Year 11 Mathematics Extension 1

Topic: Further Trigonometry (3 of 4)

Review Questions (Worked Solutions): Further Trigonometric Equations

Term 3, 2026

Solving trigonometric equations

1. Solving equations reducible to a quadratic in one ratio

Q1 2024 Caringbah Trial Q8 · **1 mark**

What is the solution of the trigonometric equation $3\sin^2 x - 4\cos x + 1 = 0$ in the domain $0 \leq x \leq \pi$?

(A) $x = \dfrac{2\pi}{3}$

(B) $x = \dfrac{2\pi}{5}$

(C) $x \approx 0.841$

(D) $x \approx 1.969$

Criteria	Marks
Correct option (C)	1

Solution.

Replace $\sin^2 x = 1 - \cos^2 x$: $3(1 - \cos^2 x) - 4\cos x + 1 = 0$, i.e. $3\cos^2 x + 4\cos x - 4 = 0$. Factorising, $(3\cos x - 2)(\cos x + 2) = 0$, so $\cos x = \frac{2}{3}$ (rejecting $\cos x = -2$). On $0 \leq x \leq \pi$, $x = \arccos \frac{2}{3} \approx 0.841$. The answer is (C).

Q2 2025 Merewether Trial Q11f · **2 marks**

Solve $\cos 2\theta = \cos^2 \theta + \cos \theta + 1$ for $0 \leq \theta \leq 2\pi$.

Criteria	Marks
Uses $\cos 2\theta = 2\cos^2 \theta - 1$ to form the quadratic $\cos^2 \theta - \cos \theta - 2 = 0$	1
Solves and gives $\theta = \pi$, rejecting $\cos \theta = 2$	1

Solution.

Using $\cos 2\theta = 2\cos^2 \theta - 1$: $2\cos^2 \theta - 1 = \cos^2 \theta + \cos \theta + 1$, so $\cos^2 \theta - \cos \theta - 2 = 0$, i.e. $(\cos \theta - 2)(\cos \theta + 1) = 0$. Rejecting $\cos \theta = 2$, we need $\cos \theta = -1$. On $0 \leq \theta \leq 2\pi$,

$$\theta = \pi.$$

Q3 2020 Gosford Prelim Q11(c) · **3 marks**

Solve the equation $\sin^2\theta = \cos\theta + 1$ where $0 \le \theta \le 2\pi$.

Criteria	Marks
Uses $\sin^2\theta = 1 - \cos^2\theta$ to form $\cos^2\theta + \cos\theta = 0$	1
Factorises to $\cos\theta(\cos\theta + 1) = 0$, giving $\cos\theta = 0$ or $\cos\theta = -1$	1
Lists all solutions $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ in the domain	1

Solution.

Replace $\sin^2\theta = 1 - \cos^2\theta$: $1 - \cos^2\theta = \cos\theta + 1$, so $\cos^2\theta + \cos\theta = 0$, i.e. $\cos\theta(\cos\theta + 1) = 0$. Then $\cos\theta = 0$ or $\cos\theta = -1$. On $0 \le \theta \le 2\pi$,

$$\theta = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}.$$

Q4 2023 Sydney Grammar Prelim Q12(a) · **3 marks**

Solve $2\cos^2 x - 3\sin x = 0$, for $0 \le x \le 2\pi$. Give your answers in exact form.

Criteria	Marks
Uses $\cos^2 x = 1 - \sin^2 x$ to form $2\sin^2 x + 3\sin x - 2 = 0$	1
Solves for $\sin x = \frac{1}{2}$, rejecting $\sin x = -2$	1
Gives both solutions $x = \frac{\pi}{6}, \frac{5\pi}{6}$ in exact form	1

Solution.

Replace $\cos^2 x = 1 - \sin^2 x$: $2(1 - \sin^2 x) - 3\sin x = 0$, so $2\sin^2 x + 3\sin x - 2 = 0$, i.e. $(2\sin x - 1)(\sin x + 2) = 0$. Rejecting $\sin x = -2$, $\sin x = \frac{1}{2}$. On $0 \le x \le 2\pi$,

$$x = \frac{\pi}{6},\ \frac{5\pi}{6}.$$

Q5 2023 Hurlstone Prelim Q13(c) · **3 marks**

Solve, correct to the nearest degree: $\sin\theta + 6\cos^2\theta = 5$ where $0° \le \theta \le 360°$.

Criteria	Marks
Uses $\cos^2\theta = 1 - \sin^2\theta$ to form $6\sin^2\theta - \sin\theta - 1 = 0$	1
Solves for $\sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{3}$	1
Gives all four solutions $\theta = 30°, 150°, 199°, 341°$	1

Solution.

Replace $\cos^2\theta = 1-\sin^2\theta$: $\sin\theta+6(1-\sin^2\theta) = 5$, so $6\sin^2\theta-\sin\theta-1 = 0$, i.e. $(2\sin\theta-1)(3\sin\theta+1) = 0$. Thus $\sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{3}$. For $\sin\theta = \frac{1}{2}$: $\theta = 30°, 150°$. For $\sin\theta = -\frac{1}{3}$ (θ in the third and fourth quadrants): $\theta \approx 199°, 341°$. Hence

$$\theta = 30°,\ 150°,\ 199°,\ 341°.$$

Q6 2024 Gosford Trial Q14a · **3 marks**

Solve $\cos 2x + \sin x - 1 = 0$ for $0 \le x \le 2\pi$.

Criteria	Marks
Uses $\cos 2x = 1 - 2\sin^2 x$ to form $2\sin^2 x - \sin x = 0$	1
Factorises to $\sin x(2\sin x - 1) = 0$, giving $\sin x = 0$ or $\sin x = \frac{1}{2}$	1
Lists all solutions $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$ in the domain	1

Solution.

Using $\cos 2x = 1 - 2\sin^2 x$: $(1 - 2\sin^2 x) + \sin x - 1 = 0$, so $-2\sin^2 x + \sin x = 0$, i.e. $\sin x(2\sin x - 1) = 0$. Then $\sin x = 0$ giving $x = 0, \pi, 2\pi$, or $\sin x = \frac{1}{2}$ giving $x = \frac{\pi}{6}, \frac{5\pi}{6}$. On $0 \le x \le 2\pi$,

$$x = 0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi,\ 2\pi.$$

Q7 2020 Sydney Tech Prelim Q13(b) · **3 marks**

Solve the following for $0 \leq \theta \leq 2\pi$: $\tan^2\theta = 2\sec^2\theta - 3$.

Criteria	Marks
Uses $\sec^2\theta = 1 + \tan^2\theta$ to reduce the equation to $\tan^2\theta = 1$	1
Solves $\tan\theta = \pm 1$	1
Lists all solutions $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ in the domain	1

Solution.

Using $\sec^2\theta = 1 + \tan^2\theta$: $\tan^2\theta = 2(1 + \tan^2\theta) - 3$, so $\tan^2\theta = 2\tan^2\theta - 1$, i.e. $\tan^2\theta = 1$ and $\tan\theta = \pm 1$. On $0 \leq \theta \leq 2\pi$,

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

Q8 2021 HSC Q11(g) · **3 marks**

By factorising, or otherwise, solve $2\sin^3 x + 2\sin^2 x - \sin x - 1 = 0$ for $0 \leq x \leq 2\pi$.

Criteria	Marks
Factorises by grouping to $(\sin x + 1)(2\sin^2 x - 1) = 0$	1
Solves $\sin x = -1$ and $\sin x = \pm\frac{1}{\sqrt{2}}$	1
Lists all solutions $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ in the domain	1

Solution.

Grouping: $2\sin^2 x(\sin x + 1) - (\sin x + 1) = (\sin x + 1)(2\sin^2 x - 1) = 0$. So $\sin x = -1$, giving $x = \frac{3\pi}{2}$; or $2\sin^2 x = 1$, i.e. $\sin x = \pm\frac{1}{\sqrt{2}}$, giving $x = \frac{\pi}{4}, \frac{3\pi}{4}$ (for $+\frac{1}{\sqrt{2}}$) and $x = \frac{5\pi}{4}, \frac{7\pi}{4}$ (for $-\frac{1}{\sqrt{2}}$). On $0 \leq x \leq 2\pi$,

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

Q9 2022 Hurlstone Prelim Q12(d) · **3 marks**

Solve $3\cos^2\theta - \sin^2\theta - 6\cos\theta = 3$, for $0° \le \theta \le 360°$.

Criteria	Marks
Uses $\sin^2\theta = 1 - \cos^2\theta$ to form $4\cos^2\theta - 6\cos\theta - 4 = 0$	1
Solves for $\cos\theta = -\frac{1}{2}$, rejecting $\cos\theta = 2$	1
Gives both solutions $\theta = 120°, 240°$	1

Solution.

Replace $\sin^2\theta = 1 - \cos^2\theta$: $3\cos^2\theta - (1 - \cos^2\theta) - 6\cos\theta = 3$, so $4\cos^2\theta - 6\cos\theta - 4 = 0$, i.e. $2\cos^2\theta - 3\cos\theta - 2 = 0$ and $(2\cos\theta + 1)(\cos\theta - 2) = 0$. Rejecting $\cos\theta = 2$, $\cos\theta = -\frac{1}{2}$. On $0° \le \theta \le 360°$,

$$\theta = 120°,\ 240°.$$

2. Solving by factorisation and identity substitution

Q10 2020 Abbotsleigh Trial Q13(d)(i) · **1 mark**

A design uses the shaded area enclosed between the curves $y = \cos^2\frac{x}{3}$ and $y = \sin^2\frac{x}{3}$. Find the x-coordinates of the points P and Q where the two curves first meet for $x > 0$.

Criteria	Marks
Reduces to $\cos\frac{2x}{3} = 0$ and states both x-coordinates	1

Solution.

The curves meet where $\cos^2\frac{x}{3} = \sin^2\frac{x}{3}$, i.e. $\cos^2\frac{x}{3} - \sin^2\frac{x}{3} = 0$. Using the double-angle result this is $\cos\frac{2x}{3} = 0$, so $\frac{2x}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ The first two positive solutions give

$$x = \frac{3\pi}{4} \quad (P), \qquad x = \frac{9\pi}{4} \quad (Q).$$

Q11 2023 Hurlstone Prelim Q6 · **1 mark**

The t-results are to be used to solve the equation $5\sin x - 2\cos x = 2$, for values of x such that $0° \leq x \leq 360°$. Which value of x needs to be considered separately because it will not be "captured" using this method?

(A) $x = 90°$

(B) $x = 180°$

(C) $x = 0°$

(D) $x = 270°$

Criteria	Marks
Correct option (B)	1

Solution.

The substitution $t = \tan\dfrac{x}{2}$ is undefined at $x = 180°$, so that value can never be produced by the t-method. Checking it directly, $5\sin 180° - 2\cos 180° = 0 - 2(-1) = 2$, so $x = 180°$ is in fact a solution and must be considered separately. The answer is (B).

Q12 2020 Fort St Trial Q14(a) · **3 marks**

Use the t-formula to solve the equation $\cos x - \sin x = 1$, where $0 \leq x \leq 2\pi$.

Criteria	Marks
Substitutes the t-results and reduces to an equation in t	1
Solves for t (obtaining $t = 0$, plus the boundary case)	1
States all solutions $x = 0,\ \frac{3\pi}{2},\ 2\pi$ over the given domain	1

Solution.

Let $t = \tan\dfrac{x}{2}$, so $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\sin x = \dfrac{2t}{1+t^2}$. Then

$$\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = 1 \implies 1 - t^2 - 2t = 1 + t^2 \implies 2t^2 + 2t = 0 \implies 2t(t+1) = 0.$$

So $t = 0$ or $t = -1$, giving $\dfrac{x}{2} = 0,\ \pi$ (from $t = 0$) or $\dfrac{x}{2} = \dfrac{3\pi}{4}$ (from $t = -1$), i.e. $x = 0,\ 2\pi,\ \dfrac{3\pi}{2}$. The t-formula misses $x = \pi$, but $x = \pi$ is not a solution here (it gives $\cos\pi - \sin\pi = -1$), so the complete solution set is

$$x = 0,\ \frac{3\pi}{2},\ 2\pi.$$

Q13 2022 Hunters Hill Trial Q12(a) · **3 marks**

Find all values of θ, such that $-\pi \leq \theta \leq \pi$, that satisfy the equation

$$3\sin\theta = \sqrt{3}\,\sin 2\theta.$$

Criteria	**Marks**
Uses $\sin 2\theta = 2\sin\theta\cos\theta$ and factorises	1
Solves $\sin\theta = 0$: $\theta = 0, \pm\pi$	1
Solves $\cos\theta = \frac{\sqrt{3}}{2}$: $\theta = \pm\frac{\pi}{6}$	1

Solution.

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$3\sin\theta = 2\sqrt{3}\,\sin\theta\cos\theta \;\Rightarrow\; \sin\theta\,(3 - 2\sqrt{3}\cos\theta) = 0.$$

Either $\sin\theta = 0$, giving $\theta = -\pi,\ 0,\ \pi$; or $\cos\theta = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$, giving $\theta = \pm\dfrac{\pi}{6}$. Over $-\pi \leq \theta \leq \pi$,

$$\theta = -\pi,\ -\frac{\pi}{6},\ 0,\ \frac{\pi}{6},\ \pi.$$

Q14 2023 Manly Trial Q13(c) · 3 marks

Use sums to products of trigonometric ratios to solve the equation

$$\cos x + \cos 2x + \cos 3x = 0 \qquad \text{where } 0 \le x \le 2\pi.$$

Criteria	Marks
Applies sum-to-product to $\cos x + \cos 3x = 2\cos 2x \cos x$	1
Factorises to $\cos 2x\,(2\cos x + 1) = 0$	1
States all solutions over $0 \le x \le 2\pi$	1

Solution.

Pairing the outer terms, $\cos x + \cos 3x = 2\cos\left(\dfrac{x+3x}{2}\right)\cos\left(\dfrac{3x-x}{2}\right) = 2\cos 2x\cos x$. Hence

$$2\cos 2x\cos x + \cos 2x = \cos 2x\,(2\cos x + 1) = 0.$$

From $\cos 2x = 0$: $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$. From $2\cos x + 1 = 0$, i.e. $\cos x = -\dfrac{1}{2}$: $x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$. So

$$x = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}.$$

Q15 2021 Riverview Trial Q1(b) · 3 marks

Solve the equation $\sin 2x = \tan x$ for the domain $[0, \pi)$.

Criteria	Marks
Writes $\tan x = \frac{\sin x}{\cos x}$ and factorises out $\sin x$	1
Reduces the second factor to $\cos 2x = 0$ (equivalently $2\cos^2 x = 1$)	1
States all solutions $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}$ over $[0, \pi)$	1

Solution.

Writing $\sin 2x = 2\sin x\cos x$ and $\tan x = \dfrac{\sin x}{\cos x}$,

$$2\sin x\cos x = \frac{\sin x}{\cos x} \implies \sin x\left(2\cos x - \frac{1}{\cos x}\right) = 0 \implies \frac{\sin x\,(2\cos^2 x - 1)}{\cos x} = 0.$$

So $\sin x = 0$ giving $x = 0$, or $2\cos^2 x - 1 = \cos 2x = 0$ giving $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$ in $[0, \pi)$. (Here $x = \dfrac{\pi}{2}$ is excluded as $\tan x$ is undefined.) Thus

$$x = 0, \frac{\pi}{4}, \frac{3\pi}{4}.$$

Q16 2023 Ascham Trial Q13(a) · **4 marks**

(i) Show that $\tan 2x + \tan x = \dfrac{\sin 3x}{\cos 2x\ \cos x}$.

(ii) Hence solve the equation $\tan 2x + \tan x = 0$ for $0 \le x \le \dfrac{\pi}{2}$.

Criteria	**Marks**
(i) Combines $\tan 2x + \tan x$ over a common denominator	1
(i) Recognises the numerator as $\sin(2x + x) = \sin 3x$	1
(ii) Sets $\sin 3x = 0$ and lists candidate solutions	1
(ii) Rejects values excluded by the domain/denominator and states $x = 0, \frac{\pi}{3}$	1

Solution.

(i) $\tan 2x + \tan x = \dfrac{\sin 2x}{\cos 2x} + \dfrac{\sin x}{\cos x} = \dfrac{\sin 2x \cos x + \cos 2x \sin x}{\cos 2x \cos x} = \dfrac{\sin(2x + x)}{\cos 2x \cos x} = \dfrac{\sin 3x}{\cos 2x \cos x}$.

(ii) From part (i), $\tan 2x + \tan x = 0$ requires $\sin 3x = 0$ (with $\cos 2x \cos x \neq 0$). So $3x = 0, \pi, 2\pi, \ldots$, i.e. $x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \ldots$ On $0 \le x \le \dfrac{\pi}{2}$ this leaves $x = 0$ and $x = \dfrac{\pi}{3}$. (The value $x = \dfrac{\pi}{4}$ is not admissible since $\cos 2x = 0$ there, and $x = \dfrac{\pi}{2}$ makes $\cos x = 0$.) Thus

$$x = 0,\ \frac{\pi}{3}.$$

Q17 2024 Hunters Hill Trial Q12(d) · **4 marks**

Solve $\sin\left(x+\frac{\pi}{3}\right)\cos\left(x-\frac{\pi}{3}\right)=\frac{1+\sqrt{3}}{4}$, for $0\le x\le 2\pi$.

Criteria	**Marks**
Applies the product-to-sum identity to the LHS	1
Simplifies to $\frac{1}{2}\sin 2x+\frac{\sqrt{3}}{4}=\frac{1+\sqrt{3}}{4}$	1
Reduces to $\sin 2x=\frac{1}{2}$	1
States all four solutions over $0\le x\le 2\pi$	1

Solution.

Using $\sin A\cos B=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$ with $A=x+\frac{\pi}{3}$, $B=x-\frac{\pi}{3}$,

$$\text{LHS}=\frac{1}{2}\left[\sin 2x+\sin\tfrac{2\pi}{3}\right]=\frac{1}{2}\sin 2x+\frac{\sqrt{3}}{4}.$$

Setting this equal to $\frac{1+\sqrt{3}}{4}$ gives $\frac{1}{2}\sin 2x=\frac{1}{4}$, so $\sin 2x=\frac{1}{2}$. Then $2x=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6}$, so

$$x=\frac{\pi}{12},\ \frac{5\pi}{12},\ \frac{13\pi}{12},\ \frac{17\pi}{12}.$$

Q18 2021 Ascham Trial Q14(c) · **4 marks**

(i) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) Let $x = \cos\theta$. Find three distinct roots of $8x^3 - 6x - 1 = 0$.

Criteria	**Marks**
(i) Expands $\cos(2\theta + \theta)$ and substitutes double-angle results	1
(i) Simplifies to $4\cos^3\theta - 3\cos\theta$	1
(ii) Uses $x = \cos\theta$ to reduce the cubic to $\cos 3\theta = \frac{1}{2}$	1
(ii) States three distinct roots	1

Solution.

(i) $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$. Using $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\cos 3\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = (2\cos^2\theta - 1)\cos\theta - 2(1 - \cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

(ii) With $x = \cos\theta$, $8x^3 - 6x - 1 = 2(4\cos^3\theta - 3\cos\theta) - 1 = 2\cos 3\theta - 1$. So $8x^3 - 6x - 1 = 0$ becomes $\cos 3\theta = \dfrac{1}{2}$, giving $3\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$, i.e. $\theta = \dfrac{\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}$. The three distinct roots are

$$x = \cos\frac{\pi}{9},\ \cos\frac{5\pi}{9},\ \cos\frac{7\pi}{9}.$$

Q19 2023 North Sydney Boys Prelim Q4 · **5 marks**

(i) Use the substitution $t = \tan\dfrac{x}{2}$ to show that

$$2\sin x - 3\cos x = \frac{3t^2 + 4t - 3}{1 + t^2}.$$

(ii) Hence solve the equation $2\sin x - 3\cos x = 2$ for $0° \le x \le 360°$. Give your answer correct to the nearest degree.

Criteria	**Marks**
(i) Substitutes $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$	1
(i) Simplifies to the required form	1
(ii) Forms and solves $t^2 + 4t - 5 = 0$ giving $t = 1$, $t = -5$	1
(ii) Converts $t = 1$ to $x = 90°$	1
(ii) Converts $t = -5$ to $x \approx 203°$	1

Solution.

(i) With $t = \tan\dfrac{x}{2}$, $\sin x = \dfrac{2t}{1 + t^2}$ and $\cos x = \dfrac{1 - t^2}{1 + t^2}$, so

$$2\sin x - 3\cos x = \frac{4t}{1 + t^2} - \frac{3(1 - t^2)}{1 + t^2} = \frac{4t - 3 + 3t^2}{1 + t^2} = \frac{3t^2 + 4t - 3}{1 + t^2}.$$

(ii) Setting the expression equal to 2: $3t^2+4t-3 = 2(1+t^2)$, so $t^2+4t-5 = 0$, i.e. $(t+5)(t-1) = 0$. Then $t = 1$ or $t = -5$. From $t = \tan\dfrac{x}{2} = 1$: $\dfrac{x}{2} = 45°$, so $x = 90°$. From $t = \tan\dfrac{x}{2} = -5$: $\dfrac{x}{2} = 180° - 78.69° = 101.31°$, so $x \approx 203°$. (Note $x = 180°$, missed by the t-method, gives $2\sin - 3\cos = 3 \neq 2$, so it is not a solution.) Hence

$$x \approx 90°,\ 203°.$$

Q20 2024 James Ruse Prelim Q9(a) · 7 marks

(i) Using compound angle results, show that $\sin 3x = 3\sin x - 4\sin^3 x$.

(ii) Consider the equation $\sin 2x = \sin 3x$. You are given that $x = \dfrac{\pi}{5}$ is a solution to this equation. Show that $x = \dfrac{3\pi}{5}$ is another solution to this equation.

(iii) Hence, find the exact value of $\cos \dfrac{3\pi}{5}$.

Criteria	Marks
(i) Expands $\sin(2x + x)$ using compound angle results	1
(i) Substitutes double-angle results and simplifies	1
(i) Reaches $3\sin x - 4\sin^3 x$	1
(ii) Verifies $\sin \frac{6\pi}{5} = \sin \frac{9\pi}{5}$ (or equivalent) so $x = \frac{3\pi}{5}$ works	1
(iii) Uses (i),(ii) to obtain $2\cos^2 \frac{3\pi}{5} - \cos \frac{3\pi}{5} = \frac{1}{2} \cdot \ldots$ reducing to a quadratic in $c = \cos \frac{3\pi}{5}$	1
(iii) Solves the quadratic $4c^2 - 2c - 1 = 0$	1
(iii) Rejects the positive root and states $\cos \frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}$	1

Solution.

(i) $\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$. Using $\sin 2x = 2\sin x \cos x$ and $\cos 2x = 1 - 2\sin^2 x$,

$$\sin 3x = 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x = 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x.$$

(ii) At $x = \dfrac{3\pi}{5}$: $\sin 2x = \sin \dfrac{6\pi}{5}$ and $\sin 3x = \sin \dfrac{9\pi}{5}$. Now $\sin \dfrac{9\pi}{5} = \sin\left(2\pi - \dfrac{\pi}{5}\right) = -\sin \dfrac{\pi}{5}$ and $\sin \dfrac{6\pi}{5} = \sin\left(\pi + \dfrac{\pi}{5}\right) = -\sin \dfrac{\pi}{5}$, so $\sin \dfrac{6\pi}{5} = \sin \dfrac{9\pi}{5}$ and $x = \dfrac{3\pi}{5}$ is indeed a solution.

(iii) Substituting $x = \dfrac{3\pi}{5}$ into $\sin 2x = \sin 3x$ and using parts (i) with $\sin 2x = 2\sin x \cos x$:

$$2\sin \tfrac{3\pi}{5} \cos \tfrac{3\pi}{5} = 3\sin \tfrac{3\pi}{5} - 4\sin^3 \tfrac{3\pi}{5}.$$

Since $\sin \dfrac{3\pi}{5} \neq 0$, divide through: $2\cos \frac{3\pi}{5} = 3 - 4\sin^2 \frac{3\pi}{5} = 3 - 4\left(1 - \cos^2 \frac{3\pi}{5}\right)$. Letting $c = \cos \dfrac{3\pi}{5}$, this gives $2c = 4c^2 - 1$, i.e. $4c^2 - 2c - 1 = 0$, so $c = \dfrac{1 \pm \sqrt{5}}{4}$. As $\dfrac{\pi}{2} < \dfrac{3\pi}{5} < \pi$, $\cos \dfrac{3\pi}{5} < 0$, so we reject the positive root:

$$\cos \frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}.$$

The auxiliary-angle method

3. Expressing and solving $a\cos x + b\sin x$

Q21 2024 Sydney Girls Trial Q3 · 1 mark

Which of the following is equivalent to $\sin x - \sqrt{3}\cos x$?

(A) $2\sin\left(x - \frac{\pi}{3}\right)$

(B) $2\sin\left(x + \frac{\pi}{3}\right)$

(C) $2\sin\left(x - \frac{\pi}{6}\right)$

(D) $2\sin\left(x + \frac{\pi}{6}\right)$

Criteria	Marks
Correct option (A)	1

Solution.

$R = \sqrt{1^2 + (\sqrt{3})^2} = 2$. Writing $\sin x - \sqrt{3}\cos x = R\sin(x - \alpha) = R\sin x\cos\alpha - R\cos x\sin\alpha$ gives $R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$, so $\tan\alpha = \sqrt{3}$ and $\alpha = \frac{\pi}{3}$. Thus $\sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$. The answer is (A).

Q22 2021 Hurlstone Trial Q5 · **1 mark**

Which of the following is an expression for $y = \sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$?

(A) $2\sin\left(x+\frac{\pi}{4}\right)$

(B) $2\sin\left(x+\frac{\pi}{3}\right)$

(C) $2\sin\left(x-\frac{\pi}{4}\right)$

(D) $2\sin\left(x-\frac{\pi}{3}\right)$

Criteria	Marks
Correct option (B)	1

Solution.

$R = \sqrt{1^2+(\sqrt{3})^2} = 2$. From $\sin x + \sqrt{3}\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$, $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$, so $\tan\alpha = \sqrt{3}$, $\alpha = \frac{\pi}{3}$. Thus $\sin x + \sqrt{3}\cos x = 2\sin\left(x+\frac{\pi}{3}\right)$. The answer is (B).

Q23 2025 Smiths Hill Trial Q1 · **1 mark**

Which of the following expressions is equivalent to $\cos x + \sin x$?

(A) $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

(B) $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$

(C) $\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

(D) $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

Criteria	Marks
Correct option (B)	1

Solution.

Write $\cos x+\sin x = R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$. Then $R\cos\alpha = 1$ and $R\sin\alpha = 1$, so $R = \sqrt{2}$ and $\alpha = \frac{\pi}{4}$, giving $\cos x+\sin x = \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$. The answer is (B). (Note $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$, so option (D) is algebraically identical to (B).)

Q24 2020 Normanhurst Boys Trial Q3 · **1 mark**

What is the smallest positive value of x for which $\sin x + \cos x$ is at its maximum?

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$

(C) π

(D) $\frac{\pi}{3}$

Criteria	Marks
Correct option (B)	1

Solution.

$\sin x + \cos x = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)$, which is a maximum when $x + \frac{\pi}{4} = \frac{\pi}{2}$, i.e. $x = \frac{\pi}{4}$. The answer is (B).

Q25 2023 St George Girls Trial Q11d · **2 marks**

Express $9\sin x + 40\cos x$ in the form $R\sin(x+\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$. Leave α correct to 3 significant figures.

Criteria	Marks
Finds $R = 41$	1
Finds $\alpha = \tan^{-1}\frac{40}{9} \approx 1.35$ and states the R-form	1

Solution.

Write $9\sin x + 40\cos x = R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$, so $R\cos\alpha = 9$ and $R\sin\alpha = 40$. Then $R = \sqrt{9^2 + 40^2} = \sqrt{1681} = 41$ and $\tan\alpha = \frac{40}{9}$, giving $\alpha = \tan^{-1}\frac{40}{9} \approx 1.35$ (radians, 3 s.f.), which lies in $\left[0, \frac{\pi}{2}\right]$. Hence $9\sin x + 40\cos x = 41\sin(x + 1.35)$.

Q26 2021 Sydney Girls Trial Q13a · **2 marks**

A particle is moving in a straight line and is oscillating backwards and forwards. At time t seconds, it has displacement x metres from a fixed point O on the line, where $x = A\cos\left(\frac{\pi}{4}t + \alpha\right)$, $A > 0$, and $0 < \alpha < \frac{\pi}{2}$. After 1 second the particle is 2 metres to the right of O, and after 3 seconds it is 4 metres to the left of O.

Given that $A\cos\alpha - A\sin\alpha = 2\sqrt{2}$ and $A\cos\alpha + A\sin\alpha = 4\sqrt{2}$, solve the equations simultaneously to find the values of A and α in exact form.

Criteria	Marks
Adds/subtracts to obtain $A\cos\alpha = 3\sqrt{2}$ and $A\sin\alpha = \sqrt{2}$	1
Finds $A = 2\sqrt{5}$ and $\alpha = \tan^{-1}\frac{1}{3}$ in exact form	1

Solution.

Adding the two equations: $2A\cos\alpha = 6\sqrt{2}$, so $A\cos\alpha = 3\sqrt{2}$. Subtracting: $2A\sin\alpha = 2\sqrt{2}$, so $A\sin\alpha = \sqrt{2}$. Then $A^2 = (A\cos\alpha)^2 + (A\sin\alpha)^2 = (3\sqrt{2})^2 + (\sqrt{2})^2 = 18 + 2 = 20$, so $A = 2\sqrt{5}$ (taking $A > 0$). Dividing, $\tan\alpha = \dfrac{A\sin\alpha}{A\cos\alpha} = \dfrac{\sqrt{2}}{3\sqrt{2}} = \dfrac{1}{3}$, so $\alpha = \tan^{-1}\dfrac{1}{3}$ (which lies in $\left(0, \frac{\pi}{2}\right)$).

Q27 2022 Trinity Grammar Trial Q14(b)(iii) · **2 marks**

An object is fired from a point O at an angle of elevation θ to the horizontal. In an earlier part it was shown that θ satisfies

$$\sin 2\theta - 3\cos 2\theta = 3.$$

Given that θ is an angle of elevation, find the value of θ.

Criteria	Marks
Writes the LHS as $\sqrt{10}\sin(2\theta - \alpha)$ and solves for 2θ	1
Selects the valid angle of elevation $\theta = \tan^{-1}3 \approx 71°34'$	1

Solution.

Writing $\sin 2\theta - 3\cos 2\theta = \sqrt{10}\,\sin(2\theta - \alpha)$ with $\tan\alpha = 3$, the equation becomes $\sin(2\theta - \alpha) = \dfrac{3}{\sqrt{10}} = \sin\alpha$. Hence $2\theta - \alpha = \alpha$ or $2\theta - \alpha = \pi - \alpha$, giving $\theta = \alpha = \tan^{-1}3$ or $\theta = \dfrac{\pi}{2}$. As θ is an angle of elevation, $\theta = 90°$ is rejected, so

$$\theta = \tan^{-1}3 \approx 71°34' \ (71.6°).$$

Q28 2025 Gosford Trial Q11a · 3 marks

(i) Express $2\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0° < \alpha < 360°$.

(ii) Hence find the range of the function $g(x) = (2\cos x - \sin x)^2$.

Criteria	Marks
(i) Finds $R = \sqrt{5}$ and $\alpha \approx 26°34'$, giving $\sqrt{5}\cos(x + 26°34')$	2
(ii) States the range $0 \leq g(x) \leq 5$	1

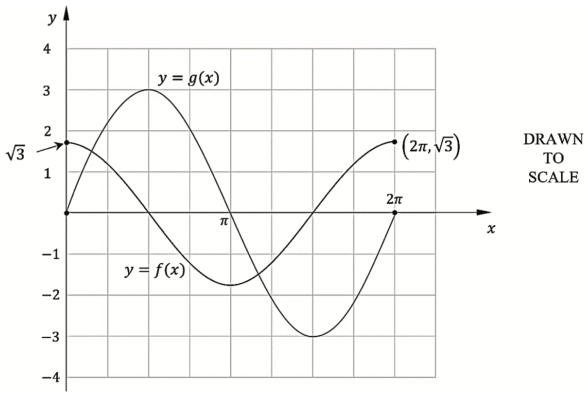
Solution.

(i) Write $2\cos x - \sin x = R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$, so $R\cos\alpha = 2$ and $R\sin\alpha = 1$. Then $R = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\tan\alpha = \dfrac{1}{2}$, so $\alpha = \tan^{-1}\dfrac{1}{2} \approx 26°34'$. Thus $2\cos x - \sin x = \sqrt{5}\cos(x + 26°34')$.

(ii) Since $2\cos x - \sin x = \sqrt{5}\cos(x + 26°34')$ ranges over $\left[-\sqrt{5}, \sqrt{5}\right]$, its square $g(x) = (2\cos x - \sin x)^2$ ranges from 0 (when $2\cos x - \sin x = 0$) up to $\left(\sqrt{5}\right)^2 = 5$. Hence the range is $0 \leq g(x) \leq 5$.

Q29 2025 St George Girls Trial Q12a · **3 marks**

The graphs of $y = f(x)$ and $y = g(x)$ are shown below, where $f(x)$ is a cosine function and $g(x)$ is a sine function. It is given that $f(x) + g(x) = R\cos(x - \theta)$, where $0 \leq \theta \leq \dfrac{\pi}{2}$. Find R and θ, where θ is in radians.



Criteria	Marks
Reads $f(x) = \sqrt{3}\cos x$ and $g(x) = \sin x$ from the graphs	1
Finds $R = 2$	1
Finds $\theta = \dfrac{\pi}{6}$	1

Solution.

From the graphs, the cosine curve has amplitude $\sqrt{3}$ (passing through $(2\pi, \sqrt{3})$) so $f(x) = \sqrt{3}\cos x$, and the sine curve has amplitude 1 so $g(x) = \sin x$. Then $f(x) + g(x) = \sqrt{3}\cos x + \sin x$. Writing this as $R\cos(x - \theta) = R\cos x\cos\theta + R\sin x\sin\theta$ gives $R\cos\theta = \sqrt{3}$ and $R\sin\theta = 1$. Hence $R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ and $\tan\theta = \dfrac{1}{\sqrt{3}}$, so $\theta = \dfrac{\pi}{6}$ (which lies in $\left[0, \dfrac{\pi}{2}\right]$). Thus $f(x) + g(x) = 2\cos\left(x - \dfrac{\pi}{6}\right)$.

Q30 2025 Hills Grammar Trial Q12(a) · 3 marks

It is given that $3\cos x + b\sin x = 5\cos(x+\theta)$, where $b > 0$ and $-90° < \theta < 90°$. Find the values of b and θ, giving your answer for θ correct to the nearest minute.

Criteria	Marks
Equates $R = 5$ to find $b = 4$	1
Sets up $\cos\theta = \frac{3}{5}$, $\sin\theta = -\frac{4}{5}$ (or equivalent)	1
Solves for $\theta = -53°8'$ to the nearest minute	1

Solution.

Expanding, $5\cos(x+\theta) = 5\cos\theta\cos x - 5\sin\theta\sin x$. Matching coefficients with $3\cos x + b\sin x$ gives $5\cos\theta = 3$ and $-5\sin\theta = b$. Since $R = \sqrt{3^2 + b^2} = 5$, we get $b^2 = 16$, so $b = 4$ (as $b > 0$). Then $\cos\theta = \dfrac{3}{5} > 0$ and $\sin\theta = -\dfrac{4}{5} < 0$, so θ is in the fourth quadrant:

$$\theta = -\tan^{-1}\frac{4}{3} = -53°7.8' \approx -53°8'.$$

Q31 2020 North Sydney Boys Trial Q13a · 4 marks

(i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\sin(\theta+\alpha)$ where $R > 0$.

(ii) Hence, solve $\cos\theta - \sqrt{3}\sin\theta = 1$ for $0 \le \theta \le 2\pi$.

Criteria	Marks
(i) Finds $R = 2$ and $\alpha = \dfrac{5\pi}{6}$, giving $2\sin\left(\theta + \dfrac{5\pi}{6}\right)$	2
(ii) Reduces to $\sin\left(\theta + \dfrac{5\pi}{6}\right) = \dfrac{1}{2}$ and finds all solutions in $[0, 2\pi]$	2

Solution.

(i) Write $\cos\theta - \sqrt{3}\sin\theta = R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$. Matching coefficients: $R\cos\alpha = -\sqrt{3}$ (from $\sin\theta$) and $R\sin\alpha = 1$ (from $\cos\theta$). Then $R = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$; since $\cos\alpha < 0$ and $\sin\alpha > 0$, α is in the second quadrant, $\alpha = \dfrac{5\pi}{6}$. Thus $\cos\theta - \sqrt{3}\sin\theta = 2\sin\left(\theta + \dfrac{5\pi}{6}\right)$.

(ii) The equation becomes $2\sin\left(\theta + \dfrac{5\pi}{6}\right) = 1$, i.e. $\sin\left(\theta + \dfrac{5\pi}{6}\right) = \dfrac{1}{2}$. Let $u = \theta + \dfrac{5\pi}{6}$; as $\theta \in [0, 2\pi]$, $u \in \left[\dfrac{5\pi}{6}, \dfrac{17\pi}{6}\right]$. In this range $\sin u = \dfrac{1}{2}$ at $u = \dfrac{13\pi}{6}$ and $u = \dfrac{17\pi}{6}$, giving $\theta = \dfrac{4\pi}{3}$ and $\theta = 2\pi$. Checking the endpoint $\theta = 0$ (where $u = \dfrac{5\pi}{6}$, $\sin u = \dfrac{1}{2}$) also satisfies the equation. Hence $\theta = 0,\ \dfrac{4\pi}{3},\ 2\pi$.

Q32 2022 Baulkham Hills Trial Q11b · 4 marks

By expressing $2\cos x + 9\sin x$ in the form $A\cos(x-\theta)$, solve

$$2\cos x + 9\sin x = 1, \qquad 0 \le x \le 2\pi.$$

Give your answer correct to two decimal places.

Criteria	Marks
Expresses $2\cos x + 9\sin x = \sqrt{85}\cos(x-\theta)$ with $\theta = \tan^{-1}\dfrac{9}{2} \approx 1.3521$	2
Reduces to $\cos(x-\theta) = \dfrac{1}{\sqrt{85}}$	1
Finds both solutions $x \approx 2.81,\ 6.17$	1

Solution.

Write $2\cos x + 9\sin x = A\cos(x-\theta) = A\cos x\cos\theta + A\sin x\sin\theta$, so $A\cos\theta = 2$ and $A\sin\theta = 9$. Then $A = \sqrt{2^2+9^2} = \sqrt{85}$ and $\tan\theta = \dfrac{9}{2}$, so $\theta = \tan^{-1}\dfrac{9}{2} \approx 1.3521$. The equation becomes $\sqrt{85}\cos(x-\theta) = 1$, i.e. $\cos(x-\theta) = \dfrac{1}{\sqrt{85}}$. So $x-\theta = \pm\cos^{-1}\dfrac{1}{\sqrt{85}} + 2k\pi = \pm 1.4619 + 2k\pi$. Taking $x = \theta \pm 1.4619 + 2k\pi$ and keeping solutions in $[0, 2\pi]$: $x = 1.3521 + 1.4619 = 2.8140 \approx 2.81$ and $x = 1.3521 - 1.4619 + 2\pi = 6.1733 \approx 6.17$. Hence $x \approx 2.81,\ 6.17$.

Q33 2020 Gosford Trial Q11b · 4 marks

(i) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta-\alpha)$, where $R > 0$ and $0 \le \alpha < 2\pi$. Give α correct to 1 decimal place.

(ii) Hence calculate the maximum value of $\dfrac{4}{12 + 6\cos\theta + 8\sin\theta}$.

Criteria	Marks
(i) Finds $R = 10$ and $\alpha = \tan^{-1}\dfrac{8}{6} \approx 0.9$, giving $10\cos(\theta - 0.9)$	2
(ii) Identifies the minimum denominator as $12 - 10 = 2$	1
(ii) States the maximum value 2	1

Solution.

(i) Write $6\cos\theta + 8\sin\theta = R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$, so $R\cos\alpha = 6$ and $R\sin\alpha = 8$. Then $R = \sqrt{6^2+8^2} = 10$ and $\tan\alpha = \dfrac{8}{6} = \dfrac{4}{3}$, so $\alpha = \tan^{-1}\dfrac{4}{3} \approx 0.9$. Thus $6\cos\theta + 8\sin\theta = 10\cos(\theta - 0.9)$.
(ii) Since $6\cos\theta + 8\sin\theta = 10\cos(\theta-\alpha)$ ranges over $[-10, 10]$, the denominator $12 + 6\cos\theta + 8\sin\theta$ ranges over $[12-10,\ 12+10] = [2, 22]$. The fraction $\dfrac{4}{12 + 6\cos\theta + 8\sin\theta}$ is largest when its denominator is smallest, i.e. equal to 2. Hence the maximum value is $\dfrac{4}{2} = 2$.

Q34 2021 North Sydney Boys Trial Q12b · **4 marks**

(i) Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 \le \alpha \le 2\pi$.

(ii) Hence solve the inequality $\sin x + \sqrt{3}\cos x \ge 1$ in the domain $[0, 2\pi]$.

Criteria	Marks
(i) Finds $R = 2$ and $\alpha = \frac{\pi}{3}$, giving $2\sin\left(x + \frac{\pi}{3}\right)$	2
(ii) Reduces to $\sin\left(x + \frac{\pi}{3}\right) \ge \frac{1}{2}$ and identifies the correct interval on $u = x + \frac{\pi}{3}$	1
(ii) States the solution $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$	1

Solution.

(i) Write $\sin x + \sqrt{3}\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$, so $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$. Then $R = 2$ and $\tan\alpha = \sqrt{3}$, so $\alpha = \frac{\pi}{3}$. Thus $\sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$.

(ii) The inequality becomes $2\sin\left(x + \frac{\pi}{3}\right) \ge 1$, i.e. $\sin\left(x + \frac{\pi}{3}\right) \ge \frac{1}{2}$. Let $u = x + \frac{\pi}{3}$; as $x \in [0, 2\pi]$, $u \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$. Now $\sin u \ge \frac{1}{2}$ on $\left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right]$; intersecting with $\left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$ gives $u \in \left[\frac{\pi}{3}, \frac{5\pi}{6}\right] \cup \left[\frac{13\pi}{6}, \frac{7\pi}{3}\right]$. Converting back with $x = u - \frac{\pi}{3}$: $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$.

Q35 2024 Trinity Grammar Trial Q11b · **4 marks**

(i) Rewrite $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(ii) Hence, or otherwise, solve $\cos x - \sqrt{3}\sin x = 1$ for $0 \le x \le 2\pi$.

Criteria	Marks
(i) Finds $R = 2$ and $\alpha = \frac{\pi}{3}$, giving $2\cos\left(x + \frac{\pi}{3}\right)$	2
(ii) Reduces to $\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ and finds all solutions in $[0, 2\pi]$	2

Solution.

(i) Write $\cos x - \sqrt{3}\sin x = R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$, so $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$. Then $R = 2$ and $\tan\alpha = \sqrt{3}$, so $\alpha = \frac{\pi}{3}$. Thus $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$.

(ii) The equation becomes $2\cos\left(x + \frac{\pi}{3}\right) = 1$, i.e. $\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$. Let $u = x + \frac{\pi}{3}$; as $x \in [0, 2\pi]$, $u \in \left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$. Here $\cos u = \frac{1}{2}$ at $u = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, giving $x = 0, \frac{4\pi}{3}, 2\pi$.

Q36 2020 Abbotsleigh Trial Q13c · 5 marks

A machine produces a sound which can be modelled using the equation $y = \cos t - \sqrt{3}\sin t$, where y gives a measure of the sound at time t seconds after the machine is switched on. The sound is audible to humans when $y \geq 1$.

(i) Express $\cos t - \sqrt{3}\sin t$ in the form $R\cos(t+\alpha)$, where $R > 0$ and $0 \leq \alpha < \dfrac{\pi}{2}$.

(ii) Hence find the time intervals during which the noise is audible to humans for $0 < t \leq 4\pi$.

Criteria	Marks
(i) Finds $R = 2$ and $\alpha = \dfrac{\pi}{3}$, giving $2\cos\left(t + \dfrac{\pi}{3}\right)$	2
(ii) Reduces the condition $y \geq 1$ to $\cos\left(t + \dfrac{\pi}{3}\right) \geq \dfrac{1}{2}$	1
(ii) States the intervals $\left[\dfrac{4\pi}{3}, 2\pi\right] \cup \left[\dfrac{10\pi}{3}, 4\pi\right]$	2

Solution.

(i) Write $\cos t - \sqrt{3}\sin t = R\cos(t+\alpha) = R\cos t\cos\alpha - R\sin t\sin\alpha$, so $R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$. Then $R = 2$ and $\tan\alpha = \sqrt{3}$, so $\alpha = \dfrac{\pi}{3}$. Thus $\cos t - \sqrt{3}\sin t = 2\cos\left(t + \dfrac{\pi}{3}\right)$.

(ii) Audible means $y \geq 1$, i.e. $2\cos\left(t + \dfrac{\pi}{3}\right) \geq 1$, so $\cos\left(t + \dfrac{\pi}{3}\right) \geq \dfrac{1}{2}$. Let $u = t + \dfrac{\pi}{3}$; as $t \in (0, 4\pi]$, $u \in \left(\dfrac{\pi}{3}, \dfrac{13\pi}{3}\right]$. Now $\cos u \geq \dfrac{1}{2}$ on $\left[2k\pi - \dfrac{\pi}{3},\ 2k\pi + \dfrac{\pi}{3}\right]$; intersecting with $\left(\dfrac{\pi}{3}, \dfrac{13\pi}{3}\right]$ gives $u \in \left[\dfrac{5\pi}{3}, \dfrac{7\pi}{3}\right] \cup \left[\dfrac{11\pi}{3}, \dfrac{13\pi}{3}\right]$. Converting back with $t = u - \dfrac{\pi}{3}$: $t \in \left[\dfrac{4\pi}{3}, 2\pi\right] \cup \left[\dfrac{10\pi}{3}, 4\pi\right]$.

Q37 2020 Girraween Trial Q15a · **6 marks**

(i) Show that $2\sin 2x - 3\cos 2x - 3\sin x + 3 = \sin x\,(6\sin x + 4\cos x - 3)$.

(ii) Express $6\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

(iii) Hence, solve $2\sin 2x - 3\cos 2x - 3\sin x + 3 = 0$ for $0 \le x < \pi$. Answer in radians, correct to 2 decimal places.

Criteria	**Marks**
(i) Uses $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 1 - 2\sin^2 x$ and factors out $\sin x$ to reach the required form	2
(ii) Finds $R = 2\sqrt{13}$ and $\alpha = \tan^{-1}\dfrac{2}{3} \approx 0.59$	2
(iii) Solves $\sin x = 0$ and $6\sin x + 4\cos x = 3$, giving $x = 0$ and $x \approx 2.12$	2

Solution.

(i) Using $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 1 - 2\sin^2 x$:

$$2\sin 2x - 3\cos 2x - 3\sin x + 3 = 4\sin x\cos x - 3(1 - 2\sin^2 x) - 3\sin x + 3 = 4\sin x\cos x + 6\sin^2 x - 3\sin x.$$

Factoring out $\sin x$: $= \sin x\,(4\cos x + 6\sin x - 3) = \sin x\,(6\sin x + 4\cos x - 3)$, as required.

(ii) Write $6\sin x + 4\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$, so $R\cos\alpha = 6$ and $R\sin\alpha = 4$. Then $R = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$ and $\tan\alpha = \dfrac{4}{6} = \dfrac{2}{3}$, so $\alpha = \tan^{-1}\dfrac{2}{3} \approx 0.59$. Thus $6\sin x + 4\cos x = 2\sqrt{13}\,\sin(x + 0.59)$.

(iii) From (i) the equation factorises as $\sin x\,(6\sin x + 4\cos x - 3) = 0$, so $\sin x = 0$ or $6\sin x + 4\cos x = 3$. On $0 \le x < \pi$, $\sin x = 0$ gives $x = 0$. Using (ii), $6\sin x + 4\cos x = 3$ becomes $2\sqrt{13}\,\sin(x + 0.59) = 3$, i.e. $\sin(x + 0.59) = \dfrac{3}{2\sqrt{13}} \approx 0.4160$. Then $x + 0.59 = 0.4291$ or $x + 0.59 = \pi - 0.4291 = 2.7125$; the first gives a negative x (rejected), the second gives $x \approx 2.12$. Hence $x = 0$ and $x \approx 2.12$.

Counting and interpreting solutions

4. Counting and interpreting solutions

Q38 2020 Sydney Tech Prelim Q1 · 1 mark

How many solutions does $-3 = 3\sin x$ have when $0° < x < 360°$?

(A) 0

(B) 1

(C) 2

(D) 3

Criteria	Marks
Correct option (B)	1

Solution.

Dividing by 3 gives $\sin x = -1$. On the open interval $0° < x < 360°$ the only angle with $\sin x = -1$ is $x = 270°$, so there is exactly 1 solution. The answer is (B).

Q39 2022 Glenwood Trial Q4 · 1 mark

How many solutions does the equation $\sin 6x - \sin 2x = 0$ have for $0 \leq x \leq 2\pi$?

(A) 5

(B) 12

(C) 14

(D) 13

Criteria	Marks
Correct option (D)	1

Solution.

Using the sum-to-product identity, $\sin 6x - \sin 2x = 2\cos 4x \ \sin 2x$, so the equation factors as $2\cos 4x \ \sin 2x = 0$.

$$\cos 4x = 0: \ 4x = \tfrac{\pi}{2} + k\pi \Rightarrow x = \tfrac{\pi}{8} + \tfrac{k\pi}{4}, \quad \text{giving 8 values in } [0, 2\pi].$$

$$\sin 2x = 0: \ 2x = k\pi \Rightarrow x = \tfrac{k\pi}{2}, \quad \text{giving 5 values } (0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}, 2\pi).$$

The $\cos 4x = 0$ roots are odd multiples of $\frac{\pi}{8}$ and the $\sin 2x = 0$ roots are multiples of $\frac{\pi}{2}$, so none coincide. Total $= 8 + 5 = 13$ solutions. The answer is (D).

Q40 2025 HSC Q5 · 1 mark

How many distinct solutions are there to the equation $\cos 5x + \sin x = 0$ for $0 \leq x \leq 2\pi$?

(A) 5

(B) 6

(C) 9

(D) 10

Criteria	Marks
Correct option (D)	1

Solution.

Write $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, so $\cos 5x = -\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} + x\right)$. Hence $5x = \pm\left(\frac{\pi}{2} + x\right) + 2k\pi$.

$$5x = \tfrac{\pi}{2} + x + 2k\pi \Rightarrow x = \tfrac{\pi}{8} + \tfrac{k\pi}{2}, \quad \text{giving 4 values in } [0, 2\pi].$$

$$5x = -\tfrac{\pi}{2} - x + 2k\pi \Rightarrow x = -\tfrac{\pi}{12} + \tfrac{k\pi}{3}, \quad \text{giving 6 values in } [0, 2\pi].$$

The two families share no common value, so there are $4 + 6 = 10$ distinct solutions. The answer is (D).

Q41 2024 Hurlstone Prelim Q5 · 1 mark

How many solutions are there to the equation $\tan 2x = 3$ when $-\pi \leq x \leq \pi$?

(A) 2

(B) 4

(C) 6

(D) 8

Criteria	Marks
Correct option (B)	1

Solution.

As x ranges over $-\pi \leq x \leq \pi$, the angle $2x$ ranges over $-2\pi \leq 2x \leq 2\pi$. Since $\tan 2x = 3 > 0$, the angle $2x$ lies in the first or third quadrant. Across a span of two full revolutions there are 4 such angles, so $2x = \arctan 3 + k\pi$ gives 4 values of x in the domain. The answer is (B).

Q42 2020 Manly Trial Q8 · 1 mark

The number of solutions to the equation $(\sin^2 x - 1)(\tan^2 x - 1) = 0$ in the domain $[0, 2\pi]$ is

(A) 2

(B) 4

(C) 6

(D) 8

Criteria	Marks
Correct option (B)	1

Solution.

A product is zero when a factor is zero, but every factor must first be *defined*. From $\tan^2 x = 1$ we get $\tan x = \pm 1$, giving $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$: four valid solutions. From $\sin^2 x = 1$ we would get $x = \frac{\pi}{2}, \frac{3\pi}{2}$, but at these angles $\tan x$ is undefined (a vertical asymptote), so the expression itself is undefined there and they are *not* solutions. Hence there are 4 solutions. The answer is (B).

Q43 2022 North Sydney Boys Trial Q7 · **1 mark**

In the range $0 \leq x \leq 2\pi$, the equation $2\sin^2 x + 2\cos^2 x = 2$

(A) has no solutions

(B) has 1 solution

(C) has 2 solutions

(D) holds for all values of x

Criteria	**Marks**
Correct option (D)	1

Solution.

Factor out 2: $2\sin^2 x + 2\cos^2 x = 2(\sin^2 x + \cos^2 x)$. By the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, so the left-hand side equals $2(1) = 2$ for every value of x. The statement is an identity, so it holds for all values of x. The answer is (D).