Year 11 Mathematics Extension 1

Topic: Further Functions (3 of 5)

Review Questions (Worked Solutions): Inverse Functions

Term 3, 2026

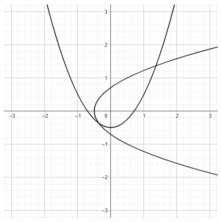
Finding and understanding inverses

1. Deciding whether a function has an inverse

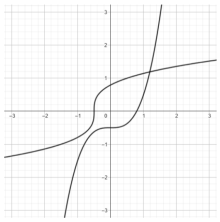
Q1 2025 Baulkham Hills Trial Q2 · **1 mark**

Which of the following best represents the graphs of a function and its inverse function?

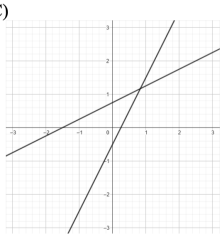
(A)



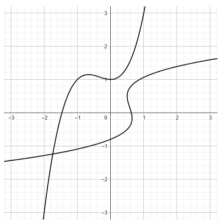
(B)



(C)



(D)



Criteria	Marks
Correct option (B)	1

Solution.

A function and its inverse are reflections of each other in the line $y = x$. The pair chosen must also be a genuine function together with a genuine inverse function, so each graph must pass the vertical line test and each must be one-to-one (pass the horizontal line test). Option (B) is the only pair whose two curves are mirror images in $y = x$ while both remain functions. The answer is (B).

Q2 2023 HSC Q14(a)(i) · **1 mark**

Let $f(x) = 2x + \ln x$, for $x > 0$. Explain why the inverse of $f(x)$ is a function.

Criteria	Marks
Shows f is strictly monotonic (hence one-to-one), so the inverse is a function	1

Solution.

Differentiating, $f'(x) = 2 + \dfrac{1}{x}$. For every $x > 0$ we have $\dfrac{1}{x} > 0$, so $f'(x) = 2 + \dfrac{1}{x} > 0$. Since $f'(x) > 0$ throughout the domain, f is strictly increasing, so it is one-to-one. A one-to-one function has an inverse that is itself a function.

Q3 2021 North Sydney Boys Trial Q7 · **1 mark**

The function $f(x) = 2x^3 - 9x^2 - 168x$ will have an inverse function with which restriction of its domain?

(A) $D = (7, \infty)$

(B) $D = (-4, 8)$

(C) $D = (-\infty, 0)$

(D) $D = \left[-\dfrac{1}{2}, \infty\right)$

Criteria	Marks
Correct option (A)	1

Solution.

$f'(x) = 6x^2 - 18x - 168 = 6(x^2 - 3x - 28) = 6(x - 7)(x + 4)$, so the turning points are at $x = -4$ and $x = 7$. The function is one-to-one only on an interval that contains no turning point in its interior, that is on $x \leq -4$ or on $x \geq 7$. Of the four choices, only $D = (7, \infty)$ lies wholly in a region where f is monotonic (here $f'(x) > 0$), so f is one-to-one there and the inverse is a function. The answer is (A).

Q4 2025 Baulkham Hills Prelim Q13(d) · **5 marks**

Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers.

(i) Write down the expression for $f'(x)$.

(ii) Given that the inverse of $f(x)$ is not a function, use part (i) to show that $b^2 - 3ac > 0$.

(iii) Consider $g(x) = \frac{1}{2}x^3 - 3x^2 + 6x - 8$. Use the result in part (ii) to show that the inverse of $g(x)$ is a function.

Criteria	**Marks**
(i) Correct $f'(x)$	1
(ii) Recognises the inverse fails to be a function when f has two turning points, i.e. $f'(x) = 0$ has two distinct real roots	1
(ii) Applies the discriminant condition to conclude $b^2 - 3ac > 0$	1
(iii) Evaluates $b^2 - 3ac = 0$ for g	1
(iii) Concludes g is one-to-one, so its inverse is a function	1

Solution.

(i) $f'(x) = 3ax^2 + 2bx + c$.

(ii) A cubic f fails to have an inverse that is a function exactly when it is not one-to-one, that is when it has two distinct turning points. Turning points occur where $f'(x) = 0$, so $3ax^2 + 2bx + c = 0$ must have two distinct real solutions. A quadratic has two distinct real roots when its discriminant is positive:

$$\Delta = (2b)^2 - 4(3a)(c) = 4b^2 - 12ac = 4\big(b^2 - 3ac\big) > 0.$$

Dividing by 4 gives $b^2 - 3ac > 0$, as required.

(iii) For $g(x) = \frac{1}{2}x^3 - 3x^2 + 6x - 8$ we have $a = \frac{1}{2}$, $b = -3$, $c = 6$. Then

$$b^2 - 3ac = (-3)^2 - 3\left(\tfrac{1}{2}\right)(6) = 9 - 9 = 0,$$

which is not greater than 0. By the contrapositive of part (ii), g does not have two turning points, so g is one-to-one and its inverse is a function. (Equivalently, $g'(x) = \frac{3}{2}x^2 - 6x + 6 = \frac{3}{2}(x-2)^2 \geq 0$, so g is increasing and therefore one-to-one.)

Q5 2025 North Sydney Girls Trial Q14(b) · **3 marks**

The function $f(x) = \dfrac{1+kx}{x^2+1}$ is continuous in the domain $-1 < x < 1$. Find the values of k such that the inverse function $f^{-1}(x)$ exists.

Criteria	**Marks**
Differentiates to obtain $f'(x)$ with numerator $-kx^2 - 2x + k$	1
Sets up the requirement that $f'(x)$ keeps a constant sign (uses the discriminant of the numerator quadratic)	1
Concludes $k \leq -1$ or $k \geq 1$	1

Solution.

For $f^{-1}(x)$ to exist, f must be one-to-one, so $f'(x)$ must keep a constant sign on the domain (either always positive or always negative). By the quotient rule,

$$f'(x) = \frac{k(x^2+1) - (1+kx)(2x)}{(x^2+1)^2} = \frac{-kx^2 - 2x + k}{(x^2+1)^2}.$$

The denominator is always positive, so the sign of $f'(x)$ is the sign of the numerator $q(x) = kx^2 + 2x - k$ (up to an overall sign). Requiring $q(x)$ to have no sign change means the quadratic $kx^2 + 2x - k$ has no real roots where the sign flips; the intended NESA solution sets the discriminant $\Delta = 2^2 - 4k^2 = 4(1 - k^2) \leq 0$, giving $k^2 \geq 1$, hence

$$k \leq -1 \quad \text{or} \quad k \geq 1.$$

Flag: the printed marking scheme states $k \leq -1$ or $k \geq 1$ using the discriminant $\Delta = 4(1 - k^2)$. Working the derivative through in full gives numerator $-kx^2 - 2x + k$, whose discriminant is $4 + 4k^2 > 0$ for every k, together with $q(-1) = 2 > 0$ and $q(1) = -2 < 0$, so a turning point actually sits inside $(-1, 1)$ for all k. The intended answer $k \leq -1$ or $k \geq 1$ is the marks-earning response; the strict analysis shows the boundary cases warrant care.

2. Domain and range of an inverse

Q6 2020 HSC Q2 · 1 mark

Given $f(x) = 1 + \sqrt{x}$, what are the domain and range of $f^{-1}(x)$?

(A) $x \geq 0, y \geq 0$

(B) $x \geq 0, y \geq 1$

(C) $x \geq 1, y \geq 0$

(D) $x \geq 1, y \geq 1$

Criteria	Marks
Correct option (C)	1

Solution.

For $f(x) = 1 + \sqrt{x}$ the domain is $x \geq 0$ and, since $\sqrt{x} \geq 0$, the range is $y \geq 1$. The inverse swaps domain and range, so f^{-1} has domain $x \geq 1$ and range $y \geq 0$. The answer is (C).

Q7 2021 Knox Trial Q3 · 1 mark

Given that $h(x) = \sqrt{2 - x}$, what are the domain and range of $h^{-1}(x)$?

(A) $x \leq 2, y \geq 0$

(B) $x \leq 2, y \leq 0$

(C) $x \geq 0, y \leq 2$

(D) $x \geq 0, y \leq -2$

Criteria	Marks
Correct option (C)	1

Solution.

For $h(x) = \sqrt{2 - x}$ the radicand must be non-negative, so $2 - x \geq 0$, giving domain $x \leq 2$; and $\sqrt{2 - x} \geq 0$, giving range $y \geq 0$. The inverse swaps these, so h^{-1} has domain $x \geq 0$ and range $y \leq 2$. The answer is (C).

Q8 2023 Abbotsleigh Trial Q1 · **1 mark**

Given $f(x) = \sqrt{x-3}$, what are the domain and range of $f^{-1}(x)$?

(A) $x \geq -3$, $y \geq 0$

(B) $x \geq -3$, $y \geq -3$

(C) $x \geq 0$, $y \geq 0$

(D) $x \geq 0$, $y \geq 3$

Criteria	Marks
Correct option (D)	1

Solution.

For $f(x) = \sqrt{x-3}$ the radicand gives $x - 3 \geq 0$, so the domain is $x \geq 3$, and $\sqrt{x-3} \geq 0$ gives range $y \geq 0$. The inverse swaps domain and range, so f^{-1} has domain $x \geq 0$ and range $y \geq 3$. The answer is (D).

Q9 2023 Sydney Tech Trial Q6 · **1 mark**

Given that $f(x) = \ln(x-2)$, what are the domain and range of $f^{-1}(x)$?

(A) $x > 2$, all real y

(B) $x < 2$, all real y

(C) all real x, $y > 2$

(D) all real x, $y < 2$

Criteria	Marks
Correct option (C)	1

Solution.

For $f(x) = \ln(x-2)$ the logarithm requires $x - 2 > 0$, so the domain is $x > 2$, and the range of a logarithm is all real y. The inverse swaps these, so f^{-1} has domain all real x and range $y > 2$. The answer is (C).

Q10 2022 St George Girls Trial Q2 · **1 mark**

Given that $f(x) = \ln(2 - x)$, what is the range of $f^{-1}(x)$?

(A) $[0, 2]$

(B) $[0, 2)$

(C) $[2, \infty)$

(D) $(-\infty, 2)$

Criteria	**Marks**
Correct option (D)	1

Solution.

The range of f^{-1} equals the domain of f. For $f(x) = \ln(2 - x)$ the logarithm requires $2 - x > 0$, that is $x < 2$, so the domain of f is $(-\infty, 2)$. Hence the range of f^{-1} is $(-\infty, 2)$. The answer is (D).

Q11 2021 Kings Trial Q11(b) · **2 marks**

The function $f(x) = \dfrac{1}{x^2 + 1}$ is defined for $x \geq 0$. State the domain and range of the inverse function $f^{-1}(x)$.

Criteria	**Marks**
Correct domain of f^{-1}: $0 < x \leq 1$	1
Correct range of f^{-1}: $y \geq 0$	1

Solution.

First find the range of f on $x \geq 0$. At $x = 0$, $f(0) = \dfrac{1}{1} = 1$, and as $x \to \infty$, $f(x) \to 0^+$ without ever reaching 0. Since f is decreasing on $x \geq 0$, the range of f is $0 < y \leq 1$. The inverse swaps domain and range, so

$$f^{-1}: \quad \text{domain } 0 < x \leq 1, \qquad \text{range } y \geq 0.$$

3. Finding an inverse (exponential and logarithmic)

Q12 2023 North Sydney Boys Trial Q4 · 1 mark

If $f(x) = e^{x+2}$, what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

Criteria	Marks
Correct option (C)	1

Solution.

Swap x and y: $x = e^{y+2}$. Take natural logs: $\ln x = y + 2$, so $y = \ln x - 2 = \log_e x - 2$. This is option **(C)**.

Q13 2022 North Sydney Girls Trial Q5 · 1 mark

What is the equation of the inverse of $f(x) = \dfrac{5 + e^{2x}}{3}$?

(A) $y = \dfrac{3}{5 + e^{2x}}$

(B) $y = e^{5-3x}$

(C) $y = \dfrac{1}{2}\ln(3x - 5)$

(D) $y = \dfrac{1}{2}\ln(5 - 3x)$

Criteria	Marks
Correct option (C)	1

Solution.

Set $y = \dfrac{5 + e^{2x}}{3}$ and swap: $x = \dfrac{5 + e^{2y}}{3}$. Then $3x = 5 + e^{2y}$, so $e^{2y} = 3x - 5$. Taking logs, $2y = \ln(3x - 5)$, giving $y = \dfrac{1}{2}\ln(3x - 5)$. This is option **(C)**.

Q14 2025 Killara Trial Q8 · **1 mark**

Given $g(x) = e^{2x} + 2$, $g^{-1}(x)$ is given by:

(A) $y = \dfrac{\ln x - 2}{2}$

(B) $y = \dfrac{\ln x}{2 \ln 2}$

(C) $y = \ln\left(x^2 - 2\right)$

(D) $y = \dfrac{\ln(x-2)}{2}$

Criteria	**Marks**
Correct option (D)	1

Solution.

Set $y = e^{2x} + 2$ and swap: $x = e^{2y} + 2$. Then $e^{2y} = x - 2$, so $2y = \ln(x-2)$, giving $y = \dfrac{\ln(x-2)}{2}$. This is option **(D)**.

Q15 2024 Girraween Trial Q8 · **1 mark**

What is the inverse function of $f(x) = e^{x^3}$?

(A) $f^{-1}(x) = 3e^x$

(B) $f^{-1}(x) = 3 \ln x$

(C) $f^{-1}(x) = \sqrt[3]{\ln x}$

(D) $f^{-1}(x) = \sqrt[3]{3 \ln x}$

Criteria	**Marks**
Correct option (C)	1

Solution.

Set $y = e^{x^3}$ and swap: $x = e^{y^3}$. Take natural logs: $\ln x = y^3$, so $y = \sqrt[3]{\ln x}$. This is option **(C)**.

Q16 2024 Hornsby Girls Trial Q12(a) · 6 marks

Consider the function $f(x) = 2^x - 3$.

(i) On the number plane provided, sketch the graph of $y = f(x)$, showing the x- and y-intercepts in exact values. **(2 marks)**

(ii) Find the equation of the inverse function $f^{-1}(x)$ as a natural logarithm function, and state its domain and range. **(3 marks)**

(iii) On the same number plane as part (i), sketch the graph of $y = f^{-1}(x)$. **(1 mark)**

Criteria	Marks
(i) Correct graph of $y = 2^x - 3$ with both intercepts in exact form	2
(ii) Correct inverse $f^{-1}(x) = \dfrac{\ln(x+3)}{\ln 2}$	1
(ii) Correct domain $x > -3$	1
(ii) Correct range (all real y)	1
(iii) Correct reflection of $y = f(x)$ in the line $y = x$	1

Solution.

(i) y-intercept: $f(0) = 2^0 - 3 = -2$, i.e. $(0, -2)$. x-intercept: $2^x - 3 = 0 \Rightarrow 2^x = 3 \Rightarrow x = \dfrac{\ln 3}{\ln 2}$, i.e. $\left(\dfrac{\ln 3}{\ln 2}, 0\right)$. The curve is the exponential $y = 2^x$ shifted down 3, with horizontal asymptote $y = -3$.

(ii) Swap x and y: $x = 2^y - 3$, so $2^y = x + 3$. Taking natural logs, $y \ln 2 = \ln(x + 3)$, giving

$$f^{-1}(x) = \frac{\ln(x+3)}{\ln 2}.$$

Domain of f^{-1} = range of f: since $2^x > 0$, $f(x) > -3$, so the domain is $x > -3$. Range of f^{-1} = domain of f = all real y.

(iii) The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$: it is a logarithmic curve with vertical asymptote $x = -3$, passing through $(-2, 0)$ and $\left(0, \dfrac{\ln 3}{\ln 2}\right)$.

Q17 2020 Manly Trial Q14(d) · 5 marks

Consider the function $f(x) = e^x - e^{-x}$.

(i) Show that $f(x)$ is increasing for all values of x. **(1 mark)**

(ii) Find the equation of the inverse function $y = f^{-1}(x)$. **(3 marks)**

(iii) Hence, or otherwise, solve $e^x - e^{-x} = 5$. Give your answer correct to two decimal places. **(1 mark)**

Criteria	Marks
(i) Shows $f'(x) = e^x + e^{-x} > 0$ for all x	1
(ii) Interchanges x and y and forms the quadratic in e^y	1
(ii) Uses the quadratic formula to make y the subject	1
(ii) Correctly chooses the positive root and states the inverse	1
(iii) Correct value $x \approx 1.65$ to two decimal places	1

Solution.

(i) $f'(x) = e^x - (-e^{-x}) = e^x + e^{-x}$. Since $e^x > 0$ and $e^{-x} > 0$ for all x, $f'(x) > 0$ for all x, so f is increasing for all values of x.

(ii) Swap x and y: $x = e^y - e^{-y}$. Let $p = e^y$ (so $p > 0$). Then $x = p - \dfrac{1}{p}$, which rearranges to

$$p^2 - xp - 1 = 0.$$

By the quadratic formula, $p = \dfrac{x \pm \sqrt{x^2+4}}{2}$. Since $p = e^y > 0$ and $\sqrt{x^2+4} > |x|$, only the + root is positive, so $e^y = \dfrac{x + \sqrt{x^2+4}}{2}$. Therefore

$$f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2+4}}{2}\right).$$

(iii) Solving $e^x - e^{-x} = 5$ is the same as evaluating $f^{-1}(5)$:

$$x = \ln\left(\frac{5 + \sqrt{5^2+4}}{2}\right) = \ln\left(\frac{5 + \sqrt{29}}{2}\right) = 1.6472\ldots \approx 1.65.$$

4. Finding the inverse of a rational function

Q18 2023 Hornsby Girls Trial Q2 · **1 mark**

Which of the following is the inverse function of $f(x) = \dfrac{1}{x+2}$, $x > -2$?

(A) $y = \dfrac{1}{x} - 2,\ x > -2$

(B) $y = \dfrac{1}{x} - 2,\ x > 0$

(C) $y = \dfrac{1}{x-2},\ x < -2$

(D) $y = \dfrac{1}{x-2},\ x < 0$

Criteria	Marks
Correct option (B)	1

Solution.

Set $y = \dfrac{1}{x+2}$ with $x > -2$, so $y > 0$. Swap x and y: $x = \dfrac{1}{y+2}$, so $y + 2 = \dfrac{1}{x}$, giving $y = \dfrac{1}{x} - 2$. The domain of the inverse is the range of f, namely $x > 0$. This is option **(B)**.

Q19 2025 HSC Q11(a) · **2 marks**

Find the inverse function, $f^{-1}(x)$, of the function $f(x) = 1 - \dfrac{1}{x-2}$.

Criteria	Marks
Interchanges x and y and isolates y	1
Correct inverse $f^{-1}(x) = \dfrac{2x-3}{x-1}$	1

Solution.

Set $y = 1 - \dfrac{1}{x-2}$ and swap x and y:

$$x = 1 - \frac{1}{y-2} \quad \Rightarrow \quad \frac{1}{y-2} = 1 - x \quad \Rightarrow \quad y - 2 = \frac{1}{1-x}.$$

So $y = 2 + \dfrac{1}{1-x} = 2 - \dfrac{1}{x-1}$. Combining over a common denominator,

$$f^{-1}(x) = \frac{2(x-1)-1}{x-1} = \frac{2x-3}{x-1}.$$

Q20 2022 Cheltenham Girls Trial Q11(e) · **4 marks**

A function is defined by $f(x) = 1 + \dfrac{3}{x-2}$, $x > 2$.

(i) Find the vertical and horizontal asymptotes for f. **(2 marks)**

(ii) Find the inverse function $f^{-1}(x)$. **(1 mark)**

(iii) State the domain of $f^{-1}(x)$. **(1 mark)**

Criteria	**Marks**
(i) Both asymptotes correct ($x = 2$ and $y = 1$)	2
(ii) Correct inverse $f^{-1}(x) = 2 + \dfrac{3}{x-1}$	1
(iii) Correct domain $x > 1$	1

Solution.

(i) As $x \to 2$, $\dfrac{3}{x-2} \to \pm\infty$, so the vertical asymptote is $x = 2$. As $x \to \pm\infty$, $\dfrac{3}{x-2} \to 0$, so $f(x) \to 1$, giving the horizontal asymptote $y = 1$.

(ii) Swap x and y: $x = 1 + \dfrac{3}{y-2}$, so $\dfrac{3}{y-2} = x - 1$, giving $y - 2 = \dfrac{3}{x-1}$. Therefore

$$f^{-1}(x) = 2 + \frac{3}{x-1}.$$

(iii) For $x > 2$ we have $x - 2 > 0$, so $\dfrac{3}{x-2} > 0$ and $f(x) = 1 + \dfrac{3}{x-2} > 1$. Hence the range of f is $y > 1$, so the domain of f^{-1} is $x > 1$.

Q21 2021 Ascham Trial Q13(a) · **3 marks**

Find $f^{-1}(x)$ if $f(x) = \dfrac{x+1}{x-2}$, clearly indicating the domain and range of $f^{-1}(x)$.

Criteria	Marks
Interchanges x and y and clears the denominator	1
Correct inverse $f^{-1}(x) = \dfrac{2x+1}{x-1}$	1
Correct domain $x \neq 1$ and range $y \neq 2$	1

Solution.

Set $y = \dfrac{x+1}{x-2}$ and swap x and y: $x = \dfrac{y+1}{y-2}$. Clear the denominator:

$$x(y-2) = y+1 \quad \Rightarrow \quad xy - 2x = y+1 \quad \Rightarrow \quad xy - y = 2x+1.$$

Factor: $y(x-1) = 2x+1$, so

$$f^{-1}(x) = \frac{2x+1}{x-1}.$$

The domain is $x \neq 1$ (the horizontal asymptote of f, $y = 1$, cannot be attained). The range is $y \neq 2$ (excluding the vertical asymptote value of f).

Q22 2024 Abbotsleigh Trial Q5 · **1 mark**

If $f(x) = \dfrac{4x}{x-2}$, what is the domain of $f^{-1}(x)$?

(A) $x \in [4, \infty)$

(B) $x \in (-\infty, 2) \cup (2, \infty)$

(C) $x \in (-\infty, 4) \cup (4, \infty)$

(D) $x \in (-\infty, \infty)$

Criteria	Marks
Correct option (C)	1

Solution.

The domain of f^{-1} is the range of f. As $x \to \pm\infty$, $f(x) = \dfrac{4x}{x-2} \to 4$, and $y = 4$ is never attained (solving $\dfrac{4x}{x-2} = 4$ gives $4x = 4x - 8$, i.e. $0 = -8$, impossible). So the range of f, and hence the domain of f^{-1}, is $(-\infty, 4) \cup (4, \infty)$. This is option **(C)**.

Q23 2023 Killara Trial Q9 · **1 mark**

Let $g(x) = \dfrac{ax+b}{cx+d}$, where a, b, c and d are positive and $ad \neq bc$. Which of the following can not be in the domain of the inverse function?

(A) $\dfrac{a}{a}$

(B) $\dfrac{a}{b}$

(C) $\dfrac{a}{c}$

(D) $\dfrac{a}{d}$

Criteria	Marks
Correct option (C)	1

Solution.

The domain of g^{-1} is the range of g. As $x \to \pm\infty$, $g(x) = \dfrac{ax+b}{cx+d} \to \dfrac{a}{c}$, which is the horizontal asymptote of g. Since $ad \neq bc$, this value is never attained, so it is excluded from the range of g and therefore cannot be in the domain of g^{-1}. The answer is $\dfrac{a}{c}$, option **(C)**.

5. Inverses of restricted-domain quadratics

Q24 2024 North Sydney Girls Trial Q5 · **1 mark**

Given that $f(x) = x^2 - 4x + 2$, where $x \geq 2$, which of the following is $f^{-1}(x)$? (A) $2 \pm \sqrt{x+2}$ (B) $2 + \sqrt{x+2}$ (C) $2 - \sqrt{x+2}$ (D) $2 - \sqrt{x-2}$.

Criteria	Marks
Correct option (B)	1

Solution.

Complete the square: $f(x) = (x-2)^2 - 2$. On $x \geq 2$ the function is increasing, so it is one-to-one. Let $y = (x-2)^2 - 2$. Then $(x-2)^2 = y + 2$, and since $x \geq 2$ we take the positive root: $x - 2 = \sqrt{y+2}$, so $x = 2 + \sqrt{y+2}$. Swapping variables, $f^{-1}(x) = 2 + \sqrt{x+2}$. The answer is **(B)**.

Q25 2022 Penrith Trial Q12(b) · **4 marks**

Consider the function $f(x) = x^2 - 3x$. **(i)** Explain why $f(x)$ does not have an inverse function. **(ii)** Find the largest possible domain, that includes $x = 0$, for the inverse of $f(x)$. **(iii)** Find an expression for $f^{-1}(x)$.

Criteria	Marks
(i) Correct explanation (fails the horizontal line test / not one-to-one)	1
(ii) Correct domain $x \leq \frac{3}{2}$	1
(iii) Correct expression for $f^{-1}(x)$	1
(iii) Correct method (complete the square and select the correct branch)	1

Solution.

(i) $y = x^2 - 3x$ is a parabola with a turning point at $x = \frac{3}{2}$. A horizontal line above the vertex cuts the curve twice, so f is not one-to-one and fails the horizontal line test. It therefore has no inverse function over its full domain.

(ii) The turning point is at $x = \frac{3}{2}$. To be one-to-one, the domain must lie entirely on one side of the vertex. The branch that includes $x = 0$ is $x \leq \frac{3}{2}$, so this is the largest possible domain: $x \leq \frac{3}{2}$.

(iii) Complete the square: $f(x) = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$. Let $y = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$, so $\left(x - \frac{3}{2}\right)^2 = y + \frac{9}{4}$. On $x \leq \frac{3}{2}$ we take the negative root: $x - \frac{3}{2} = -\sqrt{y + \frac{9}{4}}$, giving $x = \frac{3}{2} - \sqrt{y + \frac{9}{4}}$. Hence

$$f^{-1}(x) = \frac{3}{2} - \sqrt{x + \frac{9}{4}}.$$

Q26 2020 Riverview Trial Q11(a) · 7 marks

Consider the function $f(x) = x^2 - 4x + 6$. **(i)** Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **(ii)** Given that the domain is restricted to $x \geq 2$, find an expression for $f^{-1}(x)$. **(iii)** State the domain and range of $f^{-1}(x)$. **(iv)** The curve $y = f(x)$ with its restricted domain and $y = f^{-1}(x)$ intersect at P. Find the coordinates of P.

Criteria	Marks
(i) Correct explanation (parabola is not one-to-one over all reals)	1
(ii) Correct expression $f^{-1}(x) = 2 + \sqrt{x-2}$	1
(ii) Correct method (complete the square, correct branch)	1
(iii) Correct domain $x \geq 2$	1
(iii) Correct range $y \geq 2$	1
(iv) Correct coordinates of P	1
(iv) Correct method (solve the intersection)	1

Solution.

(i) $f(x) = (x-2)^2 + 2$ is a parabola with vertex $(2, 2)$. Over all real x it is not one-to-one (a horizontal line above the vertex meets it twice), so it fails the horizontal line test. The domain must be restricted to one branch of the parabola for an inverse function to exist.

(ii) With $x \geq 2$, let $y = (x-2)^2 + 2$, so $(x-2)^2 = y - 2$. Taking the positive root (as $x \geq 2$): $x - 2 = \sqrt{y-2}$, hence $x = 2 + \sqrt{y-2}$. Therefore

$$f^{-1}(x) = 2 + \sqrt{x-2}.$$

(iii) The domain of f^{-1} is the range of f on $x \geq 2$, which is $y \geq 2$; so the domain of f^{-1} is $x \geq 2$. The range of f^{-1} is the (restricted) domain of f, so $y \geq 2$.

(iv) Because f is increasing on $x \geq 2$, any intersection of $y = f(x)$ and $y = f^{-1}(x)$ lies on the line $y = x$. Solving $f(x) = x$: $x^2 - 4x + 6 = x \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$, so $x = 2$ or $x = 3$. The point $(2, 2)$ is the shared endpoint of both curves; the intersection point in the interior is $P = (3, 3)$.

Q27 2025 Manly Trial Q12(a) · 6 marks

A function is defined by $f(x) = x^2 + 8x + 10$, for $x \geq -4$. **(i)** Find the inverse function $y = f^{-1}(x)$. **(ii)** Sketch $y = f^{-1}(x)$, showing the endpoint and the x-intercept. **(iii)** Find the point(s) of intersection of $y = f(x)$ and $y = f^{-1}(x)$.

Criteria	Marks
(i) Correct expression $f^{-1}(x) = -4 + \sqrt{x+6}$	1
(i) Correct method (complete the square, correct branch)	1
(ii) Correct shape reflected in $y = x$ with endpoint $(-6, -4)$	1
(ii) Correct x-intercept $(10, 0)$	1
(iii) Correct coordinates of the intersection	1
(iii) Correct method (solve on $y = x$)	1

Solution.

(i) $f(x) = (x+4)^2 - 6$, with vertex $(-4, -6)$. On $x \geq -4$ the function is increasing (one-to-one). Let $y = (x+4)^2 - 6$, so $(x+4)^2 = y + 6$; taking the positive root ($x \geq -4$): $x + 4 = \sqrt{y+6}$, giving $x = -4 + \sqrt{y+6}$. Hence

$$f^{-1}(x) = -4 + \sqrt{x+6}.$$

(ii) f^{-1} is the reflection of f in $y = x$. Its endpoint is the reflection of the vertex $(-4, -6)$, namely $(-6, -4)$. The x-intercept occurs where $-4 + \sqrt{x+6} = 0 \Rightarrow \sqrt{x+6} = 4 \Rightarrow x + 6 = 16 \Rightarrow x = 10$, so the x-intercept is $(10, 0)$ (equivalently the y-intercept of f).

(iii) f is increasing on $x \geq -4$, so intersections with f^{-1} lie on $y = x$. Solving $f(x) = x$: $x^2 + 8x + 10 = x \Rightarrow x^2 + 7x + 10 = 0 \Rightarrow (x+2)(x+5) = 0$, so $x = -2$ or $x = -5$. Only $x = -2$ lies in the domain $x \geq -4$, giving the intersection point $(-2, -2)$.

Q28 2024 Penrith Prelim Q12(b) · 4 marks

The function $f(x) = (x+1)^2$ has domain $(-\infty, -1]$. **(i)** Write the inverse function of f in the form $y = f^{-1}(x)$ and state its domain. **(ii)** Hence sketch the inverse function, including the original function and the line $y = x$.

Criteria	Marks
(i) Correct expression $f^{-1}(x) = -1 - \sqrt{x}$	1
(i) Correct method (correct branch chosen for the restricted domain)	1
(i) Correct domain $x \geq 0$	1
(ii) Correct sketch (inverse, original, and $y = x$ shown)	1

Solution.

(i) On $(-\infty, -1]$ the function $f(x) = (x+1)^2$ is decreasing and one-to-one. Let $y = (x+1)^2$, so $(x+1)^2 = y$. Since $x \leq -1$, $x + 1 \leq 0$, so we take the negative root: $x + 1 = -\sqrt{y}$, giving $x = -1 - \sqrt{y}$. Therefore

$$f^{-1}(x) = -1 - \sqrt{x}.$$

The domain of f^{-1} is the range of f on $x \leq -1$, which is $y \geq 0$; so the domain is $x \geq 0$.

(ii) The graph of $y = f^{-1}(x) = -1 - \sqrt{x}$ starts at the endpoint $(0, -1)$ and curves down to the right (the reflection of $y = (x+1)^2$, $x \leq -1$, in $y = x$). Sketched together with the original curve and the line $y = x$, the two curves are mirror images in $y = x$.

Q29 2025 Caringbah Prelim Q7(d) · 2 marks

Consider the function $f(x) = (x-2)^2$ for $0 \leq x \leq 2$. Find the inverse function $f^{-1}(x)$ and state its domain and range.

Criteria	Marks
Correct expression $f^{-1}(x) = 2 - \sqrt{x}$	1
Correct domain $0 \leq x \leq 4$ and range $0 \leq y \leq 2$	1

Solution.

On $0 \leq x \leq 2$ the function $f(x) = (x-2)^2$ is decreasing from $f(0) = 4$ down to $f(2) = 0$, so it is one-to-one. Let $y = (x-2)^2$, so $(x-2)^2 = y$. Since $x \leq 2$, $x - 2 \leq 0$, so we take the negative root: $x - 2 = -\sqrt{y}$, giving $x = 2 - \sqrt{y}$. Therefore

$$f^{-1}(x) = 2 - \sqrt{x}.$$

The domain of f^{-1} is the range of f, which is $0 \leq x \leq 4$. The range of f^{-1} is the domain of f, which is $0 \leq y \leq 2$.

6. Evaluating an inverse value

Q30 2023 Blacktown Boys Trial Q4 · 1 mark

If $f(x) = 3 - 6x$, what is the value of $f^{-1}(2)$? (A) -9 (B) $-\frac{1}{6}$ (C) $\frac{1}{6}$ (D) 9.

Criteria	Marks
Correct option (C)	1

Solution.

$f^{-1}(2)$ is the value of x for which $f(x) = 2$. Solve $3 - 6x = 2 \Rightarrow -6x = -1 \Rightarrow x = \frac{1}{6}$. The answer is **(C)**.

Q31 2021 North Sydney Boys Trial Q8 · 1 mark

Given $f(x) = \dfrac{3}{x} - 4$, then $f^{-1}(4)$ equals (A) $-\frac{13}{4}$ (B) $\frac{13}{4}$ (C) $\frac{3}{8}$ (D) $-\frac{3}{8}$.

Criteria	Marks
Correct option (C)	1

Solution.

$f^{-1}(4)$ is the value of x for which $f(x) = 4$. Solve $\dfrac{3}{x} - 4 = 4 \Rightarrow \dfrac{3}{x} = 8 \Rightarrow x = \dfrac{3}{8}$. The answer is **(C)**.

Q32 2024 Girraween Prelim Q2 · 1 mark

Given $f(x) = \dfrac{1}{4}x^3 + 2$, what is the value of $f^{-1}(4)$? (A) 2 (B) 5 (C) 8 (D) 18.

Criteria	Marks
Correct option (A)	1

Solution.

$f^{-1}(4)$ is the value of x for which $f(x) = 4$. Solve $\dfrac{1}{4}x^3 + 2 = 4 \Rightarrow \dfrac{1}{4}x^3 = 2 \Rightarrow x^3 = 8 \Rightarrow x = 2$. The answer is **(A)**.

Q33 2025 Sydney Tech Trial Q11(f) · **2 marks**

Given the function $f(x) = \sqrt{2x+1}$ and that $f^{-1}(x)$ is the inverse function of $f(x)$, find $f^{-1}(5)$.

Criteria	Marks
Correct value $f^{-1}(5) = 12$	1
Correct method (set $f(x) = 5$ and solve)	1

Solution.

$f^{-1}(5)$ is the value of x for which $f(x) = 5$. Solve $\sqrt{2x+1} = 5$. Squaring: $2x+1 = 25 \Rightarrow 2x = 24 \Rightarrow x = 12$. (Check: $\sqrt{2(12)+1} = \sqrt{25} = 5$.) Hence $f^{-1}(5) = 12$.

Q34 2024 Manly Trial Q14(a) · **2 marks**

Given that $f(x) = \dfrac{2}{x^2+1}$, find the value of $f^{-1}\big(f(N)\big)$ for $N < 0$.

Criteria	Marks
Correct value $f^{-1}\big(f(N)\big) = -N$	1
Correct method (state $\pm$ and choose the correct branch, stating the reason for $N < 0$)	1

Solution.

Rearrange f to find the inverse. Let $y = \dfrac{2}{x^2+1}$, so $x^2 + 1 = \dfrac{2}{y}$, giving $x^2 = \dfrac{2}{y} - 1$ and $x = \pm\sqrt{\dfrac{2}{y} - 1}$.

Substituting $y = f(N) = \dfrac{2}{N^2+1}$:

$$f^{-1}\big(f(N)\big) = \pm\sqrt{\frac{2}{2/(N^2+1)} - 1} = \pm\sqrt{(N^2+1) - 1} = \pm\sqrt{N^2} = \pm|N|.$$

Since $N < 0$, $|N| = -N > 0$; the branch of the inverse that returns the original input must be chosen, so

$$f^{-1}\big(f(N)\big) = |N| = -N.$$

Q35 2025 Sydney Girls Trial Q10 · **1 mark**

The function $f(x+a)-b$ is odd, where a and b are positive. When the domain of $f(x)$ is restricted to $x \geq 2a$, $f(x)$ becomes one-to-one, with inverse $f^{-1}(x)$. Which of the following is a correct expression for $f^{-1}\big(2b-f(-1)\big)$? (A) $a+1$ (B) $a-1$ (C) $2a+1$ (D) $2a-1$.

Criteria	**Marks**
Correct option (C)	1

Solution.

Let $g(x)=f(x+a)-b$. Since g is odd, $g(-x)=-g(x)$, i.e. $f(a-x)-b=-\big(f(a+x)-b\big)$. This rearranges to

$$f(a-x)+f(a+x)=2b, \qquad \text{so} \qquad 2b-f(a+x)=f(a-x).$$

To use this on $2b-f(-1)$, write $-1=a+x$, so $x=-1-a$, and then $a-x=a-(-1-a)=2a+1$. Hence

$$2b-f(-1)=2b-f(a+x)=f(a-x)=f(2a+1).$$

Since $2a+1\geq 2a$, the value $2a+1$ lies in the restricted domain, so

$$f^{-1}\big(2b-f(-1)\big)=f^{-1}\big(f(2a+1)\big)=2a+1.$$

The answer is **(C)**.

7. Self-inverse functions

Q36 Hornsby Girls 2025 Trial Q9 · **1 mark**

Which function, $f(x)$, and its inverse function, $f^{-1}(x)$, are equivalent, i.e. $f(x) = f^{-1}(x)$?

(A) $f(x) = \sin x$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(B) $f(x) = x^2$ for $[0, \infty)$

(C) $f(x) = e^x$ for $(-\infty, \infty)$

(D) $f(x) = \dfrac{1}{x}$ for $(-\infty, 0) \cup (0, \infty)$

Criteria	Marks
Correct option (D)	1

Solution.

A function is self-inverse when its graph is symmetric about the line $y = x$, so that reflecting in $y = x$ returns the same curve. The rectangular hyperbola $y = \frac{1}{x}$ has this symmetry: solving $y = \frac{1}{x}$ for x gives $x = \frac{1}{y}$, so $f^{-1}(x) = \frac{1}{x} = f(x)$. The other options fail: $\sin x$ inverts to $\arcsin x$, x^2 (on $[0, \infty)$) inverts to $\sqrt{x}$, and e^x inverts to $\ln x$. The answer is **(D)**.

Q37 Barker 2025 Trial Q4 · **1 mark**

Which function, $f(x)$, and its inverse function, $f^{-1}(x)$, are equivalent?

(A) $f(x) = \sin|x|$ for $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(B) $f(x) = x^2$ for $[0, \infty)$

(C) $f(x) = |e^x|$ for $(-\infty, \infty)$

(D) $f(x) = \dfrac{1}{x}$ for $(-\infty, 0) \cup (0, \infty)$

Criteria	Marks
Correct option (D)	1

Solution.

The self-inverse function is the one whose graph is symmetric about $y = x$. Only $f(x) = \frac{1}{x}$ satisfies $f^{-1}(x) = \frac{1}{x} = f(x)$: solving $y = \frac{1}{x}$ for x returns $x = \frac{1}{y}$. Option (A) $\sin|x|$ is not even one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (it is even, so it fails the horizontal line test); (B) x^2 inverts to $\sqrt{x}$; (C) $|e^x| = e^x$ inverts to $\ln x$. The answer is **(D)**.

Q38 Sydney Boys 2024 Trial Q12(a) · 3 marks

The function f is defined by $f(x) = \dfrac{2x+a}{x-2}$, where a is a constant, $x > 2$.

(i) Given that f^{-1} exists, state the value that a cannot take, justifying your answer. **(2 marks)**

(ii) A function h is self-inverse if $h(x) = h^{-1}(x)$ for all x in the domain of h. State the range of values of a such that f is a self-inverse function. **(1 mark)**

Criteria	Marks
(i) Identifies $a \neq -4$ with justification (otherwise f is constant, so not one-to-one)	2
(ii) States all real a with $a \neq -4$	1

Solution.

(i) $f(x) = \dfrac{2x+a}{x-2}$ is a rational (Möbius-type) function of the form $\dfrac{Ax+B}{Cx+D}$ with $A = 2$, $B = a$, $C = 1$, $D = -2$. Such a map is one-to-one, and hence has an inverse, precisely when it does not collapse to a constant, that is when $AD - BC \neq 0$:

$$AD - BC = (2)(-2) - (a)(1) = -4 - a \neq 0 \quad \Longrightarrow \quad a \neq -4.$$

When $a = -4$, $f(x) = \dfrac{2x-4}{x-2} = \dfrac{2(x-2)}{x-2} = 2$ is constant, so it is not one-to-one and has no inverse. Thus a cannot take the value -4: $\boxed{a \neq -4}$.

(ii) Compute $f(f(x))$ and require it to equal x. Writing $u = f(x) = \dfrac{2x+a}{x-2}$,

$$f(f(x)) = \frac{2u+a}{u-2} = \frac{2\left(\frac{2x+a}{x-2}\right)+a}{\frac{2x+a}{x-2}-2} = \frac{2(2x+a)+a(x-2)}{(2x+a)-2(x-2)} = \frac{(4+a)x}{a+4} = x,$$

which holds for every x provided $a + 4 \neq 0$. So f is self-inverse for all real a with the single exclusion $a \neq -4$: $\boxed{a \in \mathbb{R},\ a \neq -4}$. (SymPy: $f(f(x)) - x$ simplifies to 0 for all tested $a \neq -4$.)

Graphs and intersections

8. Graphing a function and its inverse

Q39 Killara 2021 Trial Q12(e) · **2 marks**

Sketch the inverse $y = f^{-1}(x)$ given the function $f(x) = 2x^3 + 1$.

Criteria	**Marks**
Reflects $y = f(x)$ in the line $y = x$	1
Correct key points on the inverse (for example $(1, 0)$ and $(3, 1)$)	1

Solution.

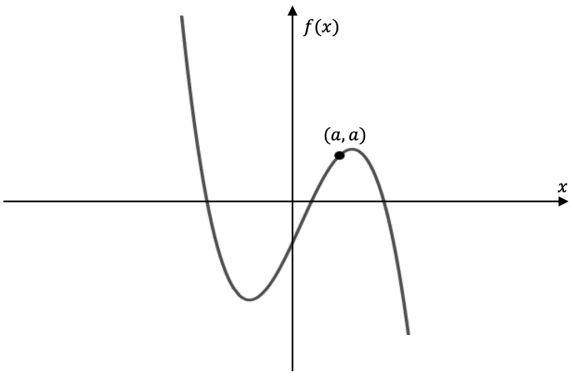
The inverse graph is the reflection of $y = f(x) = 2x^3 + 1$ in the line $y = x$. To find it algebraically, swap x and y in $y = 2x^3 + 1$ and solve:

$$x = 2y^3 + 1 \implies y^3 = \frac{x-1}{2} \implies f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}.$$

Since f is a cubic through $(0, 1)$ and $(1, 3)$, the inverse passes through the reflected points $(1, 0)$ and $(3, 1)$, and through $(1, 0)$ where f crosses the y-axis. Both graphs are increasing S-shaped cubics; the inverse is the mirror image of the original across $y = x$, meeting it on that line. (Check: $f\left(f^{-1}(x)\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$.)

Q40 Hunters Hill 2020 Trial Q11(c) · **2 marks**

The sketch shows the curve $y = f(x)$, with the point (a, a) shown. Copy or trace this and sketch the inverse relation, $y = f^{-1}(x)$, on the same axes.



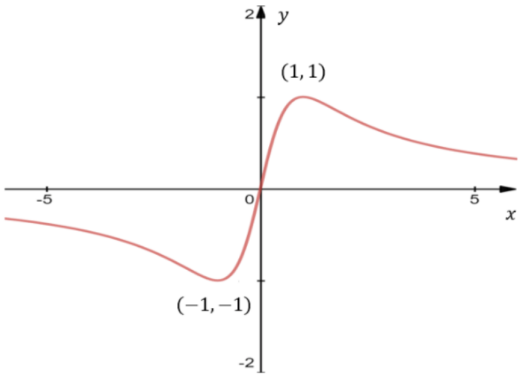
Criteria	Marks
Reflects the given curve in the line $y = x$	1
Fixed point (a, a) marked and shape correct	1

Solution.

The inverse relation is obtained by reflecting the given curve $y = f(x)$ in the line $y = x$. Every point (p, q) on $y = f(x)$ maps to (q, p) on $y = f^{-1}(x)$; in particular any x-intercept becomes a y-intercept and vice versa. The marked point (a, a) lies on $y = x$, so it is fixed by the reflection and stays exactly where it is: it is a point of intersection of f and its inverse. Draw the mirror image of the curve across $y = x$, keeping (a, a) on the diagonal.

Q41 Blacktown Boys 2022 Trial Q10 · **1 mark**

The graph of $y = f(x)$ is shown. What is the domain and range of $y = f^{-1}(x)$?



(A) Domain: all real x; Range: $[-1, 1]$

(B) Domain: $[-1, 1]$; Range: all real y

(C) Domain: $[-1, 1]$; Range: $[-1, 1]$

(D) Domain: all real x; Range: all real y

Criteria	**Marks**
Correct option (B)	1

Solution.

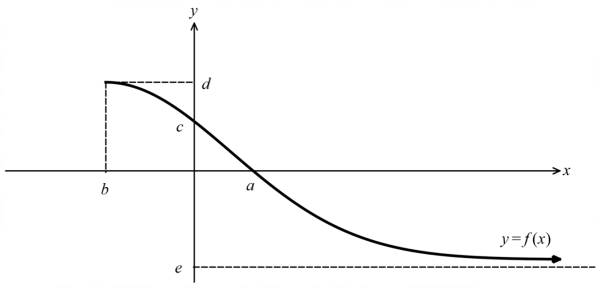
Reading the graph, $y = f(x)$ has domain all real x and range $[-1, 1]$ (it flattens towards the horizontal asymptotes $y = \pm 1$, an arctan-type shape). Taking an inverse swaps domain and range:

$$\text{domain of } f^{-1} = \text{range of } f = [-1, 1], \qquad \text{range of } f^{-1} = \text{domain of } f = \text{all real } y.$$

The answer is **(B)**.

Q42 North Sydney Girls 2023 Trial Q10 · **1 mark**

The graph of $y = f(x)$ is given. The function $g(x) = f^{-1}(x)$. Which of the following gives the range of $g(x)$?



(A) $[b, \infty)$

(B) $[b, a]$

(C) $[c, d]$

(D) $(e, d]$

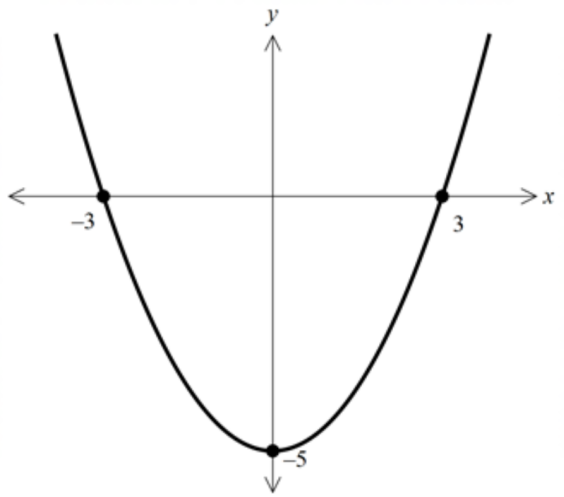
Criteria	Marks
Correct option (B)	1

Solution.

The range of $g = f^{-1}$ equals the domain of f. Reading the labelled graph, f is defined over the closed x-interval running from $x = b$ up to $x = a$ (the horizontal extent of the curve), so its domain is $[b, a]$. Hence the range of f^{-1} is $[b, a]$. The answer is **(B)**.

Q43 Cherrybrook Tech 2021 Trial Q11(c)(i) · **2 marks**

The diagram shows the graph of $y = f(x)$. Sketch $y = f^{-1}(x)$ after making an appropriate restriction on the domain of $f(x)$.



Criteria	Marks
Restricts f to a one-to-one branch	1
Reflects that branch correctly in $y = x$	1

Solution.

As drawn, $y = f(x)$ is not one-to-one (it fails the horizontal line test), so it has no inverse over its full domain. First restrict f to a single monotonic branch, for example the part to one side of the turning point, so that the restricted function passes the horizontal line test. Then reflect that branch in the line $y = x$: intercepts swap (x-intercepts become y-intercepts), and any endpoint (p, q) becomes (q, p). The resulting curve is $y = f^{-1}(x)$ for the chosen restricted domain.

9. Intersections of a function and its inverse

Q44 Glenwood 2022 Trial Q5 · **1 mark**

Consider $f(x) = x^3 + x + 8$. Which is the point of intersection of $f(x)$ and its inverse $f^{-1}(x)$?

(A) $(-1, 6)$

(B) $(-2, -2)$

(C) $(0, 0)$

(D) $(2, 2)$

Criteria	**Marks**
Correct option (B)	1

Solution.

$f(x) = x^3 + x + 8$ is strictly increasing (its derivative $3x^2 + 1 > 0$), so any intersection of f with f^{-1} lies on the line $y = x$. Set $f(x) = x$:

$$x^3 + x + 8 = x \implies x^3 + 8 = 0 \implies x = -2.$$

Then $f(-2) = (-2)^3 + (-2) + 8 = -8 - 2 + 8 = -2$, giving the point $(-2, -2)$. The answer is **(B)**.

Q45 Cherrybrook Tech 2022 Trial Q3 · **1 mark**

$f(x) = \dfrac{x^2}{x^3 - 6}$ has an inverse $f^{-1}(x)$ for $x \geq 0$. Which represents a point of intersection of $f(x)$ with $f^{-1}(x)$?

(A) $(2, 2)$

(B) $(1, -0.2)$

(C) $(3, 3)$

(D) $(-0.2, 1)$

Criteria	Marks
Correct option (A)	1

Solution.

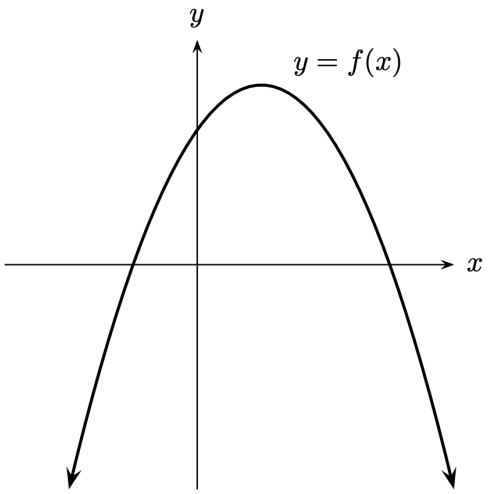
A function meets its inverse on the line $y = x$, so set $f(x) = x$ with $x \geq 0$:

$$\frac{x^2}{x^3 - 6} = x \implies x^2 = x\left(x^3 - 6\right) \implies x\left(x^3 - x - 6\right) = 0.$$

Wait: expand carefully. $x^2 = x^4 - 6x$, so $x^4 - 6x - x^2 = 0$, giving $x\left(x^3 - x - 6\right) = 0$, hence $x = 0$ or $x^3 - x - 6 = 0$. Testing $x = 2$: $8 - 2 - 6 = 0$, so $x = 2$ is a root, and $f(2) = \frac{4}{8-6} = \frac{4}{2} = 2$. The point $(2, 2)$ lies on $y = x$ and is the only listed candidate on that line. The answer is **(A)**. (Note $(1, -0.2)$ is a point on f since $f(1) = \frac{1}{-5} = -0.2$, and $(-0.2, 1)$ is its reflection on f^{-1}; neither is a common point.)

Q46 Normanhurst Boys 2025 Trial Q1 · 1 mark

The graph of $y = f(x)$ is shown (axes to the same scale). $f(x)$ is restricted to the largest domain on which $y = f(x)$ has an inverse, giving $g(x)$. How many points of intersection are there between $y = g(x)$ and $y = g^{-1}(x)$?



(A) 1

(B) 2

(C) 3

(D) 4

Criteria	Marks
Correct option (A)	1

Solution.

Restricting the parabola to its largest one-to-one domain keeps a single monotonic branch, taken from the vertex outward. That branch g is increasing (or decreasing) and continuous. Reflecting g in $y = x$ gives g^{-1}. Because g is monotonic, the two curves meet only where they cross the line $y = x$, and the restricted branch (starting at the vertex, which lies below the line $y = x$ here) crosses $y = x$ exactly once. So g and g^{-1} have a single point in common. The answer is **(A)**.

Q47 Fort St 2020 Trial Q11(e) · 4 marks

Consider $f(x) = 2 - \log_e x$.

(i) Find the equation of the inverse function $f^{-1}(x)$. **(1 mark)**

(ii) Explain why the coordinate of the point of intersection P of $y = f(x)$ and $y = f^{-1}(x)$ satisfies $e^{2-x} - x = 0$. **(3 marks)**

Criteria	Marks
(i) Correct inverse $f^{-1}(x) = e^{2-x}$	1
(ii) States the intersection lies on $y = x$	1
(ii) Sets $f(x) = x$, that is $2 - \log_e x = x$	1
(ii) Rearranges to $e^{2-x} - x = 0$	1

Solution.

(i) Let $y = 2 - \log_e x$. Swap x and y and solve for y:

$$x = 2 - \log_e y \implies \log_e y = 2 - x \implies y = e^{2-x}.$$

So $\boxed{f^{-1}(x) = e^{2-x}}$. (Check: $f\left(f^{-1}(x)\right) = 2 - \log_e\left(e^{2-x}\right) = 2 - (2 - x) = x$.)

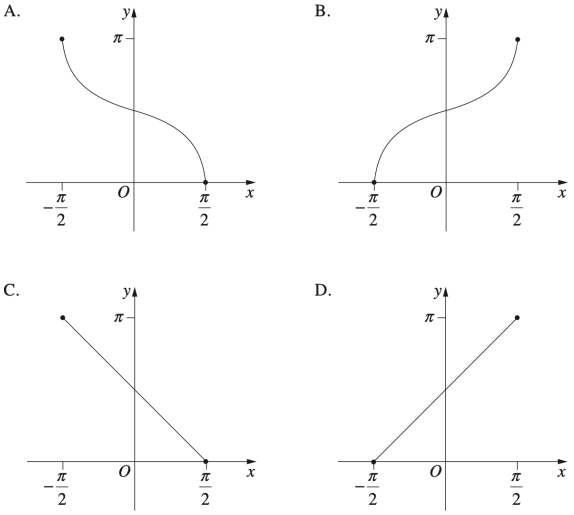
(ii) The function $f(x) = 2 - \log_e x$ is strictly decreasing on its domain $x > 0$. When a function is monotonic, any point of intersection P of $y = f(x)$ and $y = f^{-1}(x)$ must lie on the line $y = x$ (the two graphs are reflections of each other in $y = x$, so they can only meet on that mirror line). At P we therefore have $f(x) = x$:

$$2 - \log_e x = x \implies \log_e x = 2 - x \implies x = e^{2-x} \implies e^{2-x} - x = 0,$$

as required. Equivalently, setting $f(x) = f^{-1}(x)$, i.e. $2 - \log_e x = e^{2-x}$, together with $y = x$ leads to the same equation. (Numerically the root is $x \approx 1.557$, and indeed $f(1.557) \approx 1.557$, confirming P lies on $y = x$.)

Q48 2019 HSC Q10 · **1 mark**

The function $f(x) = -1 + \sqrt{1+x}$ has inverse $f^{-1}(x)$, whose graph forms part of the curve $y = x^4 - 2x^2$. The diagram shows $y = x^4 - 2x^2$. How many points do $y = f(x)$ and $y = f^{-1}(x)$ have in common?



(A) 1

(B) 2

(C) 3

(D) 4

Criteria	**Marks**
Correct option (A)	1

Solution.

The graphs of a function and its inverse are reflections in the line $y = x$. Here f has domain $x \geq -1$ with $f(-1) = -1$, so the endpoint $(-1, -1)$ lies on the line $y = x$ and is common to both graphs. Over the relevant branch the function is monotonic and stays on one side of the line $y = x$, so no further crossings occur: $(-1, -1)$ is the only point the two graphs share. The answer is **(A)**.