Year 11 Mathematics Extension 1

Topic: Further Trigonometry (2 of 4)

Review Questions (Worked Solutions): Further Trigonometric Identities

Term 3, 2026

Expanding with the compound and double angle formulae

1. Compound-angle (sum and difference) identities

Q1 2020 Abbotsleigh Trial Q6 · 1 mark

Both A and B are acute angles such that $\tan A = \dfrac{3}{5}$ and $\tan B = 4$. What is the size of angle $(A - B)$?

(A) $-135°$

(B) $45°$

(C) $135°$

(D) $-45°$

Criteria	Marks
Correct option (D)	1

Solution.

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{\frac{3}{5} - 4}{1 + \frac{3}{5} \cdot 4} = \dfrac{-\frac{17}{5}}{\frac{17}{5}} = -1$. Since $\tan(A-B) = -1$, $A - B = -45°$. The answer is (D).

Q2 2021 Caringbah Trial Q4 · **1 mark**

Given that α and β are both acute angles, evaluate $\sin(\alpha+\beta)$ if $\sin\alpha = \dfrac{8}{17}$ and $\sin\beta = \dfrac{4}{5}$.

(A) $\dfrac{108}{85}$

(B) $\dfrac{84}{85}$

(C) $\dfrac{36}{85}$

(D) $\dfrac{28}{85}$

Criteria	Marks
Correct option (B)	1

Solution.

Since α, β are acute, $\cos\alpha = \dfrac{15}{17}$ and $\cos\beta = \dfrac{3}{5}$. Then

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \tfrac{8}{17}\cdot\tfrac{3}{5} + \tfrac{15}{17}\cdot\tfrac{4}{5} = \tfrac{24+60}{85} = \tfrac{84}{85}.$$

The answer is (B).

Q3 2020 Hornsby Girls Prelim Q1 · **1 mark**

$\sin(A+3B) =$

(A) $\sin A\cos 3B - \cos A\sin 3B$

(B) $\sin 3B\cos A + \cos A\sin 3B$

(C) $\sin A\cos 3B + \cos A\sin 3B$

(D) $\sin A\cos 3B + \cos 3B\sin A$

Criteria	Marks
Correct option (C)	1

Solution.

Using the compound-angle expansion, $\sin(A+3B) = \sin A\cos 3B + \cos A\sin 3B$. The answer is (C).

Q4 2025 Normanhurst Boys Trial Q2 · **1 mark**

Given $\tan\theta = \dfrac{1}{2}$, what is the exact value of $\tan\left(\theta - \dfrac{\pi}{6}\right)$?

(A) $8 + 5\sqrt{3}$

(B) $8 - 5\sqrt{3}$

(C) $\dfrac{8 + 5\sqrt{3}}{11}$

(D) $\dfrac{8 - 5\sqrt{3}}{11}$

Criteria	Marks
Correct option (D)	1

Solution.

With $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$,

$$\tan\left(\theta - \tfrac{\pi}{6}\right) = \frac{\tan\theta - \tan\frac{\pi}{6}}{1 + \tan\theta\tan\frac{\pi}{6}} = \frac{\frac{1}{2} - \frac{1}{\sqrt{3}}}{1 + \frac{1}{2}\cdot\frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 2}{2\sqrt{3} + 1} = \frac{8 - 5\sqrt{3}}{11}.$$

The answer is (D).

Q5 2023 SCEGGS Prelim Q6 · **1 mark**

Given $\tan\theta = A$, what is the correct expression for $\tan\left(\dfrac{\pi}{4} - \theta\right)$?

(A) A

(B) $\dfrac{1}{A}$

(C) $1 - A$

(D) $\dfrac{1 - A}{1 + A}$

Criteria	Marks
Correct option (D)	1

Solution.

With $\tan\dfrac{\pi}{4} = 1$,

$$\tan\left(\tfrac{\pi}{4} - \theta\right) = \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta} = \frac{1 - A}{1 + A}.$$

The answer is (D).

Q6 2025 Baulkham Hills Prelim Q8 · **1 mark**

In triangle ABC, if $\cos A = \dfrac{1}{3}$, then which of the following equals $\cot \dfrac{A}{2}$?

(A) $\sqrt{2}$

(B) $-\sqrt{2}$

(C) $\dfrac{1}{2}$

(D) $\dfrac{1}{\sqrt{2}}$

Criteria	**Marks**
Correct option (A)	1

Solution.

Using $\cos A = 2\cos^2 \dfrac{A}{2} - 1$: $\dfrac{1}{3} = 2\cos^2 \dfrac{A}{2} - 1$, so $\cos^2 \dfrac{A}{2} = \dfrac{2}{3}$. Also $\cos A = 1 - 2\sin^2 \dfrac{A}{2}$ gives $\sin^2 \dfrac{A}{2} = \dfrac{1}{3}$.

Since A is an angle of a triangle, $\dfrac{A}{2}$ is acute, so

$$\cot \frac{A}{2} = \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} = \sqrt{\frac{2/3}{1/3}} = \sqrt{2}.$$

The answer is (A).

Q7 2020 Sydney Girls Trial Q8 · **1 mark**

Given that $\sin\left(x-\frac{\pi}{3}\right)=3\cos\left(x-\frac{\pi}{4}\right)$, find the exact value of $\tan x$.

(A) $\dfrac{\sqrt{6}+6}{6-\sqrt{2}}$

(B) $\dfrac{\sqrt{6}+6}{\sqrt{2}-6}$

(C) $\dfrac{\sqrt{2}-6}{\sqrt{6}+6}$

(D) $\dfrac{6-\sqrt{2}}{\sqrt{6}+6}$

Criteria	**Marks**
Correct option (B)	1

Solution.

Expanding both sides,

$$\tfrac{1}{2}\sin x-\tfrac{\sqrt{3}}{2}\cos x=3\left(\tfrac{\sqrt{2}}{2}\cos x+\tfrac{\sqrt{2}}{2}\sin x\right).$$

Collecting terms, $\sin x\left(\frac{1}{2}-\frac{3\sqrt{2}}{2}\right)=\cos x\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\right)$, so

$$\tan x=\frac{3\sqrt{2}+\sqrt{3}}{1-3\sqrt{2}}=\frac{\sqrt{6}+6}{\sqrt{2}-6}$$

(multiplying numerator and denominator by $\sqrt{3}$). The answer is (B).

Q8 2023 North Sydney Boys Trial Q12c · **2 marks**

By equating coefficients, find the values of P and Q in the identity

$$P(2\sin x + \cos x) + Q(2\cos x - \sin x) \equiv 7\sin x + 11\cos x.$$

Criteria	Marks
Equates coefficients of $\sin x$ and $\cos x$ to form $2P - Q = 7$ and $P + 2Q = 11$	1
Solves the simultaneous equations for $P = 5$, $Q = 3$	1

Solution.

Grouping the left-hand side, the coefficient of $\sin x$ is $2P - Q$ and the coefficient of $\cos x$ is $P + 2Q$. Equating with the right-hand side,

$$2P - Q = 7, \qquad P + 2Q = 11.$$

From the first equation $Q = 2P - 7$; substituting, $P + 2(2P - 7) = 11 \Rightarrow 5P = 25 \Rightarrow P = 5$, hence $Q = 3$. So $P = 5$ and $Q = 3$.

Q9 2020 Caringbah Prelim Q15d · **2 marks**

If $\sec x + \tan x = k$, find an expression for $\sec x$ in terms of k.

Criteria	Marks
Uses $\sec^2 x - \tan^2 x = 1$ to obtain $\sec x - \tan x = \dfrac{1}{k}$	1
Adds the two equations and solves for $\sec x = \dfrac{1}{2}\left(k + \dfrac{1}{k}\right)$	1

Solution.

Since $\sec^2 x - \tan^2 x = 1$, we have $(\sec x + \tan x)(\sec x - \tan x) = 1$, so with $\sec x + \tan x = k$,

$$\sec x - \tan x = \frac{1}{k}.$$

Adding this to $\sec x + \tan x = k$ gives $2\sec x = k + \dfrac{1}{k}$, hence

$$\sec x = \frac{1}{2}\left(k + \frac{1}{k}\right) = \frac{k^2 + 1}{2k}.$$

Q10 2021 Moriah Prelim Q6c · 3 marks

You are given that $\cos\theta = \dfrac{1}{2}$ and $\cos\beta = \dfrac{3}{5}$, and that θ and β are both acute angles.

(i) Find the exact value of $\cos(\theta + \beta)$.

(ii) If neither θ nor β is an acute angle, then the answer in part (i) is unchanged. Briefly explain why this is so.

Criteria	Marks
(i) Finds $\sin\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\beta = \dfrac{4}{5}$ from the given cosines	1
(i) Applies the compound-angle formula to obtain $\cos(\theta+\beta) = \dfrac{3-4\sqrt{3}}{10}$	1
(ii) Explains that both angles lie in the fourth quadrant, so the negative sines give the same product	1

Solution.

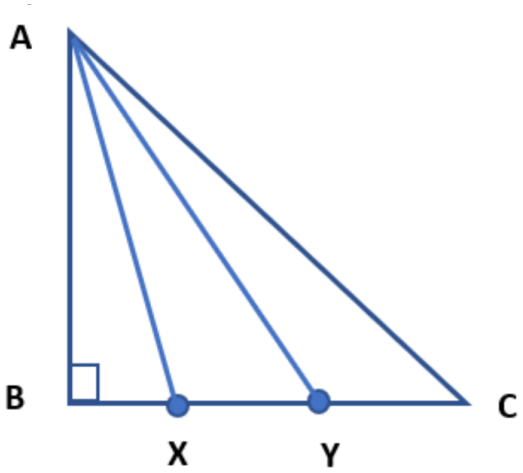
(i) Since θ, β are acute, $\sin\theta = \sqrt{1-\frac{1}{4}} = \dfrac{\sqrt{3}}{2}$ and $\sin\beta = \sqrt{1-\frac{9}{25}} = \dfrac{4}{5}$. Then

$$\cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta = \tfrac{1}{2}\cdot\tfrac{3}{5} - \tfrac{\sqrt{3}}{2}\cdot\tfrac{4}{5} = \frac{3-4\sqrt{3}}{10}.$$

(ii) If $\cos\theta = \frac{1}{2} > 0$ and $\cos\beta = \frac{3}{5} > 0$ but neither angle is acute, then each angle lies in the fourth quadrant, so $\sin\theta = -\dfrac{\sqrt{3}}{2}$ and $\sin\beta = -\dfrac{4}{5}$. The cosines are unchanged and the product $\sin\theta\sin\beta = \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{4}{5}\right)$ is the same as before, so $\cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$ is unchanged.

Q11 2023 Glenwood Prelim Q9e · **4 marks**

In the isosceles triangle below, $AB = BC$, $\angle ABC = 90°$ and points X and Y are placed on BC such that $BX = XY = YC = 1$.



Let $\angle BAX = \alpha$, $\angle XAY = \beta$ and $\angle YAC = \gamma$. Find the exact values of $\tan\alpha$, $\tan\beta$ and $\tan\gamma$.

Criteria	**Marks**
Recognises $AB = BC = 3$ and $AB \perp BC$, giving $\tan\alpha = \frac{1}{3}$	1
Uses $\tan(\alpha+\beta) = \frac{2}{3}$ to find $\tan\beta = \frac{3}{11}$	2
Uses $\tan(\alpha+\beta+\gamma) = 1$ to find $\tan\gamma = \frac{1}{5}$	1

Solution.

Since $BX = XY = YC = 1$, $BC = 3$, and as $AB = BC$ we have $AB = 3$. Because $\angle ABC = 90°$, $AB \perp BC$, so from A the perpendicular distance to BC is $AB = 3$ and B is the foot. Then, measuring from AB,

$$\tan\alpha = \tan\angle BAX = \frac{BX}{AB} = \frac{1}{3}, \quad \tan(\alpha+\beta) = \frac{BY}{AB} = \frac{2}{3}, \quad \tan(\alpha+\beta+\gamma) = \frac{BC}{AB} = 1.$$

For $\tan\beta$, using $\beta = (\alpha+\beta) - \alpha$,

$$\tan\beta = \frac{\frac{2}{3} - \frac{1}{3}}{1 + \frac{2}{3}\cdot\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{11}{9}} = \frac{3}{11}.$$

For $\tan\gamma$, using $\gamma = (\alpha+\beta+\gamma) - (\alpha+\beta)$,

$$\tan\gamma = \frac{1 - \frac{2}{3}}{1 + 1\cdot\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{5}{3}} = \frac{1}{5}.$$

So $\tan\alpha = \frac{1}{3}$, $\tan\beta = \frac{3}{11}$ and $\tan\gamma = \frac{1}{5}$.

2. Double-angle formulae

Q12 2020 Sydney Grammar Trial Q5 · 1 mark

Which expression is equivalent to $1 - \cos 2x$?

(A) $\sin^2 x - \cos^2 x$

(B) $2(\sin^2 x + 1)$

(C) $2\sin^2 x$

(D) $2(1 + \cos^2 x)$

Criteria	Marks
Correct option (C)	1

Solution.

Using $\cos 2x = 1 - 2\sin^2 x$, $1 - \cos 2x = 1 - (1 - 2\sin^2 x) = 2\sin^2 x$. The answer is **(C)**.

Q13 2020 Cheltenham Girls Trial Q1 · 1 mark

What is the value of $\sin 2x$ given that $\sin x = \dfrac{2\sqrt{3}}{4}$ and x is obtuse?

(A) $\dfrac{\sqrt{3}}{4}$

(B) $\dfrac{\sqrt{3}}{2}$

(C) $-\dfrac{\sqrt{3}}{4}$

(D) $-\dfrac{\sqrt{3}}{2}$

Criteria	Marks
Correct option (D)	1

Solution.

$\sin x = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$. Since x is obtuse, $\cos x < 0$, so $\cos x = -\sqrt{1 - \frac{3}{4}} = -\dfrac{1}{2}$.

$$\sin 2x = 2\sin x \cos x = 2 \cdot \frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}.$$

The answer is **(D)**.

Q14 2021 Glenwood Trial Q8 · **1 mark**

It is known that $\sin x = \frac{1}{3}$, where $\frac{\pi}{2} < x < \pi$. What is the value of $\cos 2x$?

(A) $-\frac{2}{9}$

(B) $\frac{7}{9}$

(C) $\frac{2}{3}$

(D) $\frac{4\sqrt{2}}{3}$

Criteria	**Marks**
Correct option (B)	1

Solution.

Using $\cos 2x = 1 - 2\sin^2 x$ (which holds in any quadrant),

$$\cos 2x = 1 - 2\left(\frac{1}{3}\right)^2 = 1 - \frac{2}{9} = \frac{7}{9}.$$

The answer is **(B)**.

Q15 2023 Hurlstone Prelim Q4 · 1 mark

It is given that $\cos\theta = \frac{3}{5}$ for $\frac{3\pi}{2} \leq \theta \leq 2\pi$. What is the value of $\sin 2\theta$ equivalent to?

(A) $-\frac{24}{25}$

(B) $\frac{24}{25}$

(C) $-\frac{4}{5}$

(D) $\frac{4}{5}$

Criteria	Marks
Correct option (A)	1

Solution.

θ lies in the fourth quadrant, so $\sin\theta < 0$: $\sin\theta = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}.$$

The answer is **(A)**.

Q16 2022 Moriah Trial Q2 · 1 mark

Which of the following is equivalent to $\frac{2\tan\theta}{1+\tan^2\theta}$?

(A) $\cos 2\theta$

(B) $\sin 2\theta$

(C) $\tan 2\theta$

(D) $\cot 2\theta$

Criteria	Marks
Correct option (B)	1

Solution.

Since $1 + \tan^2\theta = \sec^2\theta$,

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta}{\sec^2\theta} = 2\cdot\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta = 2\sin\theta\cos\theta = \sin 2\theta.$$

The answer is **(B)**.

Q17 2020 Sydney Girls Trial Q2 · **1 mark**

What is the value of $\sin 2\theta$ given that $\sin\theta = \dfrac{\sqrt{3}}{4}$ and θ is obtuse?

(A) $-\dfrac{\sqrt{39}}{8}$

(B) $-\dfrac{\sqrt{3}}{2}$

(C) $\dfrac{\sqrt{3}}{2}$

(D) $\dfrac{\sqrt{39}}{8}$

Criteria	**Marks**
Correct option (A)	1

Solution.

Since θ is obtuse, $\cos\theta < 0$: $\cos\theta = -\sqrt{1-\frac{3}{16}} = -\dfrac{\sqrt{13}}{4}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{\sqrt{3}}{4}\cdot\left(-\frac{\sqrt{13}}{4}\right) = -\frac{\sqrt{39}}{8}.$$

The answer is **(A)**.

Q18 2021 Kings Trial Q11(c) · **2 marks**

Given $\cos x = -\dfrac{1}{3}$ and $\sin x > 0$, find the exact value of $\sin 2x$.

Criteria	**Marks**
Finds $\sin x = \dfrac{2\sqrt{2}}{3}$ using $\sin x > 0$	1
Applies $\sin 2x = 2\sin x\cos x$ and simplifies	1

Solution.

Since $\sin x > 0$, $\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\frac{1}{9}} = \dfrac{2\sqrt{2}}{3}$.

$$\sin 2x = 2\sin x\cos x = 2\cdot\frac{2\sqrt{2}}{3}\cdot\left(-\frac{1}{3}\right) = -\frac{4\sqrt{2}}{9}.$$

Q19 2022 Moriah Trial Q11(d) · 3 marks

You are given that $\sin\theta = \dfrac{4}{5}$ where $90° < \theta < 180°$.

(i) Find the exact value of $\tan\theta$.

(ii) Find the exact value of $\tan 2\theta$.

Criteria	Marks
(i) Uses the obtuse constraint to get $\cos\theta = -\frac{3}{5}$ and hence $\tan\theta$	1
(ii) Substitutes $\tan\theta$ into $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$	1
(ii) Simplifies to the exact value	1

Solution.

(i) Since $90° < \theta < 180°$, $\cos\theta < 0$: $\cos\theta = -\sqrt{1-\frac{16}{25}} = -\dfrac{3}{5}$. Hence

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4/5}{-3/5} = -\frac{4}{3}.$$

(ii) Using the double-angle formula,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\frac{16}{9}} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{8}{3}\cdot\frac{9}{7} = \frac{24}{7}.$$

Q20 2022 Shore Trial Q12(b) · 3 marks

Given that $\sin\theta = \dfrac{3}{4}$ and $\dfrac{\pi}{2} \le \theta \le \pi$, determine the exact value of $\tan 2\theta$.

Criteria	Marks
Uses the second-quadrant constraint to get $\cos\theta = -\dfrac{\sqrt{7}}{4}$	1
Forms $\tan\theta = -\dfrac{3}{\sqrt{7}}$ and substitutes into $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$	1
Simplifies to the exact value $3\sqrt{7}$	1

Solution.

Since $\dfrac{\pi}{2} \le \theta \le \pi$, $\cos\theta < 0$: $\cos\theta = -\sqrt{1-\frac{9}{16}} = -\dfrac{\sqrt{7}}{4}$, so $\tan\theta = \dfrac{3/4}{-\sqrt{7}/4} = -\dfrac{3}{\sqrt{7}}$.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{3}{\sqrt{7}}\right)}{1-\frac{9}{7}} = \frac{-\frac{6}{\sqrt{7}}}{-\frac{2}{7}} = \frac{6}{\sqrt{7}}\cdot\frac{7}{2} = \frac{21}{\sqrt{7}} = 3\sqrt{7}.$$

3. Exact values of special angles

Q21 2020 Glenwood Prelim Q3 · **1 mark**

What is the exact value of $\cos 15°$?

(A) $\dfrac{\sqrt{6}}{2}$

(B) $\dfrac{\sqrt{3}}{2}$

(C) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

(D) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Criteria	Marks
Correct option (D)	1

Solution.

$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$. Then

$$\cos 15° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

This is option (D). (Options (A) and (B) exceed 1, so cannot be a cosine.)

Q22 2024 Baulkham Hills Prelim Q11a · **2 marks**

Use a compound-angle formula to find the exact value of $\cos 75°$.

Criteria	Marks
Writes $\cos 75° = \cos(45° + 30°)$ and applies the compound-angle formula	1
Substitutes exact values and simplifies	1

Solution.

$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$. Hence

$$\cos 75° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Q23 2023 Riverview Trial Q11b(i) · **2 marks**

Find the exact value of $\sin\dfrac{23\pi}{12}$. Express your answer as a single fraction with a rational denominator.

Criteria	Marks
Reduces $\dfrac{23\pi}{12}$ and applies a compound- or difference-angle formula	1
Evaluates to a single fraction with a rational denominator	1

Solution.

Since $\dfrac{23\pi}{12} = 2\pi - \dfrac{\pi}{12}$, $\sin\dfrac{23\pi}{12} = -\sin\dfrac{\pi}{12}$. Now $\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$, so

$$\sin\frac{\pi}{12} = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Therefore

$$\sin\frac{23\pi}{12} = -\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

Q24 2023 Fort St Trial Q11f · **3 marks**

Find the exact value of $\cos\dfrac{\pi}{8}$, giving your answer in simplest form.

Criteria	Marks
Uses $\cos\dfrac{\pi}{4} = 2\cos^2\dfrac{\pi}{8} - 1$ (double-angle / half-angle relation)	1
Makes $\cos^2\dfrac{\pi}{8}$ the subject and takes the square root	1
Chooses the positive root (with reason) and simplifies	1

Solution.

Using $\cos 2\theta = 2\cos^2\theta - 1$ with $\theta = \dfrac{\pi}{8}$ gives $\cos\dfrac{\pi}{4} = 2\cos^2\dfrac{\pi}{8} - 1$. Hence

$$\cos^2\frac{\pi}{8} = \frac{1+\cos\frac{\pi}{4}}{2} = \frac{1+\frac{\sqrt{2}}{2}}{2} = \frac{2+\sqrt{2}}{4}.$$

Since $\dfrac{\pi}{8}$ is in the first quadrant, $\cos\dfrac{\pi}{8} > 0$, so

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Q25 2021 Presbyterian Prelim Q7e · **3 marks**

(i) Write $\tan 2A$ in terms of $\tan A$.

(ii) Hence find the exact value of $\tan 67\frac{1}{2}^\circ$.

Criteria	**Marks**
(i) States $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$	1
(ii) Sets $A = 67.5^\circ$ so that $2A = 135^\circ$ and forms a quadratic in $\tan 67.5^\circ$	1
(ii) Solves and selects the positive root with reason	1

Solution.

(i) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$.

(ii) Let $A = 67.5^\circ$, so $2A = 135^\circ$ and $\tan 135^\circ = -1$. With $t = \tan 67.5^\circ$,

$$\frac{2t}{1-t^2} = -1 \;\Rightarrow\; 2t = t^2 - 1 \;\Rightarrow\; t^2 - 2t - 1 = 0 \;\Rightarrow\; t = 1 \pm \sqrt{2}.$$

Since 67.5° is in the first quadrant, $\tan 67.5^\circ > 0$, so

$$\tan 67\tfrac{1}{2}^\circ = 1 + \sqrt{2}.$$

Q26 2022 Ascham Trial Q13e · **4 marks**

(i) Show that $\dfrac{1-\cos 2x}{\sin 2x}+\dfrac{\sin 2x}{1+\cos 2x}=2\tan x$.

(ii) Hence, show that $\tan\dfrac{\pi}{12}=2-\sqrt{3}$.

Criteria	**Marks**
(i) Simplifies $\dfrac{1-\cos 2x}{\sin 2x}$ to $\tan x$ using the double-angle formulae	1
(i) Simplifies $\dfrac{\sin 2x}{1+\cos 2x}$ to $\tan x$ and adds to obtain $2\tan x$	1
(ii) Substitutes $x=\dfrac{\pi}{12}$ and evaluates the right-hand side using exact values	1
(ii) Simplifies to $\tan\dfrac{\pi}{12}=2-\sqrt{3}$	1

Solution.

(i) Using $1-\cos 2x=2\sin^2 x$, $1+\cos 2x=2\cos^2 x$ and $\sin 2x=2\sin x\cos x$:

$$\frac{1-\cos 2x}{\sin 2x}=\frac{2\sin^2 x}{2\sin x\cos x}=\tan x, \qquad \frac{\sin 2x}{1+\cos 2x}=\frac{2\sin x\cos x}{2\cos^2 x}=\tan x.$$

Adding gives $\tan x+\tan x=2\tan x$, as required.

(ii) Put $x=\dfrac{\pi}{12}$, so $2x=\dfrac{\pi}{6}$. Then $2\tan\dfrac{\pi}{12}=\dfrac{1-\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}+\dfrac{\sin\frac{\pi}{6}}{1+\cos\frac{\pi}{6}}$. Using the first term,

$$2\tan\frac{\pi}{12}=2\cdot\frac{1-\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}=2\cdot\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2\left(2-\sqrt{3}\right),$$

so $\tan\dfrac{\pi}{12}=2-\sqrt{3}$.

Q27 2025 Caringbah Trial Q14b · **4 marks**

It is known that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ (DO NOT PROVE THIS).

(i) Find the exact values of the solutions to the equation $8x^4 - 8x^2 + 1 = 0$, for $0 < \theta < \pi$.

(ii) Hence, or otherwise, find the exact value of $\cos\dfrac{\pi}{8}\cos\dfrac{3\pi}{8}$.

Criteria	Marks
(i) Sets $x = \cos\theta$ so the equation becomes $\cos 4\theta = 0$	1
(i) Solves $\cos 4\theta = 0$ over $0 < \theta < \pi$ and states the four exact roots	1
(ii) Identifies the required product as $\cos\dfrac{\pi}{8}\cos\dfrac{3\pi}{8}$ among the roots	1
(ii) Evaluates the product to $\dfrac{\sqrt{2}}{4}$	1

Solution.

(i) Let $x = \cos\theta$. Then $8x^4 - 8x^2 + 1 = 8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$, so the equation is $\cos 4\theta = 0$. For $0 < \theta < \pi$ we have $0 < 4\theta < 4\pi$, giving $4\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$, i.e. $\theta = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}$. Hence the solutions are

$$x = \cos\frac{\pi}{8},\ \cos\frac{3\pi}{8},\ \cos\frac{5\pi}{8},\ \cos\frac{7\pi}{8} = \pm\frac{\sqrt{2+\sqrt{2}}}{2},\ \pm\frac{\sqrt{2-\sqrt{2}}}{2}.$$

(ii) Using the product-to-sum formula,

$$\cos\frac{\pi}{8}\cos\frac{3\pi}{8} = \frac{1}{2}\left[\cos\frac{\pi}{4} + \cos\frac{\pi}{2}\right] = \frac{1}{2}\left[\frac{\sqrt{2}}{2} + 0\right] = \frac{\sqrt{2}}{4}.$$

Products, sums and simplifying

4. Products to sums and sums to products

Q28 2024 Blacktown Boys Trial Q13a · 1 mark

By expanding the left-hand side, show that

$$\sin(8x + 5x) + \sin(8x - 5x) = 2\sin 8x \cos 5x.$$

Criteria	Marks
Expands both terms with the compound-angle formula and simplifies to the required form	1

Solution.

Expanding each term with $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$,

$$\sin(8x + 5x) + \sin(8x - 5x) = (\sin 8x \cos 5x + \cos 8x \sin 5x) + (\sin 8x \cos 5x - \cos 8x \sin 5x).$$

The $\cos 8x \sin 5x$ terms cancel, leaving $2\sin 8x \cos 5x$, as required.

Q29 2020 Hunters Hill Trial Q6 · 1 mark

Which of the following expressions is equivalent to $2\sin 5\theta \cos 3\theta$?

(A) $\sin 8\theta + \sin 2\theta$

(B) $\cos 8\theta + \cos 2\theta$

(C) $\sin 8\theta - \sin 2\theta$

(D) $\cos 8\theta - \cos 2\theta$

Criteria	Marks
Correct option (A)	1

Solution.

Using $2\sin A \cos B = \sin(A + B) + \sin(A - B)$ with $A = 5\theta$, $B = 3\theta$,

$$2\sin 5\theta \cos 3\theta = \sin 8\theta + \sin 2\theta.$$

The answer is (A).

Q30 2024 Newington Trial Q2 · 1 mark

Which of the following is equivalent to $2\cos 5x \sin x$?

(A) $\cos 6x + \sin 4x$

(B) $\cos 6x - \sin 4x$

(C) $\sin 6x + \sin 4x$

(D) $\sin 6x - \sin 4x$

Criteria	Marks
Correct option (D)	1

Solution.

Using $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ with $A = 5x$, $B = x$,

$$2\cos 5x \sin x = \sin 6x - \sin 4x.$$

The answer is (D).

Q31 2024 Sydney Boys Trial Q3 · 1 mark

Which of the following is equal to $\cos\alpha - \cos\beta$?

(A) $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

(B) $2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

(C) $2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

(D) $-2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

Criteria	Marks
Correct option (D)	1

Solution.

The sum-to-product identity for a difference of cosines is

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

The answer is (D).

Q32 2023 Gosford Trial Q10 · **1 mark**

Which of the following is equal to $\cos 3\alpha \sin 2\alpha$?

(A) $\dfrac{1}{2}(\sin 5\alpha + \sin \alpha)$

(B) $\dfrac{1}{2}(\sin 5\alpha - \sin \alpha)$

(C) $\dfrac{1}{2}\left(\sin \dfrac{5\alpha}{2} + \sin \dfrac{\alpha}{2}\right)$

(D) $\dfrac{1}{2}\left(\sin \dfrac{5\alpha}{2} - \sin \dfrac{\alpha}{2}\right)$

Criteria	**Marks**
Correct option (B)	1

Solution.

Using $\cos A \sin B = \dfrac{1}{2}\big[\sin(A+B) - \sin(A-B)\big]$ with $A = 3\alpha$, $B = 2\alpha$,

$$\cos 3\alpha \sin 2\alpha = \frac{1}{2}\big[\sin 5\alpha - \sin \alpha\big].$$

The answer is (B).

Q33 2020 Hornsby Girls Trial Q12(b)(i) · **2 marks**

Use the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ to show that

$$\sin^2\left(\frac{x}{8}\right) = \frac{1}{2}\left(1 - \cos\frac{x}{4}\right).$$

Criteria	**Marks**
Applies the product-to-sum identity with $A = B = \frac{x}{8}$	1
Evaluates $\cos 0 = 1$ and simplifies to the required form	1

Solution.

Put $A = B = \frac{x}{8}$ in the given identity:

$$\sin^2\left(\frac{x}{8}\right) = \sin\frac{x}{8}\sin\frac{x}{8} = \frac{1}{2}\left[\cos\left(\frac{x}{8} - \frac{x}{8}\right) - \cos\left(\frac{x}{8} + \frac{x}{8}\right)\right].$$

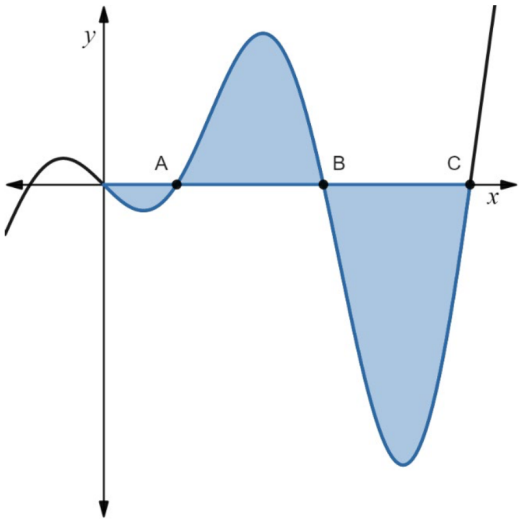
Since $\cos 0 = 1$ and $\frac{x}{8} + \frac{x}{8} = \frac{x}{4}$,

$$\sin^2\left(\frac{x}{8}\right) = \frac{1}{2}\left(1 - \cos\frac{x}{4}\right),$$

as required.

Q34 2022 Sydney Girls Trial Q14c(i) · **2 marks**

The diagram represents the graph of the function $y = \sin 5x - \sin 6x$.



Determine the exact coordinates of the point C.

Criteria	**Marks**
Factorises $\sin 5x - \sin 6x$ using the sum-to-product identity	1
Finds the first positive x-intercept $C = \left(\frac{\pi}{11}, 0\right)$	1

Solution.

Using $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ with $A = 5x$, $B = 6x$,

$$y = \sin 5x - \sin 6x = 2\cos\left(\frac{11x}{2}\right)\sin\left(-\frac{x}{2}\right) = -2\cos\left(\frac{11x}{2}\right)\sin\frac{x}{2}.$$

C is the first positive x-intercept, so $y = 0$ with $x > 0$. The factor $\sin\frac{x}{2} = 0$ gives $x = 0$, so the intercept comes from $\cos\frac{11x}{2} = 0$:

$$\frac{11x}{2} = \frac{\pi}{2} \quad \Rightarrow \quad x = \frac{\pi}{11}.$$

Hence $C = \left(\frac{\pi}{11}, 0\right)$.

Q35 2022 Baulkham Hills Trial Q14(d) · **4 marks**

(i) Show that $2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \cos x + \cos y$.

(ii) Hence find all solutions to

$$\cos 4\theta + \cos 3\theta + \cos 2\theta + \cos\theta = 0 \qquad \text{for } 0 \le \theta \le 2\pi.$$

Criteria	Marks
(i) Expands the product with the compound-angle formula to obtain $\cos x + \cos y$	1
(ii) Pairs the terms and applies the identity to factorise as $4\cos\dfrac{5\theta}{2}\cos\theta\cos\dfrac{\theta}{2}$ (or equivalent)	1
(ii) Sets each factor to zero over $0 \le \theta \le 2\pi$	1
(ii) Lists all seven solutions	1

Solution.

(i) Expanding with $\cos(P \pm Q) = \cos P\cos Q \mp \sin P\sin Q$, let $P = \dfrac{x+y}{2}$, $Q = \dfrac{x-y}{2}$, so $P+Q = x$ and $P-Q = y$:

$$2\cos P\cos Q = \cos(P+Q) + \cos(P-Q) = \cos x + \cos y,$$

as required.

(ii) Pair the terms and apply part (i):

$$(\cos 4\theta + \cos\theta) + (\cos 3\theta + \cos 2\theta) = 2\cos\frac{5\theta}{2}\cos\frac{3\theta}{2} + 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}.$$

Take out $2\cos\dfrac{5\theta}{2}$ and apply the identity once more to the bracket:

$$= 2\cos\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right) = 2\cos\frac{5\theta}{2}\cdot 2\cos\theta\cos\frac{\theta}{2} = 4\cos\frac{5\theta}{2}\cos\theta\cos\frac{\theta}{2}.$$

Setting each factor to zero over $0 \le \theta \le 2\pi$:

$$\cos\frac{5\theta}{2} = 0 \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}; \qquad \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}; \qquad \cos\frac{\theta}{2} = 0 \Rightarrow \theta = \pi.$$

Hence the solutions are

$$\theta = \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{9\pi}{5}.$$

5. Simplifying trigonometric expressions

Q36 2024 Baulkham Hills Prelim Q5 · **1 mark**

Which of the following is equivalent to $\dfrac{1-\cos 2x}{\sin 2x}$?

(A) $1-\cot 2x$

(B) 1

(C) $\cot x$

(D) $\tan x$

Criteria	**Marks**
Correct option (D)	1

Solution.

Use $\cos 2x = 1-2\sin^2 x$ and $\sin 2x = 2\sin x\cos x$:

$$\frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x.$$

The answer is (D).

Q37 2023 Hurlstone Prelim Q3 · **1 mark**

Which of the following is equivalent to $\tan\left(\frac{\pi}{4}+x\right)$?

(A) $\dfrac{\sec^2 x}{1-\tan^2 x}$

(B) $\dfrac{\cos x-\sin x}{\cos x+\sin x}$

(C) $\dfrac{\sin x+\cos x}{\sin x-\cos x}$

(D) $\dfrac{\cos x+\sin x}{\cos x-\sin x}$

Criteria	**Marks**
Correct option (D)	1

Solution.

Apply the tangent compound-angle formula with $\tan\frac{\pi}{4}=1$:

$$\tan\left(\frac{\pi}{4}+x\right)=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}=\frac{1+\tan x}{1-\tan x}.$$

Multiplying numerator and denominator by $\cos x$ gives $\dfrac{\cos x+\sin x}{\cos x-\sin x}$. The answer is (D).

Q38 2025 North Sydney Boys Trial Q2 · **1 mark**

Which of the following is equivalent to the expression $\sqrt{\dfrac{4+4\cos 2x}{1-\cos 2x}}$?

(A) $2\cot^2 x$

(B) $2\cot x$

(C) $2\tan x$

(D) $2\tan^2 x$

Criteria	Marks
Correct option (B)	1

Solution.

Use $\cos 2x = 2\cos^2 x - 1$ in the numerator and $\cos 2x = 1 - 2\sin^2 x$ in the denominator:

$$\frac{4+4\cos 2x}{1-\cos 2x} = \frac{4+4(2\cos^2 x - 1)}{1-(1-2\sin^2 x)} = \frac{8\cos^2 x}{2\sin^2 x} = 4\cot^2 x.$$

Hence $\sqrt{4\cot^2 x} = 2\cot x$ (for the domain where $\cot x > 0$). The answer is (B).

Q39 2020 St George Girls Trial Q8 · **1 mark**

Which expression is equivalent to $\cos 5x \cos 2x - \sin 6x \sin 3x$?

(A) $\cos 7x - \sin 9x$

(B) $\cos 3x - \sin 3x$

(C) $\sin 8x \sin x$

(D) $\cos 8x \cos x$

Criteria	**Marks**
Correct option (D)	1

Solution.

Convert each product to a sum. Using $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ and $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$:

$$\cos 5x \cos 2x = \tfrac{1}{2}(\cos 7x + \cos 3x), \qquad \sin 6x \sin 3x = \tfrac{1}{2}(\cos 3x - \cos 9x).$$

Subtracting,

$$\cos 5x \cos 2x - \sin 6x \sin 3x = \tfrac{1}{2}(\cos 7x + \cos 9x) = \cos 8x \cos x,$$

using the sum-to-product identity. The answer is (D).

Q40 2021 Scots Trial Q1 · 2 marks

(a) Which of the following is equivalent to $\cos(2x - y)\cos x + \sin(2x - y)\sin x$?

(A) $\cos(x - y)$

(B) $\cos(3x - y)$

(C) $\sin(x - y)$

(D) $\sin(3x - y)$

(b) Which of the following is an equivalent expression for $\dfrac{\tan(x - y) + \tan y}{1 - \tan(x - y)\tan y}$?

(A) $\tan x$

(B) $\tan(x - y)$

(C) $\tan(x - 2y)$

(D) $\dfrac{2\tan x - \tan y}{1 - \tan x \tan y + \tan^2 y}$

Criteria	Marks
Part (a): correct option (A)	1
Part (b): correct option (A)	1

Solution.

(a) This is the expansion of $\cos(A - B) = \cos A \cos B + \sin A \sin B$ with $A = 2x - y$ and $B = x$:

$$\cos(2x - y)\cos x + \sin(2x - y)\sin x = \cos\big((2x - y) - x\big) = \cos(x - y).$$

The answer is (A).

(b) This is the tangent addition formula $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ with $A = x - y$ and $B = y$:

$$\frac{\tan(x - y) + \tan y}{1 - \tan(x - y)\tan y} = \tan\big((x - y) + y\big) = \tan x.$$

The answer is (A).

Q41 2021 Sydney Grammar Prelim Q5(a) · **3 marks**

(i) Expand and simplify $(\sin\theta + \cos\theta)^2$.

(ii) Hence simplify $\dfrac{\cos\theta - \tan\theta\sin\theta}{1 + 2\sin\theta\cos\theta} + \dfrac{\tan\theta}{\sin\theta + \cos\theta}$.

Criteria	Marks
Part (i): expands and simplifies to $1 + \sin 2\theta$	1
Part (ii): rewrites the first fraction using part (i) and the numerator $\cos\theta - \tan\theta\sin\theta$	1
Part (ii): combines the two fractions and simplifies to the final form	1

Solution.

(i) Expanding and using $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$:

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta = 1 + \sin 2\theta.$$

(ii) The denominator of the first fraction is $(\sin\theta + \cos\theta)^2$ by part (i). Its numerator is

$$\cos\theta - \tan\theta\sin\theta = \cos\theta - \frac{\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta} = \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta}.$$

Hence the first fraction is

$$\frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta\,(\sin\theta + \cos\theta)^2} = \frac{\cos\theta - \sin\theta}{\cos\theta\,(\sin\theta + \cos\theta)}.$$

The second fraction is $\dfrac{\tan\theta}{\sin\theta + \cos\theta} = \dfrac{\sin\theta}{\cos\theta\,(\sin\theta + \cos\theta)}$. Adding,

$$\frac{\cos\theta - \sin\theta}{\cos\theta\,(\sin\theta + \cos\theta)} + \frac{\sin\theta}{\cos\theta\,(\sin\theta + \cos\theta)} = \frac{\cos\theta}{\cos\theta\,(\sin\theta + \cos\theta)} = \frac{1}{\sin\theta + \cos\theta}.$$

The t-formula and proving identities

6. The t-formula (half-angle) substitution

Q42 2022 Blacktown Boys Prelim Q2 · 1 mark

Which of the following gives the correct expression for $3\sin\theta - 2\cos\theta$ if $t = \tan\dfrac{\theta}{2}$?

(A) $\dfrac{6t - 2 - 2t^2}{1 + t^2}$

(B) $\dfrac{6t - 2 + 2t^2}{1 + t^2}$

(C) $\dfrac{6t - 2 - 2t^2}{1 - t^2}$

(D) $\dfrac{6t + 2 - 2t^2}{1 + t^2}$

Criteria	Marks
Correct option (B)	1

Solution.

With $t = \tan\frac{\theta}{2}$, the t-formulas give $\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$. Hence

$$3\sin\theta - 2\cos\theta = \frac{3(2t)}{1+t^2} - \frac{2(1-t^2)}{1+t^2} = \frac{6t - 2 + 2t^2}{1+t^2}.$$

The answer is (B).

Q43 2021 Knox Prelim Q3 · **1 mark**

Given that $t = \tan\dfrac{x}{2}$, which of the following is an expression for $1 + \sec x$?

(A) $\dfrac{2t}{1+t^2}$

(B) $\dfrac{2t}{1-t^2}$

(C) $\dfrac{2}{1+t^2}$

(D) $\dfrac{2}{1-t^2}$

Criteria	**Marks**
Correct option (D)	1

Solution.

With $\cos x = \dfrac{1-t^2}{1+t^2}$ we have $\sec x = \dfrac{1+t^2}{1-t^2}$, so

$$1 + \sec x = \frac{1-t^2}{1-t^2} + \frac{1+t^2}{1-t^2} = \frac{(1-t^2)+(1+t^2)}{1-t^2} = \frac{2}{1-t^2}.$$

The answer is (D).

Q44 2024 Girraween Prelim Q9(c) · **2 marks**

By making the substitution $t = \tan\dfrac{\theta}{2}$, show that

$$\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}.$$

Criteria	Marks
Writes $\operatorname{cosec}\theta$ and $\cot\theta$ in terms of t using the t-formulas	1
Simplifies to $\dfrac{1}{t} = \cot\dfrac{\theta}{2}$	1

Solution.

With $t = \tan\frac{\theta}{2}$, $\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$. Then

$$\operatorname{cosec}\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta} = \frac{1+\dfrac{1-t^2}{1+t^2}}{\dfrac{2t}{1+t^2}} = \frac{\dfrac{(1+t^2)+(1-t^2)}{1+t^2}}{\dfrac{2t}{1+t^2}} = \frac{2}{2t} = \frac{1}{t}.$$

Since $t = \tan\frac{\theta}{2}$, $\dfrac{1}{t} = \cot\dfrac{\theta}{2}$, as required.

Q45 2024 Hurlstone Prelim Q12(b) · **2 marks**

With the use of t-results, prove that

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}.$$

Criteria	Marks
Substitutes the t-formulas for $\sin\theta$ and $\cos\theta$ into the fraction	1
Simplifies the numerator and denominator to obtain $t = \tan\dfrac{\theta}{2}$	1

Solution.

With $t = \tan\frac{\theta}{2}$, $\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$. Multiplying numerator and denominator by $1+t^2$,

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \frac{(1+t^2)+2t-(1-t^2)}{(1+t^2)+2t+(1-t^2)} = \frac{2t^2+2t}{2+2t} = \frac{2t(t+1)}{2(1+t)} = t.$$

Hence the expression equals $t = \tan\dfrac{\theta}{2}$, as required.

Q46 2022 Baulkham Hills Prelim Q11(e) · 3 marks

Using $t = \tan\dfrac{x}{2}$, prove that

$$\frac{1+\sin x+\cos x}{1-\sin x+\cos x} = \tan x + \sec x.$$

Criteria	Marks
Uses the t-formulas to write $\sin x$ and $\cos x$ in terms of t	1
Simplifies the left-hand side to $\dfrac{1+t}{1-t}$	1
Shows the right-hand side $\tan x + \sec x$ equals $\dfrac{1+t}{1-t}$ (or equivalent), completing the proof	1

Solution.

With $t = \tan\frac{x}{2}$, $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$. Multiplying numerator and denominator of the left-hand side by $1+t^2$,

$$\frac{1+\sin x+\cos x}{1-\sin x+\cos x} = \frac{(1+t^2)+2t+(1-t^2)}{(1+t^2)-2t+(1-t^2)} = \frac{2+2t}{2-2t} = \frac{1+t}{1-t}.$$

For the right-hand side, $\tan x + \sec x = \dfrac{1+\sin x}{\cos x}$. Using $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$, and dividing through by $\cos^2\frac{x}{2}$,

$$\tan x + \sec x = \frac{1+\sin x}{\cos x} = \frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{1+t}{1-t}.$$

Both sides equal $\dfrac{1+t}{1-t}$, so the identity holds.

Q47 2020 Knox Prelim Q13(f) · 3 marks

By using $t = \tan\dfrac{\theta}{2}$, solve the equation $\sqrt{3}\sin\theta + \cos\theta = 1$ for $0 \le \theta \le 2\pi$.

Criteria	Marks
Substitutes the t-formulas to form an equation in t	1
Reduces to $2\sqrt{3}\,t - 2t^2 = 0$ and solves $t = 0$ or $t = \sqrt{3}$	1
Recovers all angles in $0 \le \theta \le 2\pi$: $\theta = 0,\ \dfrac{2\pi}{3},\ 2\pi$	1

Solution.

With $t = \tan\frac{\theta}{2}$, $\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$. Substituting,

$$\sqrt{3}\cdot\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1 \implies 2\sqrt{3}\,t + 1 - t^2 = 1 + t^2.$$

Hence $2\sqrt{3}\,t - 2t^2 = 0$, i.e. $2t\left(\sqrt{3} - t\right) = 0$, giving $t = 0$ or $t = \sqrt{3}$.

$$t = \tan\frac{\theta}{2} = 0 \implies \frac{\theta}{2} = 0 \text{ or } \pi \implies \theta = 0 \text{ or } 2\pi, \qquad t = \tan\frac{\theta}{2} = \sqrt{3} \implies \frac{\theta}{2} = \frac{\pi}{3} \implies \theta = \frac{2\pi}{3}.$$

Checking $\theta = \pi$ (where $\tan\frac{\theta}{2}$ is undefined) gives $\sqrt{3}(0) + (-1) = -1 \neq 1$, so it is not a solution and none is lost. For $0 \le \theta \le 2\pi$,

$$\theta = 0, \quad \frac{2\pi}{3}, \quad 2\pi.$$

Q48 2020 Barker Trial Q12(d) · 5 marks

(i) If $t = \tan A$, find an expression in terms of t for $\cos 2A$.

(ii) Use t formulas with the substitution $t = \tan x$ to prove that

$$\sec 2x + \tan 2x = \tan(x + 45°).$$

(iii) Hence or otherwise solve $\sec 2x + \tan 2x = \dfrac{4}{5}$ for $0° \le x \le 360°$. Answer to the nearest degree.

Criteria	Marks
Part (i): $\cos 2A = \dfrac{1-t^2}{1+t^2}$	1
Part (ii): writes $\sec 2x + \tan 2x = \dfrac{1+\sin 2x}{\cos 2x}$ and substitutes the t-formulas	1
Part (ii): simplifies to $\dfrac{1+t}{1-t}$ and identifies this as $\tan(x+45°)$	1
Part (iii): uses the identity to get $\tan(x+45°) = \dfrac{4}{5}$ and finds $x + 45° = 38.66° + 180°k$	1
Part (iii): states both answers $x \approx 174°,\ 354°$ in the given domain	1

Solution.

(i) With $t = \tan A$, $\cos 2A = \dfrac{1-t^2}{1+t^2}$.

(ii) Let $t = \tan x$. Then $\sin 2x = \dfrac{2t}{1+t^2}$ and $\cos 2x = \dfrac{1-t^2}{1+t^2}$, so

$$\sec 2x + \tan 2x = \frac{1}{\cos 2x} + \frac{\sin 2x}{\cos 2x} = \frac{1+\sin 2x}{\cos 2x} = \frac{1+\dfrac{2t}{1+t^2}}{\dfrac{1-t^2}{1+t^2}} = \frac{1+t^2+2t}{1-t^2} = \frac{(1+t)^2}{(1-t)(1+t)} = \frac{1+t}{1-t}.$$

Since $\tan(x+45°) = \dfrac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = \dfrac{t+1}{1-t}$, we have $\sec 2x + \tan 2x = \tan(x+45°)$.

(iii) Using the identity, $\tan(x+45°) = \dfrac{4}{5}$. The reference angle is $\tan^{-1}\dfrac{4}{5} \approx 38.66°$, so

$$x + 45° = 38.66° + 180°k \implies x = -6.34° + 180°k.$$

Taking $k = 1, 2$ places x in $0° \le x \le 360°$: $x \approx 173.66°$ and $x \approx 353.66°$. To the nearest degree,

$$x \approx 174°, \quad 354°.$$

7. Proving trigonometric identities

Q49 2020 Cherrybrook Tech Trial Q12a · **1 mark**

Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.

Criteria	Marks
Expands both compound angles and adds to reach the required form	1

Solution.

Using $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$,

$$\sin(A+B)+\sin(A-B)=(\sin A\cos B+\cos A\sin B)+(\sin A\cos B-\cos A\sin B)=2\sin A\cos B.$$

Q50 2024 Gosford Trial Q12d · **2 marks**

Show that $\dfrac{\sin 2x}{1+\cos 2x}=\tan x$.

Criteria	Marks
Substitutes $\sin 2x=2\sin x\cos x$ and $1+\cos 2x=2\cos^2 x$	1
Cancels to obtain $\tan x$	1

Solution.

Using $\sin 2x=2\sin x\cos x$ and $\cos 2x=2\cos^2 x-1$ (so $1+\cos 2x=2\cos^2 x$),

$$\frac{\sin 2x}{1+\cos 2x}=\frac{2\sin x\cos x}{2\cos^2 x}=\frac{\sin x}{\cos x}=\tan x.$$

Q51 2024 Caringbah Trial Q14a · **2 marks**

Use trigonometric identities to prove that

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)} - \cot\theta.$$

Criteria	Marks
Rationalises the surd, using $1-\cos^2\theta = \sin^2\theta$, to reach $\frac{1-\cos\theta}{\sin\theta}$	1
Shows the right-hand side equals $\frac{1-\cos\theta}{\sin\theta}$ (via $\cos(\frac{\pi}{2}-\theta) = \sin\theta$)	1

Solution.

Multiplying numerator and denominator inside the surd by $(1-\cos\theta)$,

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} = \frac{1-\cos\theta}{\sin\theta},$$

taking the positive root (for $0<\theta<\pi$, $1-\cos\theta \geq 0$ and $\sin\theta > 0$). For the right-hand side, $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, so

$$\frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)} - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

Both sides equal $\frac{1-\cos\theta}{\sin\theta}$, as required.

Q52 2022 Hurlstone Prelim Q12a · **2 marks**

Prove that $\frac{1+\cot^2\theta}{2\cot\theta} = \operatorname{cosec} 2\theta$.

Criteria	Marks
Uses $1+\cot^2\theta = \operatorname{cosec}^2\theta$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$	1
Simplifies to $\frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta$	1

Solution.

Using $1+\cot^2\theta = \operatorname{cosec}^2\theta = \frac{1}{\sin^2\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$,

$$\frac{1+\cot^2\theta}{2\cot\theta} = \frac{\frac{1}{\sin^2\theta}}{\frac{2\cos\theta}{\sin\theta}} = \frac{1}{\sin^2\theta}\cdot\frac{\sin\theta}{2\cos\theta} = \frac{1}{2\sin\theta\cos\theta} = \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta.$$

Q53 2022 SCEGGS Prelim Q9e · **2 marks**

Show that $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$.

Criteria	Marks
Writes the left-hand side over a common denominator as $\dfrac{1-\cos 2\theta}{\sin 2\theta}$	1
Substitutes $1-\cos 2\theta = 2\sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$ and simplifies to $\tan\theta$	1

Solution.

$$\operatorname{cosec} 2\theta - \cot 2\theta = \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}.$$

Using $1-\cos 2\theta = 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Q54 2024 Hurlstone Prelim Q13d · **2 marks**

Show, without using a calculator, that

$$2\cos 95^\circ\cos 55^\circ - 2\sin 50^\circ\sin 10^\circ = \tfrac{1}{2}\left(1-\sqrt{3}\right).$$

Criteria	Marks
Applies the products-to-sums formulae to both terms	1
Evaluates the resulting exact values to reach $\frac{1}{2}(1-\sqrt{3})$	1

Solution.

Using $2\cos A\cos B = \cos(A-B)+\cos(A+B)$ and $2\sin A\sin B = \cos(A-B)-\cos(A+B)$,

$$2\cos 95^\circ\cos 55^\circ = \cos 40^\circ + \cos 150^\circ, \qquad 2\sin 50^\circ\sin 10^\circ = \cos 40^\circ - \cos 60^\circ.$$

Subtracting,

$$(\cos 40^\circ + \cos 150^\circ) - (\cos 40^\circ - \cos 60^\circ) = \cos 150^\circ + \cos 60^\circ = -\frac{\sqrt{3}}{2} + \frac{1}{2} = \tfrac{1}{2}\left(1-\sqrt{3}\right).$$

Q55 2020 Hurlstone Trial Q13b · **3 marks**

(i) Show that $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

(ii) Hence, if α,β and γ are the angles of $\triangle ABC$ and $\sin\gamma=2\sin\alpha\cos\beta$, show that $\triangle ABC$ has two equal angles.

Criteria	Marks
(i) Expands the compound angles and adds to obtain $2\sin\alpha\cos\beta$	1
(ii) Uses $\gamma=\pi-(\alpha+\beta)$ to write $\sin\gamma=\sin(\alpha+\beta)$	1
(ii) Deduces $\sin(\alpha-\beta)=0$, hence $\alpha=\beta$	1

Solution.

(i) $\sin(\alpha+\beta)+\sin(\alpha-\beta)=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)+(\sin\alpha\cos\beta-\cos\alpha\sin\beta)=2\sin\alpha\cos\beta$.

(ii) Since $\alpha+\beta+\gamma=\pi$, we have $\gamma=\pi-(\alpha+\beta)$, so $\sin\gamma=\sin(\alpha+\beta)$. Given $\sin\gamma=2\sin\alpha\cos\beta$, from part (i) $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$, hence

$$\sin(\alpha+\beta)=\sin(\alpha+\beta)+\sin(\alpha-\beta)\ \Rightarrow\ \sin(\alpha-\beta)=0.$$

As α,β are angles of a triangle, $\alpha-\beta=0$, so $\alpha=\beta$ and the triangle has two equal angles.

Q56 2023 Trinity Grammar Trial Q12e · **3 marks**

Given that $\cos 2x=a+b$ and $\sin 2x=a-b$, show that $a^2+b^2=\dfrac{1}{2}$.

Criteria	Marks
Solves for a and b (or forms $\cos^2 2x+\sin^2 2x$) from the two given equations	1
Expresses a^2+b^2 in terms of $\cos 2x$ and $\sin 2x$	1
Uses $\cos^2 2x+\sin^2 2x=1$ to conclude $a^2+b^2=\dfrac{1}{2}$	1

Solution.

Adding and subtracting the two equations, $a=\dfrac{\cos 2x+\sin 2x}{2}$ and $b=\dfrac{\cos 2x-\sin 2x}{2}$. Then

$$a^2+b^2=\frac{(\cos 2x+\sin 2x)^2+(\cos 2x-\sin 2x)^2}{4}=\frac{2\cos^2 2x+2\sin^2 2x}{4}=\frac{2}{4}=\frac{1}{2},$$

using $\cos^2 2x+\sin^2 2x=1$.

Q57 2021 Scots Prelim Q10e · **3 marks**

Given that $(x+y)(x^2-xy+y^2)=x^3+y^3$, prove the trigonometric identity

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}+\tfrac{1}{2}\sin 2\theta=1.$$

Criteria	**Marks**
Factorises $\sin^3\theta+\cos^3\theta$ and cancels $(\sin\theta+\cos\theta)$	1
Simplifies the first term to $1-\sin\theta\cos\theta$ using $\sin^2\theta+\cos^2\theta=1$	1
Adds $\frac{1}{2}\sin 2\theta=\sin\theta\cos\theta$ to obtain 1	1

Solution.

With $x=\sin\theta$, $y=\cos\theta$, the given factorisation gives

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=\frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}=\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta=1-\sin\theta\cos\theta.$$

Since $\frac{1}{2}\sin 2\theta=\frac{1}{2}(2\sin\theta\cos\theta)=\sin\theta\cos\theta$,

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}+\tfrac{1}{2}\sin 2\theta=(1-\sin\theta\cos\theta)+\sin\theta\cos\theta=1.$$

Q58 2021 St George Girls Trial Q15b · **4 marks**

(i) By using $\cos 4\theta = 2\cos^2 2\theta - 1$, show that

$$8\cos^4\theta - 8\cos^2\theta + 1 - \cos 4\theta = 0.$$

(ii) Using part (i) and by letting $x = \cos\theta$ in the equation $16x^4 - 16x^2 + 2 - \sqrt{2} = 0$, show that $\cos 4\theta = \dfrac{\sqrt{2}}{2}$.

Criteria	**Marks**
(i) Substitutes $\cos 2\theta = 2\cos^2\theta - 1$ into $\cos 4\theta = 2\cos^2 2\theta - 1$	1
(i) Expands and simplifies to reach the stated result	1
(ii) Writes $16x^4 - 16x^2 + 2 = 2(8\cos^4\theta - 8\cos^2\theta + 1) = 2\cos 4\theta$	1
(ii) Sets $2\cos 4\theta - \sqrt{2} = 0$ to conclude $\cos 4\theta = \dfrac{\sqrt{2}}{2}$	1

Solution.

(i) With $\cos 2\theta = 2\cos^2\theta - 1$,

$$\cos 4\theta = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1 = 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Rearranging, $8\cos^4\theta - 8\cos^2\theta + 1 - \cos 4\theta = 0$.

(ii) Let $x = \cos\theta$. Then

$$16x^4 - 16x^2 + 2 = 2\left(8\cos^4\theta - 8\cos^2\theta + 1\right) = 2\cos 4\theta,$$

using part (i). The given equation $16x^4 - 16x^2 + 2 - \sqrt{2} = 0$ therefore becomes $2\cos 4\theta - \sqrt{2} = 0$, so $\cos 4\theta = \dfrac{\sqrt{2}}{2}$.

Q59 2021 James Ruse Prelim Q44 · **4 marks**

Given that $m = a\cos^3 x + 3a\cos x \sin^2 x$ and $n = a\sin^3 x + 3a\sin x\cos^2 x$, prove that

$$(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}.$$

Criteria	**Marks**
Forms $m+n$ and factors it as $a(\cos x + \sin x)^3$	1
Forms $m-n$ and factors it as $a(\cos x - \sin x)^3$	1
Takes the $\frac{2}{3}$ power of each to get $a^{2/3}(\cos x \pm \sin x)^2$	1
Adds and uses $(\cos x + \sin x)^2 + (\cos x - \sin x)^2 = 2$ to conclude	1

Solution.

Adding,

$$m + n = a\left(\cos^3 x + 3\cos^2 x \sin x + 3\cos x \sin^2 x + \sin^3 x\right) = a(\cos x + \sin x)^3.$$

Subtracting,

$$m - n = a\left(\cos^3 x - 3\cos^2 x \sin x + 3\cos x \sin^2 x - \sin^3 x\right) = a(\cos x - \sin x)^3.$$

Hence

$$(m+n)^{2/3} = a^{2/3}(\cos x + \sin x)^2, \qquad (m-n)^{2/3} = a^{2/3}(\cos x - \sin x)^2.$$

Adding these and expanding,

$$(m+n)^{2/3} + (m-n)^{2/3} = a^{2/3}\left[(\cos x + \sin x)^2 + (\cos x - \sin x)^2\right] = a^{2/3}\left(2\cos^2 x + 2\sin^2 x\right) = 2a^{2/3}.$$

Q60 2023 Knox Prelim Q14c · **9 marks**

(i) Show that $2\sin(k-1)\theta\cos\theta - \sin(k-2)\theta = \sin k\theta$.

(ii) Hence show that $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = \sin 5\theta$.

(iii) By letting $x = \sin\theta$ in the equation $16x^4 - 20x^2 + 5 = 0$ and by considering the roots of the equation formed, prove that

$$\sin^2\frac{\pi}{5} + \sin^2\frac{2\pi}{5} = \frac{5}{4}.$$

Criteria	Marks
(i) Applies $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ with $A = (k-1)\theta$, $B = \theta$	1
(i) Simplifies to $\sin k\theta$	1
(ii) Applies part (i) repeatedly (or otherwise) to reduce $\sin 5\theta$ to a polynomial in $\sin\theta$	2
(iii) Uses $\sin 5\theta = 0$ to show $x = \sin\theta$ satisfies $x(16x^4 - 20x^2 + 5) = 0$, identifying the roots $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}$	2
(iii) Uses the sum of roots of the quadratic in x^2 (or product/sum relations) to obtain $\frac{5}{4}$	2
(iii) States the conclusion clearly	1

Solution.

(i) By the product-to-sum formula $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ with $A = (k-1)\theta$ and $B = \theta$,

$$2\sin(k-1)\theta\cos\theta = \sin k\theta + \sin(k-2)\theta,$$

so $2\sin(k-1)\theta\cos\theta - \sin(k-2)\theta = \sin k\theta$.

(ii) Expanding $\sin 5\theta$ (e.g. by De Moivre's theorem, or by iterating part (i)) gives

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(iii) If $\sin 5\theta = 0$ then, by part (ii), $\sin\theta$ satisfies $16s^5 - 20s^3 + 5s = 0$, i.e.

$$s\left(16s^4 - 20s^2 + 5\right) = 0, \qquad s = \sin\theta.$$

The solutions of $\sin 5\theta = 0$ are $\theta = \dfrac{k\pi}{5}$. Excluding $s = 0$ (i.e. $\theta = 0, \pi$), the non-zero values of $\sin\theta$ are $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5} = \sin\frac{2\pi}{5}, \sin\frac{4\pi}{5} = \sin\frac{\pi}{5}$. Hence the roots of $16x^4 - 20x^2 + 5 = 0$ are $x = \pm\sin\frac{\pi}{5}, \pm\sin\frac{2\pi}{5}$, and x^2 takes the two values $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{2\pi}{5}$. Putting $u = x^2$, these are the roots of

$$16u^2 - 20u + 5 = 0,$$

so by the sum of roots,

$$\sin^2\frac{\pi}{5} + \sin^2\frac{2\pi}{5} = \frac{20}{16} = \frac{5}{4}.$$

Triple-angle identities

8. Triple-angle identities and cubic-root problems

Q61 2023 Glenwood Trial Q13(b)(i) · 1 mark

Consider the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. By letting $x = \cos\theta$ in the cubic equation $8x^3 - 6x - 1 = 0$, show that $\cos 3\theta = \dfrac{1}{2}$.

Criteria	Marks
Substitutes $x = \cos\theta$, halves the equation and identifies $4\cos^3\theta - 3\cos\theta$ as $\cos 3\theta = \frac{1}{2}$	1

Solution.

Let $x = \cos\theta$. Then $8\cos^3\theta - 6\cos\theta - 1 = 0$, so $8\cos^3\theta - 6\cos\theta = 1$. Dividing by 2,

$$4\cos^3\theta - 3\cos\theta = \tfrac{1}{2}.$$

Since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, it follows that $\cos 3\theta = \dfrac{1}{2}$.

Q62 2023 Manly Trial Q12(c)(i) · 2 marks

Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Criteria	Marks
Expands $\sin(2\theta + \theta)$ using the compound-angle formula	1
Substitutes the double-angle results and simplifies to the required form	1

Solution.

$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$. Using $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$,

$$\sin 3\theta = 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta = 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Q63 2025 Pymble Trial Q13(e) · 3 marks

It is known that $\cos 3\theta = 4\cos\theta\,\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)$. Using this result, or otherwise, solve

$$\sqrt{2}\,\cos\theta\left(\cos^2\theta - 3\sin^2\theta\right) = 1 \quad \text{for } 0 \le \theta \le 2\pi.$$

Criteria	Marks
Recognises $\cos\theta(\cos^2\theta - 3\sin^2\theta) = \cos 3\theta$, reducing the equation to $\cos 3\theta = \frac{1}{\sqrt{2}}$	1
Solves 3θ over the extended domain $0 \le 3\theta \le 6\pi$	1
Lists all six solutions for θ in $[0, 2\pi]$	1

Solution.

Expanding, $\cos\theta(\cos^2\theta - 3\sin^2\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$. Hence the equation becomes

$$\sqrt{2}\,\cos 3\theta = 1 \quad \Longrightarrow \quad \cos 3\theta = \frac{1}{\sqrt{2}}.$$

For $0 \le \theta \le 2\pi$ we need $0 \le 3\theta \le 6\pi$, so

$$3\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}.$$

Therefore

$$\theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}.$$

Q64 2020 Manly Trial Q12(c) · 4 marks

(i) Show that $\cos 3x = 4\cos^3 x - 3\cos x$.

(ii) Find all solutions to the equation $\cos 3x + 2\cos x = 0$ over the domain $[-2\pi, 2\pi]$.

Criteria	Marks
(i) Expands $\cos(2x + x)$ and simplifies to $4\cos^3 x - 3\cos x$	1
(ii) Substitutes part (i) and factors to $\cos x(4\cos^2 x - 1) = 0$	1
(ii) Solves $\cos x = 0$ over $[-2\pi, 2\pi]$	1
(ii) Solves $\cos x = \pm\frac{1}{2}$ over $[-2\pi, 2\pi]$	1

Solution.

(i) $\cos 3x = \cos(2x + x) = \cos 2x\cos x - \sin 2x \sin x$. With $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x\cos x$,

$$\cos 3x = (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x = 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x = 4\cos^3 x - 3\cos x.$$

(ii) Using part (i),

$$\cos 3x + 2\cos x = (4\cos^3 x - 3\cos x) + 2\cos x = 4\cos^3 x - \cos x = \cos x(4\cos^2 x - 1) = 0.$$

So $\cos x = 0$ or $\cos^2 x = \frac{1}{4}$, i.e. $\cos x = \pm\frac{1}{2}$. Over $[-2\pi, 2\pi]$:

$$\cos x = 0 :\ x = \pm\tfrac{\pi}{2},\ \pm\tfrac{3\pi}{2}; \qquad \cos x = \tfrac{1}{2} :\ x = \pm\tfrac{\pi}{3},\ \pm\tfrac{5\pi}{3}; \qquad \cos x = -\tfrac{1}{2} :\ x = \pm\tfrac{2\pi}{3},\ \pm\tfrac{4\pi}{3}.$$

Q65 2023 North Sydney Boys Prelim Q5 · **4 marks**

(i) Use the fact that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ to solve $8x^3 - 6x - 1 = 0$.

(ii) Hence, show that $\cos\dfrac{\pi}{9} = \cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9}$.

Criteria	Marks
(i) Sets $x = \cos\theta$, reduces to $\cos 3\theta = \frac{1}{2}$ and solves for 3θ	1
(i) States the three distinct roots $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$	1
(ii) Uses the sum of the roots (no x^2 term, so sum = 0)	1
(ii) Rewrites $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$, $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$ to conclude	1

Solution.

(i) Let $x = \cos\theta$. Then $8\cos^3\theta - 6\cos\theta - 1 = 0$, so $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$, i.e. $\cos 3\theta = \frac{1}{2}$. Taking $0 \le \theta \le \pi$ (to give distinct values of $\cos\theta$), $3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, so $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$ and the roots are

$$x = \cos\tfrac{\pi}{9}, \quad x = \cos\tfrac{5\pi}{9}, \quad x = \cos\tfrac{7\pi}{9}.$$

(ii) The cubic $8x^3 - 6x - 1 = 0$ has no x^2 term, so the sum of its roots is 0:

$$\cos\tfrac{\pi}{9} + \cos\tfrac{5\pi}{9} + \cos\tfrac{7\pi}{9} = 0.$$

Since $\cos\frac{5\pi}{9} = \cos\left(\pi - \frac{4\pi}{9}\right) = -\cos\frac{4\pi}{9}$ and $\cos\frac{7\pi}{9} = \cos\left(\pi - \frac{2\pi}{9}\right) = -\cos\frac{2\pi}{9}$,

$$\cos\tfrac{\pi}{9} - \cos\tfrac{4\pi}{9} - \cos\tfrac{2\pi}{9} = 0 \quad \Longrightarrow \quad \cos\tfrac{\pi}{9} = \cos\tfrac{2\pi}{9} + \cos\tfrac{4\pi}{9}.$$

Q66 2022 Fort St Trial Q16(b) · 6 marks

(i) Prove the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) Hence, find exact value expressions for three unique solutions of

$$8x^3 - 6x = -\sqrt{3}.$$

Criteria	Marks
(i) Expands $\cos(2\theta + \theta)$ using the compound-angle formula	1
(i) Substitutes double-angle results and simplifies to $4\cos^3\theta - 3\cos\theta$	1
(ii) Sets $x = \cos\theta$ and reduces to $\cos 3\theta = -\frac{\sqrt{3}}{2}$	1
(ii) Solves 3θ over a suitable range	1
(ii) Gives three distinct exact solutions	2

Solution.

(i) $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$. With $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\cos 3\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

(ii) Let $x = \cos\theta$. Then $8\cos^3\theta - 6\cos\theta = -\sqrt{3}$, so $4\cos^3\theta - 3\cos\theta = -\frac{\sqrt{3}}{2}$, i.e.

$$\cos 3\theta = -\frac{\sqrt{3}}{2}.$$

Taking $0 \le \theta \le \pi$, $3\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$, giving $\theta = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}$. Hence three unique solutions are

$$x = \cos\tfrac{5\pi}{18}, \quad x = \cos\tfrac{7\pi}{18}, \quad x = \cos\tfrac{17\pi}{18}.$$

Q67 2020 Caringbah Trial Q13(c) · 7 marks

(i) Prove the trigonometric identity

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

using the compound angle formula.

(ii) Hence find expressions for the exact values of the solutions to the equation

$$\frac{3x - x^3}{1 - 3x^2} = \sqrt{3}.$$

(iii) Hence show that $\tan\dfrac{\pi}{9}\tan\dfrac{4\pi}{9}\tan\dfrac{7\pi}{9} = -\sqrt{3}$.

Criteria	Marks
(i) Expands $\tan(2\theta+\theta)$ and substitutes $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	1
(i) Simplifies to the required single fraction	1
(ii) Sets $x = \tan\theta$, giving $\tan 3\theta = \sqrt{3}$, so $3\theta = \frac{\pi}{3} + k\pi$	1
(ii) States the three exact solutions $x = \tan\frac{\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{7\pi}{9}$	2
(iii) Rewrites the equation as a cubic in x and applies the product of roots	2

Solution.

(i) $\tan 3\theta = \tan(2\theta+\theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$. With $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$,

$$\tan 3\theta = \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1-\tan^2\theta}\cdot\tan\theta} = \frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta) - 2\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(ii) Let $x = \tan\theta$. Then the left-hand side is $\tan 3\theta$, so $\tan 3\theta = \sqrt{3}$, giving $3\theta = \frac{\pi}{3} + k\pi$, i.e. $\theta = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$ (three values with distinct $\tan\theta$). Hence

$$x = \tan\tfrac{\pi}{9}, \quad x = \tan\tfrac{4\pi}{9}, \quad x = \tan\tfrac{7\pi}{9}.$$

(iii) Clearing the denominator, $3x - x^3 = \sqrt{3}(1 - 3x^2)$, i.e. $x^3 - 3\sqrt{3}\,x^2 - 3x + \sqrt{3} = 0$. Its three roots are $\tan\frac{\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{7\pi}{9}$, and by the product of roots (−constant/leading),

$$\tan\tfrac{\pi}{9}\tan\tfrac{4\pi}{9}\tan\tfrac{7\pi}{9} = -\frac{\sqrt{3}}{1} = -\sqrt{3}.$$

Q68 2020 Riverview Trial Q13(c) · 7 marks

(i) Prove the trigonometric identity

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(ii) Use the identity from part (i) and let $x = \tan\theta$ to find the roots of the cubic equation $x^3 - 3x^2 - 3x + 1 = 0$, and hence find the exact value of $\tan\dfrac{\pi}{12}$.

Criteria	Marks
(i) Expands $\tan(2\theta + \theta)$ and substitutes $\tan 2\theta$	2
(i) Simplifies to the required single fraction	1
(ii) Rearranges the cubic to $\tan 3\theta = 1$ and finds $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$	3
(ii) Concludes $\tan\frac{\pi}{12} = 2 - \sqrt{3}$	1

Solution.

(i) $\tan 3\theta = \tan(2\theta + \theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$, and with $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$,

$$\tan 3\theta = \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan^2\theta}{1-\tan^2\theta}} = \frac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta) - 2\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(ii) Let $x = \tan\theta$. The cubic $x^3 - 3x^2 - 3x + 1 = 0$ rearranges to $x^3 - 3x = 3x^2 - 1$, i.e. $3x - x^3 = 1 - 3x^2$, so

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{3x - x^3}{1 - 3x^2} = 1.$$

Thus $3\theta = \frac{\pi}{4} + k\pi$, giving $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$, so the roots are

$$x = \tan\tfrac{\pi}{12}, \quad x = \tan\tfrac{5\pi}{12}, \quad x = \tan\tfrac{3\pi}{4} = -1.$$

The smallest root is $\tan\frac{\pi}{12}$. Since $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$ and $\tan\frac{3\pi}{4} = -1$, and the product of the roots is -1 (constant $\div$ leading, with sign), the remaining root satisfies $\tan\frac{\pi}{12} = \dfrac{-1}{(2+\sqrt{3})(-1)} = \dfrac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$. Hence

$$\tan\tfrac{\pi}{12} = 2 - \sqrt{3}.$$

Q69 2022 Normanhurst Boys Trial Q13(c) · **8 marks**

You are given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(i) Use this to show that

$$\sin^4\theta + \cos^4\theta = \frac{\cos 4\theta + 3}{4}.$$

(ii) By letting $x = \cos\theta$, show that the equation

$$8x^4 + 8\left(1 - x^2\right)^2 = 7$$

has roots $\pm\cos\dfrac{\pi}{12}$ and $\pm\cos\dfrac{5\pi}{12}$.

(iii) Deduce that $\cos\dfrac{\pi}{12}\cos\dfrac{5\pi}{12} = \dfrac{1}{4}$.

Criteria	Marks
(i) Writes $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{1}{2}\sin^2 2\theta$	1
(i) Uses $\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$ and simplifies to $\frac{\cos 4\theta+3}{4}$	2
(ii) Substitutes $x = \cos\theta$ to obtain $8(\cos^4\theta + \sin^4\theta) = 2\cos 4\theta + 6 = 7$, so $\cos 4\theta = \frac{1}{2}$	2
(ii) Solves $\cos 4\theta = \frac{1}{2}$ and identifies the four roots	1
(iii) Uses the product of the four roots (constant ÷ leading) to deduce the required value	2

Solution.

(i) $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{1}{2}\sin^2 2\theta$. Since $\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$,

$$\sin^4\theta + \cos^4\theta = 1 - \tfrac{1}{2}\cdot\frac{1-\cos 4\theta}{2} = 1 - \frac{1-\cos 4\theta}{4} = \frac{4 - 1 + \cos 4\theta}{4} = \frac{\cos 4\theta + 3}{4}.$$

(ii) Let $x = \cos\theta$, so $1 - x^2 = \sin^2\theta$. Then

$$8x^4 + 8(1-x^2)^2 = 8\cos^4\theta + 8\sin^4\theta = 8\cdot\frac{\cos 4\theta + 3}{4} = 2\cos 4\theta + 6.$$

Setting this equal to 7 gives $2\cos 4\theta + 6 = 7$, so $\cos 4\theta = \frac{1}{2}$. Taking $4\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ (i.e. $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$) and noting $x = \cos\theta$ occurs in $\pm$ pairs, the roots are

$$x = \pm\cos\tfrac{\pi}{12}, \quad x = \pm\cos\tfrac{5\pi}{12}.$$

(iii) Writing $u = x^2$, the equation $8x^4 + 8(1-x^2)^2 = 7$ becomes $16u^2 - 16u + 1 = 0$, whose roots are $u = \cos^2\frac{\pi}{12}$ and $u = \cos^2\frac{5\pi}{12}$. By the product of roots, $\cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}$. Since both cosines are positive (acute angles),

$$\cos\tfrac{\pi}{12}\cos\tfrac{5\pi}{12} = \sqrt{\tfrac{1}{16}} = \frac{1}{4}.$$