Year 11 Mathematics Extension 1

Topic: Further Trigonometry (1 of 4)

Review Questions (Worked Solutions): Trigonometry in Three Dimensions

Term 3, 2026

Trigonometry in three dimensions

1. Solving problems in three dimensions

Q1 2023 Hurlstone Trial Q11c(i) · 1 mark

In the diagram, C is the centre of a circle of radius 4 cm and $\angle ACB = \theta$ radians. The length of chord AB is l cm. Show that the length l of AB is given by $l = 4\sqrt{2 - 2\cos\theta}$.

Criteria	Marks
Applies the cosine rule in $\triangle ACB$ and simplifies to $l = 4\sqrt{2 - 2\cos\theta}$	1

Solution.

In $\triangle ACB$, $CA = CB = 4$ and $\angle ACB = \theta$. By the cosine rule,

$$l^2 = 4^2 + 4^2 - 2(4)(4)\cos\theta = 32 - 32\cos\theta = 16(2 - 2\cos\theta).$$

Since $l > 0$, $l = \sqrt{16(2 - 2\cos\theta)} = 4\sqrt{2 - 2\cos\theta}$, as required.

Q2 2022 Shore Trial Q14c(i) · 1 mark

In the triangle ABC, $AB = AC = a$ cm. The angle BAC is increasing at the rate of 2 radians/min. Let $\angle BAC = \theta$ radians and $BC = x$ cm. Show that $x = a\sqrt{2 - 2\cos\theta}$.

Criteria	Marks
Applies the cosine rule in $\triangle ABC$ and simplifies to $x = a\sqrt{2 - 2\cos\theta}$	1

Solution.

In $\triangle ABC$, $AB = AC = a$ and $\angle BAC = \theta$. By the cosine rule,

$$x^2 = a^2 + a^2 - 2(a)(a)\cos\theta = 2a^2 - 2a^2\cos\theta = a^2(2 - 2\cos\theta).$$

Since $x > 0$, $x = a\sqrt{2 - 2\cos\theta}$, as required.

Q3 2024 Trinity Grammar Trial Q14b(i) · **1 mark**

A rod AB is leaning on a metal disc of radius 1 m, with centre O. One end of the rod, A, is fixed to the ground, while the disc is tangential to the ground at C, such that $\angle ACO = \angle ATO = 90°$. Let $\angle OAC = \alpha$, $\angle BAC = \theta$ and $AC = x$. Write an expression for $\tan\alpha$ in terms of x.

Criteria	**Marks**
Uses the right angle at C to write $\tan\alpha = \dfrac{OC}{AC} = \dfrac{1}{x}$	1

Solution.

The disc is tangential to AC at C, so the radius $OC \perp AC$ and $\triangle OAC$ is right-angled at C, with $OC = 1$ (the radius) and $AC = x$. Hence

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{OC}{AC} = \frac{1}{x}.$$

Q4 2022 Knox Trial Q14b · **3 marks**

XYZ is a right-angled triangle, with the right angle at Y and M located along YZ. The lengths of XY and YZ are $\sqrt{2}$ cm and 2 cm respectively, and $\angle MXY = \angle ZXM$. Find the exact length of MY.

Criteria	**Marks**
Writes $\tan 2\theta = \dfrac{YZ}{XY} = \sqrt{2}$ and applies the double-angle formula to get a quadratic in $\tan\theta$	1
Solves for $\tan\theta = \sqrt{3} - 1$ (rejecting the negative root since θ is acute)	1
Uses $MY = XY\tan\theta$ to obtain $MY = (\sqrt{3} - 1)$ cm	1

Solution.

Let $\angle MXY = \angle ZXM = \theta$, so $\angle YXZ = 2\theta$. In right-angled $\triangle XYZ$,

$$\tan 2\theta = \frac{YZ}{XY} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Using $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ with $t = \tan\theta$,

$$\frac{2t}{1 - t^2} = \sqrt{2} \implies \sqrt{2}\,t^2 + 2t - \sqrt{2} = 0 \implies t^2 + \sqrt{2}\,t - 1 = 0.$$

Then $t = \dfrac{-\sqrt{2} \pm \sqrt{2 + 4}}{2} = \dfrac{-\sqrt{2} \pm \sqrt{6}}{2}$. Since θ is acute, $\tan\theta > 0$, so

$$\tan\theta = \frac{\sqrt{6} - \sqrt{2}}{2}.$$

Finally, $MY = XY\tan\theta = \sqrt{2} \cdot \dfrac{\sqrt{6} - \sqrt{2}}{2} = \dfrac{\sqrt{12} - 2}{2} = \dfrac{2\sqrt{3} - 2}{2} = \sqrt{3} - 1.$

$$\therefore\ MY = (\sqrt{3} - 1) \text{ cm}.$$

Q5 2021 Sydney Grammar Trial Q14d · 3 marks

The diagram shows two concentric circles with radii R and r, $R > r > 0$. In the annulus between these circles n unit squares are arranged. One diagonal of each square lies on a radius of the larger circle, and the two circles intersect each square at the two vertices at the ends of this diagonal. Each square subtends an angle of 2θ at the common centre. **(i)** Show that $r = \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\tan\theta} - 1\right)$. **(ii)** Find a similar expression for R and hence show that $\dfrac{R}{r} = \dfrac{1+\tan\theta}{1-\tan\theta}$.

Criteria	Marks
(i) Uses the half-diagonal $\frac{1}{\sqrt{2}}$ and the half-angle θ to obtain $r = \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\tan\theta} - 1\right)$	1
(ii) Obtains $R = \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\tan\theta} + 1\right)$	1
(ii) Forms $\dfrac{R}{r}$ and simplifies to $\dfrac{1+\tan\theta}{1-\tan\theta}$	1

Solution.

(i) Each unit square has diagonal of length $\sqrt{2}$, so its half-diagonal is $\dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$. The diagonal lies along a radius; its midpoint is at distance $r + \dfrac{1}{\sqrt{2}}$ from the centre, and the two side vertices sit a perpendicular distance $\dfrac{1}{\sqrt{2}}$ from that radius. The half-angle subtended is θ, so

$$\tan\theta = \frac{1/\sqrt{2}}{r + 1/\sqrt{2}} \implies r + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\tan\theta} \implies r = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right).$$

(ii) The outer end of the diagonal is at distance $R = r + \sqrt{2}$ from the centre. Hence

$$R = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1\right) + \sqrt{2} = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} - 1 + 2\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\tan\theta} + 1\right).$$

Therefore

$$\frac{R}{r} = \frac{\dfrac{1}{\tan\theta} + 1}{\dfrac{1}{\tan\theta} - 1} = \frac{1+\tan\theta}{1-\tan\theta},$$

after multiplying numerator and denominator by $\tan\theta$.

Q6 2022 Girraween Prelim Q8b · **8 marks**

A canal has two parallel sides. A vertical pole CD of height h metres stands with its base at C on one edge of the canal. A and B are two points on the other edge such that $AB = 28$ metres and $\angle ACB = 60°$. From A and B the angles of elevation of the top D of the pole are α and β respectively, where $\tan\alpha = \dfrac{1}{3}$ and $\tan\beta = \dfrac{1}{8}$. **(i)** Use the cosine rule to show the pole height is 4 metres. **(ii)** Find the area of $\triangle ACB$. **(iii)** Hence find the width of the canal in simplest exact form.

Criteria	**Marks**
(i) Uses the elevations to write $AC = 3h$ and $BC = 8h$	1
(i) Substitutes into the cosine rule in $\triangle ACB$ with $\angle ACB = 60°$	1
(i) Solves $49h^2 = 784$ to obtain $h = 4$ metres	2
(ii) Computes the area $\frac{1}{2}(AC)(BC)\sin 60° = 96\sqrt{3}$ m^2	2
(iii) Uses area $= \frac{1}{2}(AB)(\text{width})$ to obtain width $= \dfrac{48\sqrt{3}}{7}$ m	2

Solution.

(i) The pole CD is vertical of height h, so from A: $\tan\alpha = \dfrac{h}{AC} = \dfrac{1}{3}$, giving $AC = 3h$. From B: $\tan\beta = \dfrac{h}{BC} = \dfrac{1}{8}$, giving $BC = 8h$. Applying the cosine rule in $\triangle ACB$ with $\angle ACB = 60°$ and $AB = 28$:

$$AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos 60°$$

$$28^2 = (3h)^2 + (8h)^2 - 2(3h)(8h)\left(\tfrac{1}{2}\right) = 9h^2 + 64h^2 - 24h^2 = 49h^2.$$

So $784 = 49h^2$, hence $h^2 = 16$ and $h = 4$ metres (as $h > 0$).

(ii) With $h = 4$, $AC = 12$ and $BC = 32$. The area of $\triangle ACB$ is

$$\frac{1}{2}(AC)(BC)\sin\angle ACB = \frac{1}{2}(12)(32)\sin 60° = 192 \cdot \frac{\sqrt{3}}{2} = 96\sqrt{3}\text{ m}^2.$$

(iii) The width of the canal is the perpendicular distance from C to line AB. Taking AB as the base,

$$\text{Area} = \frac{1}{2}(AB)(\text{width}) \implies 96\sqrt{3} = \frac{1}{2}(28)(\text{width}),$$

so width $= \dfrac{2(96\sqrt{3})}{28} = \dfrac{192\sqrt{3}}{28} = \dfrac{48\sqrt{3}}{7}$ metres.