}{3}}{\binom{n+1}{3}}.$$

This ratio simplifies neatly: $\dfrac{\binom{n}{3}}{\binom{n+1}{3}} = \dfrac{n-2}{n+1}$ (the three red-draw probabilities $\frac{n}{n+1} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n-1}$ telescope). Setting this equal to $\frac{5}{8}$:

$$\frac{n-2}{n+1} = \frac{5}{8} \implies 8(n-2) = 5(n+1) \implies 8n - 16 = 5n + 5 \implies 3n = 21 \implies n = 7.$$

Check: with $n = 7$, $P = \dfrac{\binom{7}{3}}{\binom{8}{3}} = \dfrac{35}{56} = \dfrac{5}{8}$. ✓

Q111 2022 Normanhurst Boys Trial Q11(c) · **3 marks**

A school has twenty prefects. Jun and Alex are two of those prefects. A committee of seven prefects is to be formed to organise a function.

(i) How many different committees can be formed?

(ii) What is the probability that the committee contains Jun but does not contain Alex?

Criteria	**Marks**
(i) Correct total $\binom{20}{7} = 77520$	1
(ii) Correct favourable count $\binom{18}{6}$ (Jun in, Alex out, remaining 6 from other 18)	1
(ii) Correct probability $\frac{91}{380}$	1

Solution.

(i) A committee of 7 is chosen from 20 prefects:

$$\binom{20}{7} = 77520.$$

(ii) Require Jun on the committee and Alex off it. Place Jun in the committee (this fixes 1 of the 7 places) and exclude Alex. The remaining 6 members are chosen from the other $20 - 2 = 18$ prefects:

$$\text{favourable} = \binom{18}{6} = 18564.$$

$$P(\text{Jun in, Alex out}) = \frac{18564}{77520} = \frac{91}{380}.$$