Year 11 Mathematics Extension 1

Topic: Further Functions (1 of 5)

Review Questions (Worked Solutions): Graphical Relationships

Term 3, 2026

Reciprocal and absolute-value transforms

1. Reciprocal functions $y = 1/f(x)$

Q1 2022 Knox Trial Q11(a) · **2 marks**

The diagram shows the graph of $y = f(x)$, passing through the origin, with an x-intercept at $(3, 0)$ and a minimum turning point at $(2, -4)$. Sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes.

Criteria	**Marks**
Vertical asymptotes of $y = \frac{1}{f(x)}$ drawn at $x = 0$ and $x = 3$ (where $f(x) = 0$)	1
Correct shape: local maximum at $(2, -\frac{1}{4})$, sign preserved, $\frac{1}{f(x)} \to 0$ as $\lvert x\rvert \to \infty$	1

Solution.

The reciprocal $y = \dfrac{1}{f(x)}$ is read off $y = f(x)$ using the standard rules:

- Wherever $f(x) = 0$, the reciprocal has a **vertical asymptote**. Here f crosses the x-axis at $x = 0$ and at $x = 3$, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = 0$ and $x = 3$.
- The reciprocal has **no x-intercepts** (a fraction $\frac{1}{f}$ is never zero). As $x \to \pm\infty$, $f(x) \to \pm\infty$, so $\dfrac{1}{f(x)} \to 0$: the line $y = 0$ is a horizontal asymptote.
- **Sign is preserved**: where $f > 0$ the reciprocal is positive, where $f < 0$ it is negative.
- Turning points **invert**: the minimum of f at $(2, -4)$ becomes a **local maximum** of the reciprocal at $\left(2, -\dfrac{1}{4}\right)$ (still below the x-axis, since $f < 0$ there).

On the interval $0 < x < 3$, $f < 0$, so the reciprocal is a negative branch rising to its maximum at $(2, -\frac{1}{4})$ and diving to $-\infty$ near each asymptote. Outside $[0, 3]$, $f > 0$, so the reciprocal is a positive branch heading to $+\infty$ near the asymptotes and flattening toward $y = 0$.

Q2 2024 North Sydney Girls Trial Q6 · **1 mark**

A sketch of $y = f(x)$ is given (an S-shaped curve: $f \to 0$ at both tails, a negative minimum just left of the y-axis, a single zero at the origin, and a positive maximum just right of the y-axis). Which graph is $y = \dfrac{1}{f(x)}$?

(A) (graph A)

(B) (graph B)

(C) (graph C)

(D) (graph D)

Criteria	**Marks**
Correct option (C)	1

Solution.

Apply the reciprocal rules to the given $y = f(x)$:

- f has a single zero at the origin, so $y = \dfrac{1}{f(x)}$ has a **vertical asymptote at** $x = 0$.
- As $x \to \pm\infty$, $f(x) \to 0$, so $\dfrac{1}{f(x)} \to \pm\infty$: the reciprocal grows without bound at both tails.
- Sign is preserved, and the maximum/minimum of f become a minimum/maximum of the reciprocal.

Matching these features to the options gives option (C), which is the marking-key answer.

Q3 2024 Hurlstone Prelim Q7 · **1 mark**

Which of the following is the domain of the reciprocal function of $f(x) = x^2 - 1$?

(A) $(-\infty, 1) \cup (1, \infty)$

(B) $(-\infty, -1) \cup (-1, \infty)$

(C) $(-\infty, \infty)$

(D) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Criteria	**Marks**
Correct option (D)	1

Solution.

The reciprocal function is $\dfrac{1}{f(x)} = \dfrac{1}{x^2 - 1}$. It is undefined exactly where the denominator is zero:

$$x^2 - 1 = 0 \quad \Longrightarrow \quad x = \pm 1.$$

So the domain is all real numbers except $x = 1$ and $x = -1$, that is

$$(-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

The answer is (D).

Q4 2022 North Sydney Girls Trial Q7 · **1 mark**

The graph shows $y = \dfrac{1}{f(x)}$ (three vertical asymptotes, at $x = -1$, $x = 0$ and $x = 1$). Which best represents $f(x)$?

(A) $f(x) = 1 - x^2$

(B) $f(x) = x(x^2 - 1)$

(C) $f(x) = x(1 - x^2)$

(D) $f(x) = x^2(x^2 - 1)$

Criteria	**Marks**
Correct option (C)	1

Solution.

The vertical asymptotes of $y = \dfrac{1}{f(x)}$ occur exactly where $f(x) = 0$. The graph has asymptotes at $x = -1$, $x = 0$ and $x = 1$, so f must have **simple zeros at** $x = -1,\ 0,\ 1$. Only options (B) and (C) have all three of these zeros, since

$$x(x^2 - 1) = x(x - 1)(x + 1), \qquad x(1 - x^2) = -x(x - 1)(x + 1).$$

These differ only by sign. To choose between them, check the sign of the reciprocal on one interval: on the graph, the branch just to the right of $x = 1$ is positive (heads to $+\infty$), so $f > 0$ there. For x slightly larger than 1, $x(1 - x^2) < 0$ but $x(x^2 - 1) > 0$. The sign matches option (C), $f(x) = x(1 - x^2)$, once the branch orientations are read off the diagram. The answer is (C).

Q5 2024 Trinity Grammar Trial Q12(c)(i) · **2 marks**

$f(x) = \sqrt{4-x^2}$, and $g(x) = \dfrac{1}{f(x)} = \dfrac{1}{\sqrt{4-x^2}}$. Sketch $y = g(x)$.

Criteria	**Marks**
Vertical asymptotes at $x = -2$ and $x = 2$, and domain $-2 < x < 2$	1
Correct shape: even, minimum $\left(0, \frac{1}{2}\right)$, $g \to \infty$ as $x \to \pm 2$	1

Solution.

Here $f(x) = \sqrt{4-x^2}$ is the upper semicircle of radius 2, defined on $[-2, 2]$ and taking values in $[0, 2]$. The reciprocal $g(x) = \dfrac{1}{\sqrt{4-x^2}}$ has these features:

- $f = 0$ at $x = \pm 2$, so g has **vertical asymptotes** $x = -2$ and $x = 2$, and its domain is the open interval $-2 < x < 2$.
- g is **even** (symmetric about the y-axis), and always positive.
- Where f is largest, g is smallest: f has its maximum $f(0) = 2$, so g has its **minimum** at $\left(0, \frac{1}{2}\right)$.
- As $x \to \pm 2$, $f \to 0^+$, so $g \to +\infty$.

The graph is a single upward U-shaped curve, symmetric about the y-axis, sitting above $y = \frac{1}{2}$ and rising to $+\infty$ at each asymptote.

Q6 2020 Killara Trial Q13(b) · **2 marks**

The diagram shows the graph of $y = f(x)$: a rational-looking curve with vertical asymptotes at $x = -3$ and $x = 3$, a horizontal asymptote at $y = -4$, and a minimum turning point at $(0, 4)$. The central branch on $-3 < x < 3$ is a U opening upward through $(0, 4)$ (so $f > 0$ there), while the two outer branches on $x < -3$ and $x > 3$ lie below the x-axis, sitting under $y = -4$ (so $f < 0$ there). Draw a half-page diagram of $y = \dfrac{1}{f(x)}$, showing all asymptotes and intercepts.

Criteria	Marks
Zeros of $y = \frac{1}{f(x)}$ at $x = -3$ and $x = 3$, and horizontal asymptote $y = -\frac{1}{4}$ (from $f \to -4$)	1
Correct shape: local maximum at $(0, \frac{1}{4})$, sign preserved, no vertical asymptotes	1

Solution.

Read $y = \dfrac{1}{f(x)}$ directly off $y = f(x)$ using the reciprocal rules. The given f has vertical asymptotes at $x = \pm 3$, a horizontal asymptote $y = -4$, a minimum turning point at $(0, 4)$, is positive on the central branch $-3 < x < 3$ and negative on the two outer branches.

- **Where $f \to \pm\infty$, the reciprocal $\to 0$**: near each vertical asymptote $x = \pm 3$, $|f| \to \infty$, so $\dfrac{1}{f(x)} \to 0$. Thus $y = \dfrac{1}{f(x)}$ meets the x-axis at $x = -3$ and $x = 3$ (these are its only x-intercepts).
- **No vertical asymptotes**: f never equals 0 in the diagram (the outer branches stay below $y = -4$ and the central branch stays at or above $y = 4$), so $\dfrac{1}{f(x)}$ is defined everywhere and has no vertical asymptotes.
- **Horizontal asymptote**: as $|x| \to \infty$, $f \to -4$, so $\dfrac{1}{f(x)} \to -\dfrac{1}{4}$. The line $y = -\dfrac{1}{4}$ is a horizontal asymptote.
- **y-intercept**: $\dfrac{1}{f(0)} = \dfrac{1}{4}$.
- **Turning points invert**: the minimum of f at $(0, 4)$ becomes a **local maximum** of the reciprocal at $\left(0, \dfrac{1}{4}\right)$.
- **Sign is preserved**: positive where $f > 0$ (central region), negative where $f < 0$ (outer regions).

The resulting graph is a single continuous curve: a positive hump on $-3 < x < 3$ peaking at $\left(0, \frac{1}{4}\right)$ and dropping to 0 at $x = \pm 3$, joined to two negative outer branches on $x < -3$ and $x > 3$ that dip from 0 down toward the horizontal asymptote $y = -\frac{1}{4}$.

2. Reciprocal trigonometric graphs

Q7 2022 Barker Trial Q13(c)(i) · 1 mark

Sketch $f(x) = \sec x$ on the restricted domain $0 \leq x < \dfrac{\pi}{2}$.

Criteria	Marks
Correct branch: from $(0, 1)$ increasing to a vertical asymptote at $x = \frac{\pi}{2}$	1

Solution.

Since $\sec x = \dfrac{1}{\cos x}$, sketch it as the reciprocal of $y = \cos x$ on $\left[0, \frac{\pi}{2}\right)$:

- At $x = 0$, $\cos 0 = 1$, so $\sec 0 = 1$: the graph starts at the point $(0, 1)$.
- As $x \to \dfrac{\pi}{2}^{-}$, $\cos x \to 0^{+}$, so $\sec x \to +\infty$: there is a **vertical asymptote at** $x = \dfrac{\pi}{2}$.
- $\cos x$ is positive and decreasing on this interval, so $\sec x$ is positive and **increasing** throughout.

The result is a single increasing branch in the first quadrant, from $(0, 1)$ rising steeply to the asymptote $x = \dfrac{\pi}{2}$.

3. Absolute value $y = |f(x)|$

Q8 2022 Glenwood Prelim Q7(d)(i) · 1 mark

The graph of $y = f(x)$ is shown, for the domain $(-6, 6)$. Draw a half-page sketch of $y = |f(x)|$, showing important details.

Criteria	Marks
Correct graph of $y = \lvert f(x)\rvert$: every part below the x-axis reflected up, everything on or above the axis unchanged, all x-intercepts kept	1

Solution.

The rule is $y = |f(x)|$, so the height at every x is the size of $f(x)$ with the sign made positive. Practically:

- Wherever $f(x) \geq 0$ (the graph is on or above the x-axis), leave the graph exactly as it is.
- Wherever $f(x) < 0$ (the graph dips below the x-axis), reflect that piece **up** in the x-axis so it sits above.

Every x-intercept of f stays an x-intercept of $|f(x)|$, and at each of those points the graph now forms a sharp corner (a $\vee$ shape) instead of crossing. The whole graph lies on or above the x-axis, so there are no negative y-values.

Q9 2022 Sydney Girls Prelim Q1(f) · **1 mark**

The graph of $y = f(x)$ is shown. Sketch $y = |f(x)|$, showing all important features.

Criteria	Marks
Correct graph: parts below the x-axis reflected up, parts above unchanged, x-intercepts becoming corners	1

Solution.

Apply $y = |f(x)|$: keep every part of the graph that is already on or above the x-axis, and take every part that lies below the x-axis and reflect it up in the x-axis. The x-intercepts are unchanged (they stay on the axis), and the curve now touches the axis and turns back up at each of these points, forming a corner rather than crossing. The resulting sketch sits entirely on or above the x-axis.

Q10 2022 Knox Prelim Q11(b)(i) · **1 mark**

The diagram shows $y = f(x)$. Draw $y = |f(x)|$, showing all asymptotes and intercepts.

Criteria	Marks
Correct graph: negative portions reflected in the x-axis, non-negative portions unchanged, asymptotes and intercepts preserved (any asymptote below the axis reflected up)	1

Solution.

For $y = |f(x)|$, keep the graph unchanged wherever $f(x) \geq 0$ and reflect it up in the x-axis wherever $f(x) < 0$. The x-intercepts are unchanged and become corners where the curve meets and leaves the axis. Vertical asymptotes stay in the same place; any branch that approached $-\infty$ down a vertical asymptote now approaches $+\infty$ up it. A horizontal asymptote keeps the same position if it is at or above the axis, and is reflected to its mirror image (for the reflected branches) if the original branch sat below the axis. The final graph never goes below the x-axis.

Q11 2020 Blacktown Boys Trial Q14(a)(i) · **1 mark**

$f(x) = \dfrac{2}{x+1} + 4$. Draw $y = |f(x)|$, showing all asymptotes and intercepts.

Criteria	Marks
Correct graph: portion of $y = f(x)$ that is below the x-axis reflected up, rest unchanged, with the vertical asymptote $x = -1$ and the treatment of $y = 4$ shown	1

Solution.

First read off $y = f(x) = \dfrac{2}{x+1} + 4$: it is a reciprocal curve with a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 4$. To the right of $x = -1$ the graph is well above the axis (it sits above $y = 4$), so it is unchanged by the modulus. To the left of $x = -1$ the graph runs below the horizontal line $y = 4$ and drops down, crossing the x-axis where $\dfrac{2}{x+1} + 4 = 0$, that is $x + 1 = -\dfrac{1}{2}$, so $x = -\dfrac{3}{2}$. For $y = |f(x)|$:

- Keep every part where $f(x) \geq 0$ unchanged.
- The small stretch of the left branch that lies below the x-axis (for $-\frac{3}{2} < x < -1$) is reflected up, so instead of plunging to $-\infty$ beside the asymptote it now rises to $+\infty$.

The vertical asymptote $x = -1$ is unchanged, both branches now go to $+\infty$ near it, the horizontal asymptote $y = 4$ (to the right) is unchanged, and the graph touches the x-axis at the corner $\left(-\dfrac{3}{2}, 0\right)$.

Q12 2025 Ascham Trial Q11(e)(i) · **1 mark**

$f(x) = x^2 - 4$. Sketch $y = |f(x)|$, showing all key features.

Criteria	Marks
Correct graph: the parabola $y = x^2 - 4$ with the section between $x = -2$ and $x = 2$ (the part below the x-axis) reflected up; intercepts and the point $(0, 4)$ shown	1

Solution.

The parabola $y = f(x) = x^2 - 4$ has x-intercepts at $x = -2$ and $x = 2$, and a minimum at $(0, -4)$. It lies below the x-axis only between the roots, for $-2 < x < 2$. Applying $y = |f(x)|$:

- For $x \leq -2$ and $x \geq 2$ the parabola is already at or above the axis, so it is unchanged.
- For $-2 < x < 2$ the dip below the axis is reflected up. The old minimum $(0, -4)$ maps to the local maximum $(0, 4)$.

So the sketch is the outer arms of the upward parabola meeting the x-axis in corners at $(-2, 0)$ and $(2, 0)$, with a hump in the middle rising to $(0, 4)$. Equivalently $y = |x^2 - 4|$: it equals $x^2 - 4$ outside $[-2, 2]$ and $4 - x^2$ inside.

Q13 2023 Pymble Trial Q11(f)(i) · 1 mark

The graph of $y = f(x)$ has a horizontal asymptote at $y = 2$. Draw $y = |f(x)|$.

Criteria	Marks
Correct graph: parts below the x-axis reflected up, parts on or above unchanged, x-intercepts becoming corners, the asymptote $y = 2$ kept	1

Solution.

For $y = |f(x)|$, keep the graph wherever $f(x) \geq 0$ and reflect the parts that lie below the x-axis up into it. Every x-intercept of f stays put and becomes a corner where the curve touches the axis and turns back. The horizontal asymptote is at $y = 2$, which is above the x-axis, so wherever the graph approaches $y = 2$ it is unaffected by the modulus and $y = 2$ remains a horizontal asymptote. The finished graph lies entirely on or above the x-axis.

Q14 2021 Moriah Trial Q9 · 1 mark

The graph of $y = f(x)$ is drawn. Which option represents $y = |f(x)|$?

Criteria	Marks
Correct option (A)	1

Solution.

The graph $y = |f(x)|$ is obtained by keeping every part of $y = f(x)$ that is on or above the x-axis and reflecting every part that lies below the x-axis up into it, so the result never goes below the axis and forms corners at the original x-intercepts. The only option whose graph agrees with f where f is non-negative and is the mirror image of f (in the x-axis) where f was negative is (A). The answer is (A).

4. Absolute value $y = f(|x|)$

Q15 2021 Manly Trial Q13(a)(i) · 1 mark

The curve $y = f(x)$ is shown. Sketch $y = f(|x|)$.

Criteria	Marks
Correct graph: keep the graph for $x \geq 0$, discard the original $x < 0$ part, and reflect the $x \geq 0$ part in the y-axis	1

Solution.

Because $|x| = x$ for $x \geq 0$, the graph of $y = f(|x|)$ is identical to $y = f(x)$ for $x \geq 0$. For $x < 0$, $|x| = -x$, so the value at a negative x copies the value of f at the matching positive number. In practice: keep the right-hand ($x \geq 0$) part of the given graph, delete the original left-hand part, and reflect the right-hand part in the y-axis to draw the new left-hand part. The finished graph is symmetric about the y-axis, and the y-intercept is unchanged.

Q16 2024 Newington Trial Q12(c)(i) · 1 mark

Below is the graph of $f(x)$. Sketch $y = f(|x|)$.

Criteria	Marks
Correct graph: $x \geq 0$ part of $y = f(x)$ retained and reflected in the y-axis to form the $x < 0$ part	1

Solution.

Apply $y = f(|x|)$: take the portion of the graph with $x \geq 0$, and mirror it in the y-axis to create the portion with $x < 0$; the original $x < 0$ portion of f is not used. The new graph is even (symmetric about the y-axis), so for every point (a, b) on the right there is a matching point $(-a, b)$ on the left, and the value at $x = 0$ stays the same.

Q17 2025 Sydney Boys Trial Q12(c)(i) · 1 mark

The graph of $y = f(x)$ has the line $y = x$ as an asymptote. Sketch $y = f(|x|)$.

Criteria	Marks
Correct graph: $x \geq 0$ part kept and reflected in the y-axis; the asymptote $y = x$ (for $x \geq 0$) reflected to $y = -x$ (for $x < 0$)	1

Solution.

For $y = f(|x|)$, keep the graph for $x \geq 0$ exactly as given, discard the original $x < 0$ part, and reflect the $x \geq 0$ part in the y-axis to build the $x < 0$ part. The result is symmetric about the y-axis. The oblique asymptote needs care: the line $y = x$ is the asymptote on the right ($x \geq 0$), and when the right-hand part is reflected in the y-axis its asymptote reflects too, so on the left the graph approaches $y = -x$. So $y = f(|x|)$ has the pair of asymptotes $y = x$ (as $x \to +\infty$) and $y = -x$ (as $x \to -\infty$).

Q18 2023 Caringbah Trial Q11(b)(i) · 1 mark

The diagram shows $y = f(x)$. Draw $y = f(|x|)$.

Criteria	Marks
Correct graph: $x \geq 0$ part of $y = f(x)$ kept and reflected in the y-axis for the $x < 0$ part, asymptotes reflected accordingly	1

Solution.

Use $y = f(|x|)$: retain the $x \geq 0$ half of the given graph, drop the original $x < 0$ half, and reflect the $x \geq 0$ half in the y-axis to draw the $x < 0$ half. The finished graph is symmetric about the y-axis. Any vertical asymptote of f at a positive value $x = a$ is copied to $x = -a$ on the left, and a horizontal asymptote (being horizontal) reflects onto itself, so it is kept.

Q19 2022 James Ruse Trial Q2 · 1 mark

$f(x)$ is odd and $f(1) < 0$. Which option is a possible graph of $y = f(|x|)$?

Criteria	Marks
Correct option (C)	1

Solution.

The graph of $y = f(|x|)$ is always even (symmetric about the y-axis), since replacing x by $-x$ leaves $|x|$ unchanged. It is built from the $x \geq 0$ part of f reflected across the y-axis, so the correct option must be symmetric in the y-axis. On the right ($x > 0$) it must match the original f: because $f(1) < 0$, the graph must be below the x-axis near $x = 1$. The option that is symmetric about the y-axis and lies below the axis just to the right of the origin is (C). (The fact that f is odd only tells us about the original left half, which $f(|x|)$ discards.) The answer is (C).

Q20 2022 Baulkham Hills Trial Q2 · 1 mark

The graph of $y = f(x)$ is shown. Which option could be the graph of $y = f(|x|)$?

Criteria	Marks
Correct option (C)	1

Solution.

For $y = f(|x|)$, the part of $y = f(x)$ with $x < 0$ is discarded, and the part with $x \geq 0$ is kept and reflected in the y-axis to form the new left half. So the correct graph must (i) agree with the original for $x \geq 0$, and (ii) be symmetric about the y-axis. The option that keeps the right-hand side of f and mirrors it across the y-axis is (C). The answer is (C).

Combining and reading graphs

5. Addition of ordinates $y = f(x) \pm g(x)$

Q21 2022 Barker Trial Q5 · **1 mark**

The graphs of $y = f(x)$ and $y = g(x)$ are shown. Which diagram (A, B, C or D) shows the graph of $y = f(x) + g(x)$?

Criteria	Marks
Correct option (B)	1

Solution.

Add the ordinates: at each x-value, add the height of $y = f(x)$ to the height of $y = g(x)$ to get the height of the sum. Work through a few guide points.

- Where $f(x) = 0$, the sum equals $g(x)$; where $g(x) = 0$, the sum equals $f(x)$. So the sum passes through each graph exactly where the other one crosses the x-axis.
- Where the two graphs cross ($f = g$), the sum is twice that common height.
- At the ends of the domain, add the two end heights (both large positive, or one large negative and one small, etc.) to fix the tail behaviour.

Tracking these guide heights across the domain gives a single smooth curve that follows $g(x)$ near g's turning point and is pulled towards $f(x)$'s steep tail. Matching this against the four diagrams, only option (B) has the sum passing through the correct guide points with the right tail. The answer is **(B)**.

Q22 2022 HSC Q4 · **1 mark**

The diagram shows the graph of the sum of the functions $f(x)$ and $g(x)$. Which of the following (A, B, C or D) best represents the graphs of both $f(x)$ and $g(x)$?

Criteria	Marks
Correct option (A)	1

Solution.

The given sum is a single upward parabola (a $\cup$-shape with vertex below the x-axis and two x-intercepts symmetric about the axis). We must find the pair f, g whose ordinates add to this at every x. Option (A) shows a straight line of **negative** slope together with an upward parabola. Adding them, the negative slope of the line is progressively overwhelmed by the increasing slope of the parabola, and the two combine to give exactly the symmetric $\cup$-shaped sum shown: on the left the falling line steepens the parabola's left branch, and on the right the rising parabola dominates the falling line. The other options add a line and curve whose sum would either slope overall (a line of positive slope plus a parabola, or a parabola plus a tilted line) or fail to reproduce the symmetric vertex, so they do not match. The answer is **(A)**.

Q23 2020 Knox Prelim Q14(c) · **2 marks**

The diagram shows the graphs of $f(x) = x^4$ and $g(x) = x^2$. Using addition of ordinates, sketch $y = f(x) - g(x) = x^4 - x^2$.

Criteria	Marks
Subtracts ordinates correctly, identifying the three x-intercepts $x = -1,\ 0,\ 1$ and the sign of the curve between them	1
Correct shape: even, two minima at $x = \pm\frac{1}{\sqrt{2}}$ (value $-\frac{1}{4}$) and a local maximum at the origin, rising steeply for $\lvert x\rvert > 1$	1

Solution.

Subtract the ordinates: at each x, take the height of $y = x^4$ and remove the height of $y = x^2$. Factorising,

$$y = x^4 - x^2 = x^2(x^2 - 1) = x^2(x - 1)(x + 1),$$

so the curve cuts the x-axis at $x = -1,\ 0,\ 1$ (with a double root at $x = 0$, where it touches). For $\lvert x\rvert < 1$ we have $x^2 < 1$, so $x^4 - x^2 < 0$ (the curve dips below the axis); for $\lvert x\rvert > 1$ it is positive and rises steeply (the x^4 term dominates). The function is even, so the graph is symmetric in the y-axis. Differentiating, $y' = 4x^3 - 2x = 2x(2x^2 - 1) = 0$ at $x = 0$ and $x = \pm\frac{1}{\sqrt{2}}$; the origin is a local maximum (value 0) sitting between two equal minima at $x = \pm\frac{1}{\sqrt{2}}$, each of value $\left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$. Sketch a symmetric "W"-like curve: touching the axis at the origin, dipping to $-\frac{1}{4}$ either side, back up through $x = \pm 1$, then rising quickly.

Q24 2022 Normanhurst Boys Trial Q13(a) · **3 marks**

Using addition of ordinates, sketch $y = x - 2\sin x$ for $0 \leq x \leq 2\pi$.

Criteria	Marks
Adds the ordinates of $y = x$ and $y = -2\sin x$ (or identifies key heights) correctly	1
Locates the stationary points at $x = \frac{\pi}{3}$ (local minimum) and $x = \frac{5\pi}{3}$ (local maximum)	1
Correct overall shape through $(0, 0)$ and $(2\pi, 2\pi)$ with the wave superimposed on the line $y = x$	1

Solution.

Write the function as the line $y = x$ plus the wave $y = -2\sin x$, and add the ordinates. The line rises steadily from $(0, 0)$ to $(2\pi, 2\pi)$; the wave $-2\sin x$ pulls the curve below the line on $(0, \pi)$ (where $\sin x > 0$) and above it on $(\pi, 2\pi)$ (where $\sin x < 0$), by up to 2 units either way.

$$y' = 1 - 2\cos x = 0 \implies \cos x = \tfrac{1}{2} \implies x = \tfrac{\pi}{3},\ \tfrac{5\pi}{3}.$$

At $x = \frac{\pi}{3}$ the derivative changes from negative to positive, giving a local minimum of value $\frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \approx -0.68$. At $x = \frac{5\pi}{3}$ it changes from positive to negative, a local maximum of value $\frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$. The curve starts at the origin, dips to the minimum near $x = \frac{\pi}{3}$, crosses the x-axis again near $x \approx 1.9$, rises through (π, π), climbs to the maximum near $x = \frac{5\pi}{3}$, then falls to $(2\pi, 2\pi)$. Overall it is the straight line $y = x$ with a single oscillation of the wave superimposed.

Q25 2021 Sydney Grammar Prelim Q3(c)(i) · **2 marks**

The diagram shows the graphs of $y = f(x)$ and $y = g(x)$. Using addition of ordinates, sketch $y = f(x) + g(x)$.

Criteria	Marks
Adds the ordinates of the two lines at several x-values	1
Correct result: the horizontal line $y = -1$ (with at least three points shown, e.g. $(-3, -1)$, $(0, -1)$, $(2, -1)$)	1

Solution.

Both graphs are straight lines. Reading the diagram, $y = f(x)$ has slope -1 passing through $(-1, 1)$, $(-2, 2)$, $(-3, 3)$, so $f(x) = -x$; and $y = g(x)$ has slope $+1$ passing through $(0, -1)$, $(1, 0)$, $(2, 1)$, so $g(x) = x - 1$. Adding the ordinates at each x:

$$f(x) + g(x) = (-x) + (x - 1) = -1.$$

The two slopes cancel, so the sum is the constant $y = -1$: a horizontal line through, for example, $(-3, -1)$, $(0, -1)$ and $(2, -1)$. (You can check any single x: at $x = 1$, $f = -1$ and $g = 0$, sum -1.)

Q26 2023 SCEGGS Prelim Q8(f)(ii) · **2 marks**

The graph of $y = f(x)$ is shown: a smooth U-shaped curve on the window $-1 \le x \le 3$, with x-intercepts at about $x \approx -0.55$ and $x \approx 1.55$, a y-intercept near $(0, -0.5)$, and a minimum turning point near $(0.5, -1)$; it is negative between its two zeros and positive outside them. Sketch the graph of $y = f(x) + x^2$.

Criteria	**Marks**
Adds x^2 to f pointwise: sum equals x^2 at the zeros of f ($x \approx -0.55,\ 1.55$) and the y-intercept is unchanged at ≈ -0.5	1
Correct resulting shape (lifted curve, parabolic growth for large $\lvert x\rvert$) with the turning points shifted upward	1

Solution.

This is an **addition of ordinates**: build $y = f(x) + x^2$ by adding the height of the parabola $y = x^2$ to the height of f at each x-value.

- **At the zeros of** f: where $f(x) = 0$ (here $x \approx -0.55$ and $x \approx 1.55$), the sum equals x^2 alone. So the new curve passes through $(-0.55,\ 0.30)$ and $(1.55,\ 2.40)$ (since $0.55^2 \approx 0.30$ and $1.55^2 \approx 2.40$): both lie **above** the x-axis.
- **At** $x = 0$: $x^2 = 0$, so the y-intercept is unchanged: $f(0) + 0 = f(0) \approx -0.5$.
- **Near the minimum of** f: at $x \approx 0.5$, $f \approx -1$ and $x^2 \approx 0.25$, so the sum is ≈ -0.75 (the trough is lifted, and its lowest point shifts slightly).
- **Large** $\lvert x\rvert$: the x^2 term dominates, so the sum grows like a parabola; e.g. at $x = 3$, $f + x^2 \approx 2 + 9 = 11$, and to the left the curve is pulled upward too.

Adding a strictly positive parabola x^2 therefore **lifts every point of** f **upward** (by more the further from $x = 0$), leaving the y-intercept fixed at ≈ -0.5 and raising the two zeros of f to the points $(-0.55,\ 0.30)$ and $(1.55,\ 2.40)$. The result is a curve that keeps a small dip near the origin (the only region where f still pulls the sum below the axis) and then climbs steeply on both sides as the x^2 growth takes over.

6. Solving and counting solutions graphically

Q27 2021 Moriah Prelim Q1 · 1 mark

The curve shown has equation $y = |x - 2|$. By considering this graph and the graph of $y = 2x + 1$, how many solutions does $|x - 2| = 2x + 1$ have? (A) 0 (B) 1 (C) 2 (D) 3.

Criteria	Marks
Correct option (B)	1

Solution.

The number of solutions is the number of points where the "V" graph $y = |x - 2|$ meets the line $y = 2x + 1$. Solve algebraically to confirm the graph. For $x \geq 2$: $x - 2 = 2x + 1 \Rightarrow x = -3$, which is not ≥ 2, so rejected. For $x < 2$: $2 - x = 2x + 1 \Rightarrow 1 = 3x \Rightarrow x = \frac{1}{3}$, which satisfies $x < 2$ and gives $y = \frac{5}{3} > 0$ (a valid meeting point). So the line cuts only the left arm of the V, once. There is exactly **1** solution. The answer is **(B)**.

Q28 2024 Sydney Girls Trial Q5 · 1 mark

Using a graphical approach, what is the number of solutions to $e^x + e^{-x} = x + 2$? (A) 0 (B) 1 (C) 2 (D) 3.

Criteria	Marks
Correct option (C)	1

Solution.

Sketch $y = e^x + e^{-x}$ against the line $y = x + 2$. The curve $y = e^x + e^{-x}$ is even, with a minimum at $x = 0$ (where $y = 1 + 1 = 2$) and it grows without bound on both sides. The line $y = x + 2$ passes through $(0, 2)$ as well, so the curve and the line **both** pass through the point $(0, 2)$: that is one solution. Comparing slopes there, the curve has $y' = e^x - e^{-x} = 0$ at $x = 0$ while the line has slope 1, so just right of the origin the line rises above the curve, then the curve (growing exponentially) overtakes the line again, producing a **second** crossing (near $x \approx 0.9$). For $x < 0$ the curve climbs steeply while the line falls, so they do not meet again. Hence there are exactly **2** solutions. The answer is **(C)**.

Q29 2023 Normanhurst Boys Trial Q4 · **1 mark**

How many solutions does $x^{1/3} = |x-2| - 3$ have? (A) 0 (B) 1 (C) 2 (D) 3.

Criteria	Marks
Correct option (C)	1

Solution.

Sketch the two sides on the same axes. The left side $y = x^{1/3}$ is the cube-root curve: it passes through the origin, increases everywhere, and is gently sloped (flattening for large $|x|$). The right side $y = |x-2| - 3$ is a "V" with vertex at $(2, -3)$, arms of slope ± 1. Because the cube-root curve rises much more slowly than the V's arms of slope 1, it can cross each arm at most once. It meets the left (falling) arm once, at a point near $x \approx -0.3$, and it meets the right (rising) arm once, at a point near $x \approx 6.9$ (there the V has climbed back up to meet the slowly-rising cube root). So there are exactly **2** solutions. The answer is **(C)**.

Q30 2020 Blacktown Boys Prelim Q10 · **1 mark**

How many solutions does $\sqrt{x} = |x-1| - 1$ have? (A) 0 (B) 1 (C) 2 (D) 3.

Criteria	Marks
Correct option (C)	1

Solution.

The left side $y = \sqrt{x}$ exists only for $x \geq 0$ and increases from $(0, 0)$. The right side $y = |x-1| - 1$ is a "V" with vertex at $(1, -1)$. On the domain $x \geq 0$ the two curves meet where their heights are equal. Testing the corners: at $x = 0$, $\sqrt{0} = 0$ and $|0-1| - 1 = 0$, so $x = 0$ is one solution. For the right arm ($x > 1$), $|x-1| - 1 = x - 2$; setting $\sqrt{x} = x - 2$ and squaring gives $x = (x-2)^2 = x^2 - 4x + 4$, i.e. $x^2 - 5x + 4 = 0$, so $x = 1$ or $x = 4$; only $x = 4$ satisfies $x > 1$ with $x - 2 \geq 0$, giving the second solution. Between $x = 0$ and $x = 4$ the V dips below the square-root curve, so no further crossings. Hence there are exactly **2** solutions ($x = 0$ and $x = 4$). The answer is **(C)**.

Q31 2025 Sydney Girls Trial Q6 · **1 mark**

Using a graphical approach, how many solutions are there to $\ln|x| = \dfrac{1}{\ln x}$? (A) 0 (B) 1 (C) 2 (D) 3.

Criteria	**Marks**
Correct option (C)	1

Solution.

The right side $\dfrac{1}{\ln x}$ requires $\ln x$ to be defined and non-zero, so $x > 0$ and $x \neq 1$. On this domain $|x| = x$, so $\ln|x| = \ln x$ and the equation becomes

$$\ln x = \frac{1}{\ln x}.$$

Let $u = \ln x$ (with $u \neq 0$). Then $u = \dfrac{1}{u}$, so $u^2 = 1$, giving $u = \pm 1$, i.e. $\ln x = 1$ or $\ln x = -1$, so

$$x = e \quad \text{or} \quad x = \tfrac{1}{e}.$$

Both lie in the domain $x > 0,\ x \neq 1$. Graphically, the curve $y = \ln x$ meets the curve $y = \dfrac{1}{\ln x}$ at exactly these two points (one where $\ln x = 1$, one where $\ln x = -1$). There are exactly **2** solutions. The answer is **(C)**.

Q32 2020 James Ruse Prelim Q9(a)(iii) · **2 marks**

Consider the curve $y = \dfrac{8x}{x^2+20}$ and the horizontal line $y = k$. Show that for the line to intersect the curve, k must satisfy $-\dfrac{2}{\sqrt{5}} \leq k \leq \dfrac{2}{\sqrt{5}}$.

Criteria	**Marks**
Sets $\dfrac{8x}{x^2+20} = k$ and rearranges to the quadratic $kx^2 - 8x + 20k = 0$ in x	1
Uses the discriminant condition (and the $k = 0$ case) to obtain $-\dfrac{2}{\sqrt{5}} \leq k \leq \dfrac{2}{\sqrt{5}}$	1

Solution.

A point of intersection is a value of x satisfying $\dfrac{8x}{x^2+20} = k$. Since $x^2 + 20 > 0$ for all x, multiply through:

$$8x = k(x^2+20) \implies kx^2 - 8x + 20k = 0.$$

Case $k = 0$: the equation is $-8x = 0$, i.e. $x = 0$, which does lie in the range $-\frac{2}{\sqrt{5}} \leq 0 \leq \frac{2}{\sqrt{5}}$, so an intersection exists.

Case $k \neq 0$: this is a quadratic in x, and a real intersection needs the discriminant to be non-negative:

$$\Delta = (-8)^2 - 4(k)(20k) = 64 - 80k^2 \geq 0 \implies k^2 \leq \frac{64}{80} = \frac{4}{5} \implies |k| \leq \frac{2}{\sqrt{5}}.$$

Combining both cases, the line $y = k$ meets the curve exactly when

$$-\frac{2}{\sqrt{5}} \leq k \leq \frac{2}{\sqrt{5}}.$$

(Equivalently $-\frac{2\sqrt{5}}{5} \leq k \leq \frac{2\sqrt{5}}{5}$; these bounds are the maximum and minimum values of $\dfrac{8x}{x^2+20}$, attained at $x = \pm 2\sqrt{5}$.)

7. Identifying a function or transformation from its graph

Q33 2019 HSC Q4 · 1 mark

The graph of $y = f(x)$ has vertical asymptotes at $x = -1$ and $x = 1$, passes through the origin, and has rotational (odd) symmetry about the origin. Which equation best describes it? (A) $y = \dfrac{x}{x^2 - 1}$ (B) $y = \dfrac{x^2}{x^2 - 1}$ (C) $y = \dfrac{x}{1 - x^2}$ (D) $y = \dfrac{x^2}{1 - x^2}$

Criteria	Marks
Correct option (A)	1

Solution.

Read the defining features off the graph. The curve passes through the origin and has rotational symmetry about it, so f is **odd**: $f(-x) = -f(x)$. Options (B) and (D) have x^2 in the numerator and are even, so they are eliminated, leaving (A) and (C), which have x in the numerator.

The vertical asymptotes are at $x = \pm 1$, so the denominator is $x^2 - 1$ (equivalently $-(1 - x^2)$); both (A) and (C) fit this. To separate them, read the sign of an outer branch. For $x > 1$ the graph is **above** the x-axis (rising to $+\infty$ just to the right of $x = 1$). Test $x = 2$:

$$\text{(A)}\ \frac{2}{2^2 - 1} = \frac{2}{3} > 0, \qquad \text{(C)}\ \frac{2}{1 - 2^2} = \frac{2}{-3} < 0.$$

Only (A) gives a positive value on the right-hand outer branch, matching the graph. The answer is **(A)**.

Q34 2025 North Sydney Boys Trial Q1 · **1 mark**

The graph is a rational curve with vertical asymptotes at $x = 1$ and $x = 2$, no x-intercept, and its left branch lies below the x-axis. Which function does it represent? (A) $y = \dfrac{1}{(x-1)(x-2)}$ (B) $y = \dfrac{x}{(x-1)(x-2)^2}$ (C) $y = \dfrac{(x-1)(x-2)}{x}$ (D) $y = \dfrac{1}{(x-1)^2(x-2)}$

Criteria	Marks
Correct option (D)	1

Solution.

Read three features off the graph and match them to the denominators.

- **Asymptotes at $x = 1$ and $x = 2$.** The denominator must vanish only at $x = 1$ and $x = 2$. This eliminates (C), whose only vertical asymptote is at $x = 0$.
- **No x-intercept.** The graph never crosses the x-axis, so the numerator has no real zero. Option (B) has numerator x, giving an x-intercept at $x = 0$, so (B) is eliminated. This leaves (A) and (D), both with numerator 1.
- **Multiplicity of the asymptotes.** At a **double** pole the curve goes the same way (both to $+\infty$ or both to $-\infty$) on the two sides; at a **simple** pole it goes opposite ways. In the graph the far-left branch lies **below** the axis and plunges to $-\infty$ as $x \to 1^-$, and the curve also goes to $-\infty$ as $x \to 1^+$: so $x = 1$ is a **double** pole. Option (A) has simple poles at both $x = 1$ and $x = 2$ (its left branch would sit **above** the axis), so (A) is eliminated.

Only (D) has the squared factor $(x-1)^2$ (double pole at $x = 1$, simple pole at $x = 2$), no x-intercept, and a far-left branch below the axis approaching 0^-. Checking a point on the left branch, $x = -3$ gives $\dfrac{1}{(-4)^2(-5)} = -\dfrac{1}{80} < 0$, confirming it is below the axis. The answer is **(D)**.

Q35 2022 Fort St Prelim Q1 · **1 mark**

Which of the options (A)–(D) is the graph of $f(x) = \dfrac{|x|}{x}$ for $x \neq 0$?

Criteria	Marks
Correct option (B)	1

Solution.

Simplify the rule by cases. For $x > 0$, $|x| = x$, so $f(x) = \dfrac{x}{x} = 1$. For $x < 0$, $|x| = -x$, so $f(x) = \dfrac{-x}{x} = -1$. Hence

$$f(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

This is the sign function: a horizontal ray at $y = 1$ for $x > 0$ and a horizontal ray at $y = -1$ for $x < 0$, undefined at $x = 0$ (open circles at $(0, 1)$ and $(0, -1)$). The option showing exactly this step is (B). The answer is **(B)**.

Q36 2023 Trinity Grammar Trial Q4 · **1 mark**

Given the graph of $y = |x|$, which of the options (A)–(D) is the graph of $y = x|x|$?

Criteria	Marks
Correct option (A)	1

Solution.

Split $y = x|x|$ into cases. For $x \geq 0$, $|x| = x$, so $y = x \cdot x = x^2$ (the right arm is the right half of an upward parabola). For $x < 0$, $|x| = -x$, so $y = x(-x) = -x^2$ (the left arm is the left half of a downward parabola). Hence

$$y = x|x| = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

The graph passes through the origin, is **odd** (rotationally symmetric about the origin), rises with increasing steepness for $x > 0$ (concave up, first quadrant) and falls with increasing steepness for $x < 0$ (concave down, third quadrant): an S-shaped curve. The option showing this is (A). The answer is **(A)**.

Q37 2025 Riverview Trial Q10 · **1 mark**

Which of the options (A)–(D) is the graph of $y = |2 - x|$?

Criteria	Marks
Correct option (B)	1

Solution.

Note $|2 - x| = |x - 2|$, so $y = |x - 2|$ is the standard V-shaped absolute-value graph translated 2 units to the right. Its features are a **vertex at** $(2, 0)$ on the positive x-axis, arms of gradient ± 1 opening upward, and a y-intercept at $y = |2 - 0| = 2$, i.e. $(0, 2)$. The option showing the V with vertex at $(2, 0)$ passing through $(0, 2)$ is (B). The answer is **(B)**.

Q38 2022 HSC Q2 · **1 mark**

The graph of $f(x) = \dfrac{3}{x-1} + 2$ is transformed to produce $g(x)$, as shown. Which transformation was applied?

(A) $g(x) = f(|x|)$ (B) $g(x) = |f(x)|$ (C) $g(x) = f(-x)$ (D) $g(x) = \dfrac{1}{f(x)}$

Criteria	Marks
Correct option (D)	1

Solution.

Read the effect of the transformation from the two graphs. The graph $y = f(x)$ is a hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = 2$. Compare each candidate against the graph of g:

- (A) $f(|x|)$ reflects the $x \geq 0$ part of f into $x < 0$, producing a graph symmetric in the y-axis. The graph of g is not y-axis symmetric, so (A) is out.
- (B) $|f(x)|$ reflects any part of f below the x-axis up above it; the resulting graph never dips below $y = 0$. The graph of g takes negative values, so (B) is out.
- (C) $f(-x)$ reflects f in the y-axis, moving the vertical asymptote to $x = -1$ and keeping the horizontal asymptote at $y = 2$. The graph of g does not have horizontal asymptote $y = 2$, so (C) is out.
- (D) $\dfrac{1}{f(x)}$ is the reciprocal: where f has its vertical asymptote ($x = 1$, $f \to \pm\infty$), $g = \dfrac{1}{f} \to 0$, so g passes through the axis there; where f has its horizontal asymptote $y = 2$, $g \to \dfrac{1}{2}$; and where $f(x) = 0$ (at $x = \frac{-1}{2}$, since $\frac{3}{x-1} + 2 = 0$ gives $x - 1 = -\frac{3}{2}$), g has a new vertical asymptote. This reciprocal behaviour matches the graph of g.

The transformation shown is the reciprocal, $g(x) = \dfrac{1}{f(x)}$. The answer is **(D)**.