Year 11 Mathematics Extension 1

Topic: The Binomial Theorem (1 of 2)
Review Questions (Worked Solutions)
Term 3, 2026

Expanding and finding terms

1. Expanding a binomial

Q1 2025 Smiths Hill Trial Q2 · 1 mark

Which of the following is the correct expansion of $(3x-1)^3$?

(A) $27x^3+27x^2+9x+1$

(B) $9x^3-27x^2+12x-1$

(C) $27x^3-27x^2+9x-1$

(D) $27x^3-27x^2+12x-9$

Criteria	Marks
Correct answer (C)	1

Solution.

Using $(A+B)^3=A^3+3A^2B+3AB^2+B^3$ with $A=3x$, $B=-1$:

$$(3x-1)^3=(3x)^3+3(3x)^2(-1)+3(3x)(-1)^2+(-1)^3=27x^3-27x^2+9x-1.$$

This is option **(C)**.

Q2 2021 Kings Trial Q2 · 1 mark

Expand $(2x-1)^3$.

Criteria	Marks
Correct expansion	1

Solution.

With $A=2x$, $B=-1$ in $(A+B)^3=A^3+3A^2B+3AB^2+B^3$:

$$(2x-1)^3=(2x)^3+3(2x)^2(-1)+3(2x)(-1)^2+(-1)^3=8x^3-12x^2+6x-1.$$

Q3 2025 Sydney Grammar Trial Q11(b) · **2 marks**

Expand and simplify $(x-2)^3$.

Criteria	Marks
Applies the cube expansion with correct terms	1
Correct simplified answer	1

Solution.

With $A = x$, $B = -2$:

$$(x-2)^3 = x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3 = x^3 - 6x^2 + 12x - 8.$$

Q4 2021 HSC Q11(b) · **2 marks**

Expand and simplify $(2a-b)^4$.

Criteria	Marks
Uses the binomial coefficients $1, 4, 6, 4, 1$ with correct powers	1
Correct simplified answer	1

Solution.

Using $(A+B)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$ with $A = 2a$, $B = -b$:

$$\begin{aligned}(2a-b)^4 &= (2a)^4 + 4(2a)^3(-b) + 6(2a)^2(-b)^2 + 4(2a)(-b)^3 + (-b)^4 \\ &= 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4.\end{aligned}$$

Q5 2023 Girraween Prelim Q10(a) · **2 marks**

Write the expansion of $(2x-y)^5$.

Criteria	Marks
Uses the binomial coefficients $1, 5, 10, 10, 5, 1$ with correct powers	1
Correct simplified answer	1

Solution.

Using $\binom{5}{k}$ for $k = 0, \ldots, 5$ with $A = 2x$, $B = -y$:

$$\begin{aligned}(2x-y)^5 &= (2x)^5 + 5(2x)^4(-y) + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)(-y)^4 + (-y)^5 \\ &= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5.\end{aligned}$$

Q6 2021 Sydney Grammar Prelim Q2(c) · **3 marks**

Use a binomial expansion to find integers a and b such that $(1-\sqrt{2})^5 = a + b\sqrt{2}$.

Criteria	**Marks**
Sets up the binomial expansion of $(1-\sqrt{2})^5$	1
Evaluates each term, using $(\sqrt{2})^2 = 2$, etc.	1
Correct values $a = 41$, $b = -29$	1

Solution.

Expand with $A = 1$, $B = -\sqrt{2}$ and coefficients $1, 5, 10, 10, 5, 1$:

$$\begin{aligned}(1-\sqrt{2})^5 &= 1 + 5(-\sqrt{2}) + 10(-\sqrt{2})^2 + 10(-\sqrt{2})^3 + 5(-\sqrt{2})^4 + (-\sqrt{2})^5 \\ &= 1 - 5\sqrt{2} + 10(2) - 10(2\sqrt{2}) + 5(4) - 4\sqrt{2} \\ &= (1 + 20 + 20) + (-5 - 20 - 4)\sqrt{2} \\ &= 41 - 29\sqrt{2}.\end{aligned}$$

Hence $a = 41$ and $b = -29$.

Q7 2023 Blacktown Boys Prelim Q11(b) · **2 marks**

Show through expansion that $(3-\sqrt{5})^4 = 376 - 168\sqrt{5}$.

Criteria	Marks
Sets up the binomial expansion with correct coefficients and powers	1
Simplifies to the required result	1

Solution.

Expand with $A = 3$, $B = -\sqrt{5}$ and coefficients $1, 4, 6, 4, 1$:

$$\begin{aligned}(3-\sqrt{5})^4 &= 3^4 + 4(3)^3(-\sqrt{5}) + 6(3)^2(-\sqrt{5})^2 + 4(3)(-\sqrt{5})^3 + (-\sqrt{5})^4 \\ &= 81 - 108\sqrt{5} + 6(9)(5) + 12(-5\sqrt{5}) + 25 \\ &= 81 - 108\sqrt{5} + 270 - 60\sqrt{5} + 25 \\ &= (81 + 270 + 25) + (-108 - 60)\sqrt{5} \\ &= 376 - 168\sqrt{5}. \qquad \blacksquare\end{aligned}$$

2. Finding a specific term

Q8 2023 Girraween Prelim Q5 · **1 mark**

What is the seventh term in the expansion of $(x + 3y)^{11}$?

(A) $\binom{11}{7}(3)^7 x^7 y^6$

(B) $\binom{11}{7}(3)^7 x^6 y^7$

(C) $\binom{11}{6}(3)^6 x^6 y^5$

(D) $\binom{11}{6}(3)^6 x^5 y^6$

Criteria	Marks
Correct option (D)	1

Solution.

The general term is $T_{k+1} = \binom{11}{k} x^{11-k}(3y)^k$. The seventh term is $k = 6$:

$$T_7 = \binom{11}{6} x^5 (3y)^6 = \binom{11}{6}(3)^6 x^5 y^6.$$

This is **(D)**. (Numerically $\binom{11}{6}(3)^6 = 462 \times 729 = 336\,798$, so $T_7 = 336\,798\,x^5y^6$.)

Q9 2024 Glenwood Prelim Q3 · **1 mark**

What is the sixth term in the expansion of $(2x + 3)^8$?

(A) $\binom{8}{4}(2x)^4(3)^4$

(B) $\binom{8}{4}(2x)^3(3)^5$

(C) $\binom{8}{5}(2x)^2(3)^7$

(D) $\binom{8}{5}(2x)^3(3)^5$

Criteria	**Marks**
Correct option (D)	1

Solution.

The general term is $T_{k+1} = \binom{8}{k}(2x)^{8-k}(3)^k$. The sixth term is $k = 5$:

$$T_6 = \binom{8}{5}(2x)^3(3)^5.$$

This is **(D)**. (Numerically $\binom{8}{5} \cdot 2^3 \cdot 3^5 = 56 \times 8 \times 243 = 108\,864$, so $T_6 = 108\,864\,x^3$.)

Q10 2023 North Sydney Boys Prelim QE1 · **2 marks**

Find the term in x^5 in the expansion of $\left(2x - \frac{1}{x^2}\right)^8$.

Criteria	**Marks**
Writes the general term of the expansion	1
Identifies k and evaluates the x^5 term correctly	1

Solution.

The general term is

$$T_{k+1} = \binom{8}{k}(2x)^{8-k}\left(-\frac{1}{x^2}\right)^k = \binom{8}{k}(-1)^k\, 2^{8-k}\, x^{8-3k}.$$

For the term in x^5, set $8 - 3k = 5$, so $k = 1$. Then

$$T_2 = \binom{8}{1}(-1)^1\, 2^7\, x^5 = 8 \times (-1) \times 128\, x^5 = -1024x^5.$$

The term in x^5 is $\boxed{-1024x^5}$.

Q11 2023 North Sydney Boys Trial Q2 · **1 mark**

Find the fifth term in the expansion of $\left(3x^2 - \dfrac{1}{2x}\right)^8$.

(A) $\binom{8}{4}\left(\dfrac{3}{2}\right)^4 x^4$

(B) $\binom{4}{8}\left(\dfrac{2}{3}\right)^4 x^{-4}$

(C) $\binom{8}{4}\left(\dfrac{2}{3}\right)^4 x^{-4}$

(D) $\binom{4}{8}\left(\dfrac{3}{2}\right)^4 x^4$

Criteria	**Marks**
Correct option (A)	1

Solution.

The general term is $T_{k+1} = \binom{8}{k}(3x^2)^{8-k}\left(-\dfrac{1}{2x}\right)^k$. The fifth term is $k = 4$:

$$T_5 = \binom{8}{4}(3x^2)^4\left(-\frac{1}{2x}\right)^4 = \binom{8}{4}3^4 x^8 \cdot \frac{1}{2^4}x^{-4} = \binom{8}{4}\left(\frac{3}{2}\right)^4 x^4.$$

This is **(A)**. (Numerically $\binom{8}{4}\left(\frac{3}{2}\right)^4 = 70 \times \frac{81}{16} = \frac{2835}{8}$, so $T_5 = \frac{2835}{8}x^4$.)

Q12 2025 Baulkham Hills Prelim Q4 · **1 mark**

Which of the following is the coefficient of the fourth term in the expansion of $(3x-4)^6$?

(A) -34560

(B) 34560

(C) 25920

(D) -25920

Criteria	**Marks**
Correct option (A)	1

Solution.

The general term is $T_{k+1} = \binom{6}{k}(3x)^{6-k}(-4)^k$. The fourth term is $k=3$:

$$T_4 = \binom{6}{3}(3x)^3(-4)^3 = 20 \times 27x^3 \times (-64) = -34560x^3.$$

The coefficient is -34560, which is **(A)**.

Q13 2022 Hurlstone Trial Q12(c) · **3 marks**

Consider the expansion of $\left(x+\frac{1}{x}\right)^8$.

(i) Find the general term of $\left(x+\frac{1}{x}\right)^8$.

(ii) Find the coefficient of x^2 in $\left(x+\frac{1}{x}\right)^8$.

Criteria	Marks
(i) Correct general term	1
(ii) Sets $8-2k=2$ to find $k=3$	1
(ii) Correct coefficient $\binom{8}{3}=56$	1

Solution.

(i) The general term is

$$T_{k+1}=\binom{8}{k}x^{8-k}\left(\frac{1}{x}\right)^k=\binom{8}{k}x^{8-2k}.$$

(ii) For the term in x^2, set $8-2k=2$, so $k=3$. The coefficient is

$$\binom{8}{3}=56.$$

The coefficient of x^2 is 56.

3. Finding a coefficient

Q14 2022 Abbotsleigh Trial Q11(c) · **1 mark**

Find the coefficient of y^4 in $(1+2y)^6$.

Criteria	Marks
Correct coefficient 240	1

Solution.

The general term is $\binom{6}{k}(1)^{6-k}(2y)^k=\binom{6}{k}2^k\,y^k$. Set $k=4$:

$$\binom{6}{4}2^4=15\times 16=240.$$

The coefficient of y^4 is $\boxed{240}$.

Q15 2022 Barker Trial Q11(b) · **2 marks**

Determine the coefficient of x^3 in the expansion of $(2-3x)^8$.

Criteria	Marks
Sets up the correct general term and identifies $k=3$	1
Evaluates the coefficient -48384 correctly	1

Solution.

The general term is $\binom{8}{k}(2)^{8-k}(-3x)^k = \binom{8}{k}2^{8-k}(-3)^k x^k$. Set $k=3$:

$$\binom{8}{3}2^5(-3)^3 = 56\times 32\times(-27) = -48384.$$

The coefficient of x^3 is $\boxed{-48384}$.

Q16 2024 Knox Trial Q11(a) · **2 marks**

Find the coefficient of x^2 in the expansion of $\left(2x+\dfrac{1}{x}\right)^{10}$.

Criteria	Marks
Correct general term and finds k giving x^2 (namely $k=4$)	1
Evaluates the coefficient 13440 correctly	1

Solution.

The general term is

$$\binom{10}{k}(2x)^{10-k}\left(\frac{1}{x}\right)^k = \binom{10}{k}2^{10-k}x^{10-2k}.$$

For the power of x to be 2: $10-2k=2$, so $k=4$. The coefficient is

$$\binom{10}{4}2^6 = 210\times 64 = 13440.$$

The coefficient of x^2 is $\boxed{13440}$.

Q17 2025 Killara Trial Q5 · **1 mark**

The coefficient of a^5b^4 in the expansion of $\left(3a - \frac{b}{3}\right)^9$ is:

(A) -42

(B) 3

(C) 42

(D) 378

Criteria	**Marks**
Correct option **(D)** 378	1

Solution.

The general term is

$$\binom{9}{k}(3a)^{9-k}\left(-\frac{b}{3}\right)^k = \binom{9}{k}3^{9-k}(-1)^k 3^{-k}\, a^{9-k}b^k.$$

The term in a^5b^4 has $k = 4$ (so $9 - k = 5$). Its coefficient is

$$\binom{9}{4}3^5(-1)^4 3^{-4} = 126 \times 3^{5-4} = 126 \times 3 = 378.$$

So the coefficient is 378, and the answer is **(D)**.

Q18 2022 Cheltenham Girls Trial Q13(a) · **2 marks**

Determine the coefficient of x^4 in the expansion of $\left(\frac{2}{x}+3x\right)^6$.

Criteria	Marks
Correct general term and finds k giving x^4 (namely $k=5$)	1
Evaluates the coefficient 2916 correctly	1

Solution.

The general term is

$$\binom{6}{k}\left(\frac{2}{x}\right)^{6-k}(3x)^k=\binom{6}{k}2^{6-k}3^k\,x^{k-(6-k)}=\binom{6}{k}2^{6-k}3^k\,x^{2k-6}.$$

For the power of x to be 4: $2k-6=4$, so $k=5$. The coefficient is

$$\binom{6}{5}2^1 3^5=6\times 2\times 243=2916.$$

The coefficient of x^4 is $\boxed{2916}$.

Q19 2022 North Sydney Boys Prelim C1 · **2 marks**

Find the coefficient of x^2 in the expansion of $\left(x^2+\frac{2}{x}\right)^7$.

Criteria	Marks
Correct general term and finds k giving x^2 (namely $k=4$)	1
Evaluates the coefficient 560 correctly	1

Solution.

The general term is

$$\binom{7}{k}(x^2)^{7-k}\left(\frac{2}{x}\right)^k=\binom{7}{k}2^k\,x^{2(7-k)-k}=\binom{7}{k}2^k\,x^{14-3k}.$$

For the power of x to be 2: $14-3k=2$, so $k=4$. The coefficient is

$$\binom{7}{4}2^4=35\times 16=560.$$

The coefficient of x^2 is $\boxed{560}$.

Q20 2021 North Sydney Boys Trial Q11(d) · 3 marks

Find the coefficient of x^8 in the expansion of $\left(\frac{x}{4}-\frac{4}{x}\right)^{14}$.

Criteria	Marks
Writes the correct general term	1
Finds k giving x^8 (namely $k=3$)	1
Evaluates the coefficient $-\frac{91}{16384}$ correctly	1

Solution.

The general term is

$$\binom{14}{k}\left(\frac{x}{4}\right)^{14-k}\left(-\frac{4}{x}\right)^{k}=\binom{14}{k}\left(\frac{1}{4}\right)^{14-k}(-4)^{k}\,x^{14-2k}.$$

For the power of x to be 8: $14-2k=8$, so $k=3$. The coefficient is

$$\binom{14}{3}\left(\frac{1}{4}\right)^{11}(-4)^{3}=364\times\frac{1}{4^{11}}\times(-64)=\frac{-23296}{4194304}=-\frac{91}{16384}.$$

The coefficient of x^8 is $\boxed{-\frac{91}{16384}}$.

Q21 2022 North Sydney Boys Trial Q9 · **1 mark**

Let n be a positive integer. The coefficient of x^3y^5 in the expansion of $(1 + xy + y^2)^n$ equals:

(A) n

(B) $2n$

(C) $\binom{n}{3}\binom{n}{5}$

(D) $4\binom{n}{4}$

Criteria	**Marks**
Correct option **(D)** $4\binom{n}{4}$	1

Solution.

Each factor of $(1 + xy + y^2)^n$ contributes either 1, xy, or y^2. To obtain x^3y^5 we must take xy exactly three times (this supplies the x^3 and three of the y's), y^2 once (supplying the remaining y^2), and 1 from the other $n - 4$ factors. The number of ways to assign these choices among the n factors is the multinomial

$$\frac{n!}{3!\,1!\,(n-4)!} = \binom{n}{3}\binom{n-3}{1} = \frac{n(n-1)(n-2)(n-3)}{6}.$$

Since $4\binom{n}{4} = \dfrac{4\,n(n-1)(n-2)(n-3)}{24} = \dfrac{n(n-1)(n-2)(n-3)}{6}$, the coefficient equals $4\binom{n}{4}$, so the answer is $\boxed{\textbf{(D)}}$.

4. The term independent of x

Q22 2025 Hurlstone Trial Q1 · 1 mark

What is the term independent of x in the expansion of $\left(x+\frac{2}{x}\right)^6$? (A) 160 (B) 40 (C) 80 (D) 20

Criteria	Marks
Selects (A): correct term independent of x	1

Solution.

The general term is

$$T_{r+1}=\binom{6}{r}x^{6-r}\left(\frac{2}{x}\right)^r=\binom{6}{r}2^r\,x^{6-2r}.$$

The term is independent of x when the power of x is zero: $6-2r=0$, so $r=3$. Then

$$\binom{6}{3}2^3=20\times 8=160.$$

The answer is **(A)** 160.

Q23 2025 St George Girls Trial Q11(b) · 2 marks

Find the constant term in the expansion of $\left(x^2+\frac{2}{x}\right)^6$.

Criteria	Marks
Correct general term and equates the power of x to zero to find $r=4$	1
Evaluates the constant term correctly	1

Solution.

The general term is

$$T_{r+1}=\binom{6}{r}\left(x^2\right)^{6-r}\left(\frac{2}{x}\right)^r=\binom{6}{r}2^r\,x^{12-2r-r}=\binom{6}{r}2^r\,x^{12-3r}.$$

The constant (independent-of-x) term needs $12-3r=0$, so $r=4$. Then

$$\binom{6}{4}2^4=15\times 16=240.$$

The constant term is $\boxed{240}$.

Q24 2025 Abbotsleigh Trial Q11(d) · **2 marks**

Find the constant term in the expansion of $\left(2x - \frac{1}{x}\right)^{10}$.

Criteria	Marks
Correct general term and equates the power of x to zero to find $r = 5$	1
Evaluates the constant term correctly (including the sign)	1

Solution.

The general term is

$$T_{r+1} = \binom{10}{r}(2x)^{10-r}\left(-\frac{1}{x}\right)^{r} = \binom{10}{r}2^{10-r}(-1)^{r}\,x^{10-2r}.$$

The constant term needs $10 - 2r = 0$, so $r = 5$. Then

$$\binom{10}{5}2^{5}(-1)^{5} = 252 \times 32 \times (-1) = -8064.$$

The constant term is $\boxed{-8064}$.

Q25 2023 Glenwood Prelim Q10(a) · **3 marks**

Find the term independent of x in the expansion of $\left(3x + \frac{1}{x^2}\right)^{9}$.

Criteria	Marks
Correct expression for the general term	1
Equates the power of x to zero to find $r = 3$	1
Evaluates the term independent of x correctly	1

Solution.

The general term is

$$T_{r+1} = \binom{9}{r}(3x)^{9-r}\left(\frac{1}{x^2}\right)^{r} = \binom{9}{r}3^{9-r}\,x^{9-r-2r} = \binom{9}{r}3^{9-r}\,x^{9-3r}.$$

The term is independent of x when $9 - 3r = 0$, so $r = 3$. Then

$$\binom{9}{3}3^{9-3} = \binom{9}{3}3^{6} = 84 \times 729 = 61236.$$

The term independent of x is $\boxed{61236}$.

Q26 2022 Manly Trial Q11(c) · 3 marks

Find the term independent of x in the expansion of $\left(3x^2 - \frac{1}{x}\right)^{12}$.

Criteria	Marks
Correct expression for the general term	1
Equates the power of x to zero to find $r = 4$	1
Evaluates the term independent of x correctly	1

Solution.

The general term is

$$T_{r+1} = \binom{12}{r}\left(3x^2\right)^{12-r}\left(-\frac{1}{x}\right)^r = \binom{12}{r}3^{12-r}(-1)^r x^{24-2r-r} = \binom{12}{r}3^{12-r}(-1)^r x^{24-3r}.$$

The term is independent of x when $24 - 3r = 0$, so $r = 8$. Then

$$\binom{12}{8}3^{12-8}(-1)^8 = \binom{12}{8}3^4 = 495 \times 81 = 40095.$$

The term independent of x is $\boxed{40095}$.

Q27 2020 North Sydney Boys Trial Q11(b) · 3 marks

Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{10}$.

Criteria	Marks
Correct expression for the general term	1
Equates the power of x to zero to find $r = 5$	1
Evaluates the term independent of x correctly (including the sign)	1

Solution.

The general term is

$$T_{r+1} = \binom{10}{r}\left(x^2\right)^{10-r}\left(-\frac{1}{x^2}\right)^r = \binom{10}{r}(-1)^r x^{20-2r-2r} = \binom{10}{r}(-1)^r x^{20-4r}.$$

The term is independent of x when $20 - 4r = 0$, so $r = 5$. Then

$$\binom{10}{5}(-1)^5 = 252 \times (-1) = -252.$$

The term independent of x is $\boxed{-252}$.

Q28 2020 James Ruse Prelim Q1 · **1 mark**

For what positive values of n does the expansion of $\left(x^2 - \dfrac{2}{x^3}\right)^n$ have a constant term? (A) Odd values of n (B) Even values of n (C) Multiples of 3 (D) Multiples of 5

Criteria	Marks
Selects (D): the expansion has a constant term exactly when n is a multiple of 5	1

Solution.

The general term is

$$T_{r+1} = \binom{n}{r}\left(x^2\right)^{n-r}\left(-\frac{2}{x^3}\right)^r = \binom{n}{r}(-2)^r\, x^{2(n-r)-3r} = \binom{n}{r}(-2)^r\, x^{2n-5r}.$$

A constant term exists when the power of x is zero for some integer r with $0 \le r \le n$, i.e. $2n - 5r = 0$, so $r = \dfrac{2n}{5}$. This is a whole number exactly when 5 divides $2n$; since $\gcd(2, 5) = 1$, this means 5 divides n. So a constant term occurs precisely for multiples of 5 (e.g. $n = 5$ gives $r = 2$). The answer is **(D) Multiples of** 5.

5. Coefficients in a product

Q29 2022 Blacktown Boys Prelim Q12(c) · **2 marks**

Find the coefficient of x^8 in the expansion of $(3 - 4x)(1 + x)^{10}$.

Criteria	Marks
Identifies which terms of $(1 + x)^{10}$ produce x^8 (the x^8 and x^7 terms)	1
Correct coefficient -345	1

Solution.

The general term of $(1+x)^{10}$ is $\binom{10}{k}x^k$. The product $(3-4x)(1+x)^{10}$ gives an x^8 term from $3\times$(the x^8 term) and $-4x\times$(the x^7 term):

$$3\binom{10}{8} - 4\binom{10}{7} = 3(45) - 4(120) = 135 - 480 = -345.$$

The coefficient of x^8 is $\boxed{-345}$.

Q30 2023 Ascham Trial Q12(b) · **2 marks**

Find the coefficient of x^2 in the expansion of $(2+x^2)(3-2x)^5$.

Criteria	Marks
Finds the relevant terms of $(3-2x)^5$ (the x^2 term and the constant term)	1
Correct coefficient 2403	1

Solution.

The general term of $(3-2x)^5$ is $\binom{5}{k}3^{5-k}(-2x)^k$. The product $(2+x^2)(3-2x)^5$ gives an x^2 term from $2\times$ (the x^2 term) and $x^2\times$ (the constant term):

$$\begin{aligned}2\binom{5}{2}3^3(-2)^2+\binom{5}{0}3^5 &= 2(10)(27)(4)+243\\ &= 2(1080)+243=2403.\end{aligned}$$

The coefficient of x^2 is $\boxed{2403}$.

Q31 2023 Sydney Grammar Prelim Q13(a) · **2 marks**

Find the coefficient of x^2 in the expansion of $(1+x+x^2)(2+x)^5$.

Criteria	Marks
Finds the x^2, x^1 and x^0 terms of $(2+x)^5$	1
Correct coefficient 192	1

Solution.

The general term of $(2+x)^5$ is $\binom{5}{k}2^{5-k}x^k$. The product $(1+x+x^2)(2+x)^5$ gives an x^2 term from $1\times$ (the x^2 term), $x\times$ (the x^1 term) and $x^2\times$ (the constant term):

$$\begin{aligned}\binom{5}{2}2^3+\binom{5}{1}2^4+\binom{5}{0}2^5 &= 10(8)+5(16)+1(32)\\ &= 80+80+32=192.\end{aligned}$$

The coefficient of x^2 is $\boxed{192}$.

Q32 2022 St George Girls Trial Q11(b) · 3 marks

Find the coefficient of x^9 in the expansion of $(1+2x)(2+x)^{12}$.

Criteria	Marks
Writes the general term of $(2+x)^{12}$ as $\binom{12}{k}2^{12-k}x^k$	1
Identifies the x^9 and x^8 terms as the contributors	1
Correct coefficient 17600	1

Solution.

The general term of $(2+x)^{12}$ is $\binom{12}{k}2^{12-k}x^k$. The product $(1+2x)(2+x)^{12}$ gives an x^9 term from $1\times$ (the x^9 term) and $2x\times$ (the x^8 term):

$$\begin{aligned}\binom{12}{9}2^3+2\binom{12}{8}2^4 &= 220(8)+2(495)(16)\\ &= 1760+2(7920)\\ &= 1760+15840=17600.\end{aligned}$$

The coefficient of x^9 is $\boxed{17600}$.

Q33 2023 Girraween Prelim Q10(e) · 3 marks

Find the coefficient of x^5 in the expansion of $(1-3x+2x^3)(1-2x)^6$.

Criteria	Marks
Writes the general term of $(1-2x)^6$ as $\binom{6}{k}(-2)^k x^k$	1
Identifies the x^5, x^4 and x^2 terms as the contributors	1
Correct coefficient -792	1

Solution.

The general term of $(1-2x)^6$ is $\binom{6}{k}(-2x)^k=\binom{6}{k}(-2)^k x^k$. The product $(1-3x+2x^3)(1-2x)^6$ gives an x^5 term from $1\times$ (the x^5 term), $-3x\times$ (the x^4 term) and $2x^3\times$ (the x^2 term):

$$\begin{aligned}\binom{6}{5}(-2)^5-3\binom{6}{4}(-2)^4+2\binom{6}{2}(-2)^2 &= 6(-32)-3(15)(16)+2(15)(4)\\ &= -192-720+120=-792.\end{aligned}$$

The coefficient of x^5 is $\boxed{-792}$.

Q34 2025 Baulkham Hills Trial Q12(c) · **2 marks**

Find the coefficient of x^2y^6 in the expansion of $\left(1-\frac{y}{x}\right)(x+y)^8$.

Criteria	Marks
Finds the x^2y^6 term and the x^3y^5 term of $(x+y)^8$	1
Correct coefficient -28	1

Solution.

The general term of $(x+y)^8$ is $\binom{8}{k}x^{8-k}y^k$. Writing $\left(1-\frac{y}{x}\right)(x+y)^8 = (x+y)^8 - \frac{y}{x}(x+y)^8$, the x^2y^6 term comes from:

- the x^2y^6 term of $(x+y)^8$ (i.e. $k=6$): coefficient $\binom{8}{6} = 28$;
- $-\frac{y}{x}$ times the x^3y^5 term of $(x+y)^8$ (i.e. $k=5$), since $-\frac{y}{x}\cdot x^3y^5 = -x^2y^6$: coefficient $-\binom{8}{5} = -56$.

Adding: $28 + (-56) = -28$. The coefficient of x^2y^6 is $\boxed{-28}$.

Q35 2024 James Ruse Trial Q12(b) · 3 marks

Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^5\left(2x^2-\frac{3}{x}\right)^6$.

Criteria	Marks
Writes the general term of each factor and the total power of x	1
Finds the pairs of terms whose powers of x cancel	1
Correct constant term -36180	1

Solution.

A general term of $\left(x+\frac{1}{x}\right)^5$ is $\binom{5}{a}x^{5-a}x^{-a}=\binom{5}{a}x^{5-2a}$. A general term of $\left(2x^2-\frac{3}{x}\right)^6$ is $\binom{6}{b}(2x^2)^{6-b}\left(-\frac{3}{x}\right)^b=\binom{6}{b}2^{6-b}(-3)^b x^{12-3b}$. The product has power of x equal to $(5-2a)+(12-3b)=17-2a-3b$. For the term independent of x we need

$$17-2a-3b=0, \qquad 0\le a\le 5,\ 0\le b\le 6,$$

which gives $(a,b)=(1,5)$ and $(a,b)=(4,3)$. The corresponding contributions are:

$$(a,b)=(1,5): \quad \binom{5}{1}\cdot\binom{6}{5}2^1(-3)^5=5\times(-2916)=-14580,$$
$$(a,b)=(4,3): \quad \binom{5}{4}\cdot\binom{6}{3}2^3(-3)^3=5\times(-4320)=-21600.$$

Adding: $-14580+(-21600)=-36180$. The term independent of x is $\boxed{-36180}$.

6. Finding an unknown from a coefficient condition

Q36 2021 Girraween Trial Q2 · 1 mark

In the expansion of $(2x+k)^6$, the coefficients of x and x^2 are equal. What is the value of k?

(A) 5

(B) 6

(C) 11

(D) 12

Criteria	Marks
Correct option (A)	1

Solution.

The general term of $(2x+k)^6$ is $\binom{6}{r}(2x)^{6-r}k^r = \binom{6}{r}2^{6-r}k^r x^{6-r}$.

Coefficient of x^1 (put $6-r=1$, i.e. $r=5$): $\binom{6}{5}2^1k^5 = 12k^5$.

Coefficient of x^2 (put $6-r=2$, i.e. $r=4$): $\binom{6}{4}2^2k^4 = 60k^4$.

Setting them equal (with $k \neq 0$):

$$12k^5 = 60k^4 \implies k = 5.$$

The answer is **(A)** 5.

Q37 2020 North Sydney Boys Trial Q2 · **1 mark**

The coefficient of x^3 in the expansion of $(a+x)^6$ is 1280. What is the value of a?

(A) 64

(B) 4

(C) 40

(D) $\sqrt[3]{32}$

Criteria	**Marks**
Correct option (B)	1

Solution.

The term in x^3 of $(a+x)^6$ is $\binom{6}{3}a^3x^3 = 20a^3x^3$, so the coefficient is $20a^3$.

$$20a^3 = 1280 \implies a^3 = 64 \implies a = 4.$$

The answer is **(B)** 4.

Q38 2023 Hunters Hill Trial Q11(a) · 3 marks

Find the value of the constant a such that $8750x^4$ is a term in the binomial expansion of $(a+2x)^7$.

Criteria	Marks
Writes the general term of $(a+2x)^7$	1
Identifies the x^4 term and forms $560a^3 = 8750$	1
Solves for $a = \frac{5}{2}$	1

Solution.

The general term of $(a+2x)^7$ is

$$\binom{7}{r}a^{7-r}(2x)^r = \binom{7}{r}a^{7-r}2^r x^r.$$

The x^4 term has $r=4$: $\binom{7}{4}a^3 2^4 x^4 = 35\cdot 16\cdot a^3x^4 = 560a^3x^4$.

This term equals $8750x^4$, so

$$560a^3 = 8750 \implies a^3 = \frac{8750}{560} = \frac{125}{8} \implies a = \frac{5}{2}.$$

Q39 2021 Sydney Grammar Prelim Q6(a) · 3 marks

In the expansion of $(2+x)^n$, the coefficients of x^5 and x^6 are equal. Find the value of n.

Criteria	Marks
Writes the coefficients of x^5 and x^6 as $\binom{n}{5}2^{n-5}$ and $\binom{n}{6}2^{n-6}$	1
Forms the equation and simplifies the binomial ratio	1
Solves for $n = 17$	1

Solution.

The coefficient of x^r in $(2+x)^n$ is $\binom{n}{r}2^{n-r}$. Equating the coefficients of x^5 and x^6:

$$\binom{n}{5}2^{n-5} = \binom{n}{6}2^{n-6}.$$

Dividing both sides by $\binom{n}{5}2^{n-6}$ and using $\frac{\binom{n}{6}}{\binom{n}{5}} = \frac{n-5}{6}$:

$$2 = \frac{n-5}{6} \implies n-5 = 12 \implies n = 17.$$

Q40 2019 HSC Q13(b) · 3 marks

In the expansion of $(5x+2)^{20}$, the coefficients of x^k and x^{k+1} are equal. What is the value of k?

Criteria	Marks
Writes the coefficient of x^k as $\binom{20}{k}5^k 2^{20-k}$	1
Forms the equation for equal coefficients and reduces the ratio	1
Solves for $k = 14$	1

Solution.

The coefficient of x^r in $(5x+2)^{20}$ is $\binom{20}{r}5^r 2^{20-r}$. Setting the coefficients of x^k and x^{k+1} equal:

$$\binom{20}{k}5^k 2^{20-k} = \binom{20}{k+1}5^{k+1}2^{19-k}.$$

Dividing by $\binom{20}{k}5^k 2^{19-k}$ and using $\dfrac{\binom{20}{k+1}}{\binom{20}{k}} = \dfrac{20-k}{k+1}$:

$$2 = \frac{20-k}{k+1}\cdot 5 \implies 2(k+1) = 5(20-k) \implies 2k+2 = 100-5k \implies 7k = 98 \implies k = 14.$$

Q41 2025 Caringbah Trial Q14(a) · 3 marks

Given that k is a non-zero constant and n is a positive integer, then

$$(1+kx)^n = 1 + 40x + 120k^2x^2 + \cdots$$

Find the values of k and n.

Criteria	Marks
Compares the x^2 coefficient to get $\binom{n}{2} = 120$, hence $n(n-1) = 240$	1
Solves for $n = 16$	1
Uses the x coefficient $nk = 40$ to get $k = \frac{5}{2}$	1

Solution.

Expanding $(1+kx)^n = 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + \cdots = 1 + nk\,x + \dfrac{n(n-1)}{2}k^2x^2 + \cdots$.

Comparing the x^2 coefficients: $\dfrac{n(n-1)}{2}k^2 = 120k^2$. Since $k \neq 0$,

$$\frac{n(n-1)}{2} = 120 \implies n(n-1) = 240 \implies n = 16 \quad (n > 0).$$

Comparing the x coefficients: $nk = 40$, so $16k = 40 \implies k = \dfrac{5}{2}$.

Therefore $n = 16$ and $k = \dfrac{5}{2}$.

Q42 2023 Hornsby Girls Trial Q11(d) · 3 marks

In the expansion of $(1+2x)^n$, the coefficients of x^6 and x^8 are in the ratio $7:3$. Find the value of n.

Criteria	Marks
Writes the coefficients of x^6 and x^8 as $\binom{n}{6}2^6$ and $\binom{n}{8}2^8$ and forms the ratio	1
Simplifies the binomial ratio to the quadratic $(n-6)(n-7)=6$	1
Solves and selects $n=9$ (rejecting $n=4$, since x^8 requires $n\geq 8$)	1

Solution.

The general term of $(1+2x)^n$ is $\binom{n}{r}(2x)^r=\binom{n}{r}2^r x^r$, so the coefficient of x^6 is $\binom{n}{6}2^6$ and the coefficient of x^8 is $\binom{n}{8}2^8$. The ratio condition gives

$$\frac{\binom{n}{6}2^6}{\binom{n}{8}2^8}=\frac{7}{3}.$$

Writing the coefficients as factorials, $\binom{n}{6}=\dfrac{n!}{6!\,(n-6)!}$ and $\binom{n}{8}=\dfrac{n!}{8!\,(n-8)!}$, so

$$\frac{\binom{n}{6}}{\binom{n}{8}}=\frac{8!\,(n-8)!}{6!\,(n-6)!}=\frac{8\cdot 7}{(n-6)(n-7)}=\frac{56}{(n-6)(n-7)}.$$

Therefore

$$\frac{56}{(n-6)(n-7)}\cdot\frac{1}{2^2}=\frac{7}{3}\implies\frac{14}{(n-6)(n-7)}=\frac{7}{3}\implies(n-6)(n-7)=6.$$

Expanding, $n^2-13n+36=0$, i.e. $(n-9)(n-4)=0$, so $n=9$ or $n=4$. Since the expansion must contain an x^8 term we need $n\geq 8$, so

$$n=9.$$

Q43 2021 Knox Prelim Q13(d) · 3 marks

In the expansion of $(ax+b)^6$, where a and b are constants, the coefficient of x^4 is 2160 and the coefficient of x^5 is -576. Find a and b.

Criteria	Marks
Writes the x^4 and x^5 coefficients as $15a^4b^2$ and $6a^5b$	1
Divides the two equations to relate a and b	1
Solves for $(a,b)=(2,-3)$ or $(-2,3)$	1

Solution.

The general term of $(ax+b)^6$ is $\binom{6}{r}(ax)^{6-r}b^r = \binom{6}{r}a^{6-r}b^r x^{6-r}$.

Coefficient of x^5 (put $6-r=5$, i.e. $r=1$): $\binom{6}{1}a^5b = 6a^5b$.

Coefficient of x^4 (put $6-r=4$, i.e. $r=2$): $\binom{6}{2}a^4b^2 = 15a^4b^2$.

So $15a^4b^2 = 2160$ and $6a^5b = -576$. Dividing the second by the first:

$$\frac{6a^5b}{15a^4b^2} = \frac{-576}{2160} \implies \frac{2a}{5b} = -\frac{4}{15} \implies a = -\frac{2}{3}b.$$

Substituting into $15a^4b^2 = 2160$ (so $a^4b^2 = 144$):

$$15\left(\frac{2}{3}b\right)^4 b^2 = 2160 \implies 15\cdot\frac{16}{81}b^6 = 2160 \implies b^6 = 729 \implies b = \pm 3.$$

If $b=3$ then $a=-2$; if $b=-3$ then $a=2$. Checking $6a^5b=-576$ confirms both:

$$(a,b)=(2,-3) \quad \text{or} \quad (a,b)=(-2,3).$$

Coefficients and identities

7. Manipulating binomial coefficients

Q44 2023 SCEGGS Prelim Q10(a) · **2 marks**

Prove Pascal's identity, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, given that n and k are natural numbers with $n > k$.

Criteria	**Marks**
Writes the two right-hand terms in factorial form and takes a common denominator	1
Simplifies to $\dfrac{n!}{k!(n-k)!} = \binom{n}{k}$ to complete the proof	1

Solution.

Start from the right-hand side and write each term with factorials:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}.$$

Take a common denominator of $k!\,(n-k)!$. Multiply the first fraction by $\dfrac{k}{k}$ and the second by $\dfrac{n-k}{n-k}$:

$$= \frac{k\,(n-1)!}{k!\,(n-k)!} + \frac{(n-k)\,(n-1)!}{k!\,(n-k)!} = \frac{(n-1)!\,\big[k + (n-k)\big]}{k!\,(n-k)!} = \frac{(n-1)!\,n}{k!\,(n-k)!}.$$

Since $n\,(n-1)! = n!$, this equals

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k},$$

which is the left-hand side. Hence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. ■

Q45 2021 SCEGGS Prelim Q5(e) · **2 marks**

Use factorial notation to show that $\binom{8}{5} = \binom{7}{4} + \binom{7}{5}$.

Criteria	Marks
Writes $\binom{7}{4}$ and $\binom{7}{5}$ in factorial form and combines over the common denominator $5!\,3!$	1
Simplifies to $\dfrac{8!}{5!\,3!} = \binom{8}{5}$	1

Solution.

Write the two right-hand terms with factorials:

$$\binom{7}{4} + \binom{7}{5} = \frac{7!}{4!\,3!} + \frac{7!}{5!\,2!}.$$

Use the common denominator $5!\,3!$. Multiply the first fraction by $\dfrac{5}{5}$ and the second by $\dfrac{3}{3}$:

$$= \frac{5 \cdot 7!}{5!\,3!} + \frac{3 \cdot 7!}{5!\,3!} = \frac{7!\,(5+3)}{5!\,3!} = \frac{8 \cdot 7!}{5!\,3!} = \frac{8!}{5!\,3!} = \binom{8}{5}.$$

Hence $\binom{8}{5} = \binom{7}{4} + \binom{7}{5}$ (both sides equal 56). ■

Q46 2021 Presbyterian Prelim Q7(c) · **2 marks**

Show that $\binom{n+1}{r} \div \binom{n}{r-1} = \dfrac{n+1}{r}$.

Criteria	Marks
Writes both coefficients in factorial form and forms the quotient	1
Cancels the factorials to obtain $\dfrac{n+1}{r}$	1

Solution.

Write each coefficient with factorials:

$$\binom{n+1}{r} = \frac{(n+1)!}{r!\,(n+1-r)!}, \qquad \binom{n}{r-1} = \frac{n!}{(r-1)!\,(n-r+1)!}.$$

Dividing is multiplying by the reciprocal (note $n+1-r = n-r+1$):

$$\binom{n+1}{r} \div \binom{n}{r-1} = \frac{(n+1)!}{r!\,(n-r+1)!} \times \frac{(r-1)!\,(n-r+1)!}{n!}.$$

The $(n-r+1)!$ factors cancel, and using $(n+1)! = (n+1)\,n!$ and $r! = r\,(r-1)!$:

$$= \frac{(n+1)\,n!}{r\,(r-1)!} \times \frac{(r-1)!}{n!} = \frac{n+1}{r}.$$

Hence $\binom{n+1}{r} \div \binom{n}{r-1} = \dfrac{n+1}{r}$. ■

Q47 2023 Barker Prelim Q12(d) · **2 marks**

Prove that $\binom{2n}{2} - 2\binom{n}{2} = n^2$.

Criteria	**Marks**
Writes $\binom{2n}{2} = \frac{2n(2n-1)}{2}$ and $\binom{n}{2} = \frac{n(n-1)}{2}$	1
Simplifies the difference to n^2	1

Solution.

Use $\binom{m}{2} = \frac{m(m-1)}{2}$ for each term:

$$\binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1) = 2n^2 - n,$$

$$2\binom{n}{2} = 2 \cdot \frac{n(n-1)}{2} = n(n-1) = n^2 - n.$$

Subtracting,

$$\binom{2n}{2} - 2\binom{n}{2} = (2n^2 - n) - (n^2 - n) = 2n^2 - n - n^2 + n = n^2.$$

Hence $\binom{2n}{2} - 2\binom{n}{2} = n^2$. ■

Q48 2023 HSC Q12(d) · **2 marks**

It is known that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for all integers such that $1 \leq r \leq n-1$. (Do NOT prove this.)

Find ONE possible set of values for p and q such that

$$\binom{2022}{80} + \binom{2022}{81} + \binom{2023}{1943} = \binom{p}{q}.$$

Criteria	**Marks**
Uses Pascal's identity to combine $\binom{2022}{80} + \binom{2022}{81} = \binom{2023}{81}$ (and/or uses symmetry on the third term)	1
Completes to $\binom{2024}{81}$, giving $p = 2024$, $q = 81$	1

Solution.

Combine the first two terms using the given identity (Pascal's rule) with $n = 2023$, $r = 81$:

$$\binom{2022}{80} + \binom{2022}{81} = \binom{2023}{81}.$$

For the third term, use the symmetry $\binom{n}{r} = \binom{n}{n-r}$. Since $2023 - 1943 = 80$,

$$\binom{2023}{1943} = \binom{2023}{80}.$$

The left-hand side is now $\binom{2023}{81} + \binom{2023}{80}$. Applying Pascal's rule again with $n = 2024$, $r = 81$:

$$\binom{2023}{80} + \binom{2023}{81} = \binom{2024}{81}.$$

Therefore $p = 2024$ and $q = 81$ is one possible set of values. (Since $\binom{2024}{81} = \binom{2024}{1943}$, the pair $p = 2024$, $q = 1943$ also works.)

Q49 2025 HSC Q13(e) · 3 marks

The Pascal's triangle relation can be expressed as $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ (Do NOT prove this).

(i) Show that $\binom{m}{R} = \binom{m+1}{R+1} - \binom{m}{R+1}$. (1 mark)

(ii) Hence, or otherwise, prove that

$$\binom{2000}{2000} + \binom{2001}{2000} + \binom{2002}{2000} + \cdots + \binom{2050}{2000} = \binom{2051}{2001}.$$ (2 marks)

Criteria	Marks
(i) Substitutes $n = m+1, r = R+1$ into the given relation and rearranges to the required form	1
(ii) Applies part (i) to write each term as a difference (telescoping setup)	1
(ii) Cancels the telescoping sum to reach $\binom{2051}{2001}$	1

Solution.

(i) In the given relation $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, put $n = m+1$ and $r = R+1$:

$$\binom{m+1}{R+1} = \binom{m}{R} + \binom{m}{R+1}.$$

Rearranging for $\binom{m}{R}$ gives

$$\binom{m}{R} = \binom{m+1}{R+1} - \binom{m}{R+1}. \quad \blacksquare$$

(ii) Each term on the left has lower index 2000. Apply part (i) with $R = 2000$ (so $R+1 = 2001$) and m running from 2000 to 2050:

$$\binom{m}{2000} = \binom{m+1}{2001} - \binom{m}{2001}.$$

So the sum becomes a telescoping difference:

$$\sum_{m=2000}^{2050} \binom{m}{2000} = \left(\binom{2001}{2001} - \binom{2000}{2001}\right) + \left(\binom{2002}{2001} - \binom{2001}{2001}\right) + \left(\binom{2003}{2001} - \binom{2002}{2001}\right) + \cdots + \left(\binom{2051}{2001} - \binom{2050}{2001}\right).$$

All intermediate terms cancel in pairs, leaving only the last positive term and the first negative term:

$$= \binom{2051}{2001} - \binom{2000}{2001}.$$

Now $\binom{2000}{2001} = 0$ (you cannot choose 2001 objects from 2000), so the sum is

$$\binom{2000}{2000} + \binom{2001}{2000} + \cdots + \binom{2050}{2000} = \binom{2051}{2001}. \qquad \blacksquare$$

8. Summing and evaluating coefficients by substitution

Q50 2020 North Sydney Boys Trial Q14(a) · 2 marks

Show that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Criteria	Marks
States the expansion of $(1+x)^n$ and substitutes $x = 1$	1
Concludes the sum equals 2^n	1

Solution.

The binomial theorem gives

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Substituting $x = 1$ makes every power of x equal to 1:

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Since $(1+1)^n = 2^n$,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n. \qquad \blacksquare$$

Q51 2021 Sydney Girls Prelim Q3(a) · **1 mark**

Given the binomial expansion $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$, find the value of

$$\binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \cdots + \binom{100}{100}.$$

Criteria	**Marks**
Substitutes $x = 1$, $n = 100$ to obtain 2^{100}	1

Solution.

Put $x = 1$ in $(1+x)^{100}$. Every power of x becomes 1, so

$$\binom{100}{0} + \binom{100}{1} + \binom{100}{2} + \cdots + \binom{100}{100} = (1+1)^{100} = 2^{100}.$$

Q52 2023 Blacktown Boys Prelim Q10 · **1 mark**

By considering the binomial expansion of $(1+x)^{10}$, what is the value of

$$5\binom{10}{1}+5^2\binom{10}{2}+5^3\binom{10}{3}+\cdots+5^{10}\binom{10}{10}?$$

(A) 4^{10}

(B) 6^{10}

(C) $4^{10}-1$

(D) $6^{10}-1$

Criteria	Marks
Correct answer (D)	1

Solution.

The binomial theorem gives

$$(1+x)^{10}=\binom{10}{0}+\binom{10}{1}x+\binom{10}{2}x^2+\cdots+\binom{10}{10}x^{10}.$$

Substitute $x=5$:

$$6^{10}=(1+5)^{10}=\binom{10}{0}+5\binom{10}{1}+5^2\binom{10}{2}+\cdots+5^{10}\binom{10}{10}.$$

The required sum omits the $k=0$ term $\binom{10}{0}=1$, so

$$5\binom{10}{1}+5^2\binom{10}{2}+\cdots+5^{10}\binom{10}{10}=6^{10}-1.$$

This is option **(D)**.

Q53 2024 Girraween Prelim Q11(c) · 5 marks

(i) Write down the binomial expansion of $(1+x)^n$.

(ii) Hence, show that $\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n-1$.

(iii) By substituting a suitable value for x in the expansion $(1+x)^n$, show that

$$1+2\binom{n}{1}+4\binom{n}{2}+\cdots+2^{n-1}\binom{n}{n-1}=3^n-2^n.$$

Criteria	Marks
(i) Correct expansion of $(1+x)^n$	1
(ii) Substitutes $x=1$ to get 2^n	1
(ii) Subtracts the $\binom{n}{0}$ term to reach 2^n-1	1
(iii) Substitutes $x=2$ into $(1+x)^n$	1
(iii) Removes the last term $2^n\binom{n}{n}$ to reach 3^n-2^n	1

Solution.

(i) By the binomial theorem,

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n.$$

(ii) Substitute $x=1$:

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}.$$

Since $\binom{n}{0}=1$, subtract 1 from both sides:

$$\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n-1. \qquad \blacksquare$$

(iii) Substitute $x=2$ into the expansion in (i):

$$3^n=(1+2)^n=\binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots+2^{n-1}\binom{n}{n-1}+2^n\binom{n}{n}.$$

Now $\binom{n}{0}=1$ (the first term) and the final term is $2^n\binom{n}{n}=2^n$. Moving that last term to the left,

$$1+2\binom{n}{1}+4\binom{n}{2}+\cdots+2^{n-1}\binom{n}{n-1}=3^n-2^n. \qquad \blacksquare$$

Q54 2021 Scots Prelim Q10(b) · 3 marks

(i) Using the binomial theorem, obtain an expression for $(1+x)^{2n} + (1-x)^{2n}$.

(ii) Hence, show that $1 + \binom{20}{2} + \binom{20}{4} + \cdots + \binom{20}{20} = 2^{19}$.

Criteria	Marks
(i) Expands $(1+x)^{2n}$ and $(1-x)^{2n}$	1
(i) Adds to leave only the even-power terms	1
(ii) Substitutes $x = 1$, $n = 10$ to reach 2^{19}	1

Solution.

(i) By the binomial theorem,

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n}x^{2n},$$
$$(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \cdots + \binom{2n}{2n}x^{2n}.$$

Adding, the odd-power terms cancel and the even-power terms double:

$$(1+x)^{2n} + (1-x)^{2n} = 2\left[\binom{2n}{0} + \binom{2n}{2}x^2 + \binom{2n}{4}x^4 + \cdots + \binom{2n}{2n}x^{2n}\right].$$

(ii) Take $2n = 20$ (so $n = 10$) and substitute $x = 1$. The left side is $2^{20} + 0^{20} = 2^{20}$, giving

$$2^{20} = 2\left[\binom{20}{0} + \binom{20}{2} + \binom{20}{4} + \cdots + \binom{20}{20}\right].$$

Dividing by 2 and using $\binom{20}{0} = 1$,

$$1 + \binom{20}{2} + \binom{20}{4} + \cdots + \binom{20}{20} = 2^{19}. \qquad \blacksquare$$

Q55 2023 Sydney Tech Trial Q13(e) · 3 marks

Given that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n \qquad \text{(do not show this)},$$

and by selecting two appropriate values of x, show that

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1} \qquad \text{when } n \text{ is odd.}$$

Criteria	Marks
Substitutes $x = 1$ to get $\sum \binom{n}{k} = 2^n$	1
Substitutes $x = -1$ to get the alternating sum $= 0$	1
Adds the two results and halves to reach 2^{n-1}	1

Solution.

Substitute the two values $x = 1$ and $x = -1$ into $(1+x)^n$.

With $x = 1$:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}. \tag{1}$$

With $x = -1$:

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}. \tag{2}$$

Adding (1) and (2), the odd-index terms cancel and the even-index terms double:

$$2^n = 2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\right].$$

Since n is odd, the largest even index is $n-1$, so the bracket ends at $\binom{n}{n-1}$. Dividing by 2,

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1}. \qquad \blacksquare$$

Q56 2024 Cheltenham Girls Trial Q14(d) · **3 marks**

By considering the binomial expansion $(1+x)^n$, prove that

$$\binom{n}{1} + 3\binom{n}{2} + 9\binom{n}{3} + \cdots + 3^{n-1}\binom{n}{n} = \frac{1}{3}\left(2^{2n} - 1\right).$$

Criteria	Marks
Expands $(1+x)^n$ and substitutes $x = 3$	1
Uses $\binom{n}{0} = 1$ to obtain $2^{2n} - 1 = 3\binom{n}{1} + 9\binom{n}{2} + \cdots + 3^n\binom{n}{n}$	1
Divides by 3 to reach the required identity	1

Solution.

By the binomial theorem,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

Substitute $x = 3$, noting $(1+3)^n = 4^n = 2^{2n}$:

$$2^{2n} = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + \cdots + 3^n\binom{n}{n}.$$

Since $\binom{n}{0} = 1$, subtract 1 from both sides:

$$2^{2n} - 1 = 3\binom{n}{1} + 9\binom{n}{2} + 27\binom{n}{3} + \cdots + 3^n\binom{n}{n}.$$

Every term on the right has a factor of 3; dividing through by 3,

$$\frac{1}{3}\left(2^{2n} - 1\right) = \binom{n}{1} + 3\binom{n}{2} + 9\binom{n}{3} + \cdots + 3^{n-1}\binom{n}{n}. \qquad \blacksquare$$

9. Identities via differentiation and integration

Q57 2023 Penrith Trial Q14(d) · **2 marks**

By differentiating both sides of the expansion of $(1+x)^n$ with respect to x, prove that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n\,2^{n-1}.$$

Criteria	Marks
Differentiates the expansion to get $n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}$	1
Substitutes $x = 1$ to obtain the required identity	1

Solution.

Start from the binomial expansion

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Differentiating both sides with respect to x:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

Now substitute $x = 1$:

$$n\,2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n},$$

which is the required identity.

Q58 2021 Knox Trial Q14(c)(i) · 4 marks

By considering the expansion of $(1+x)^{2n}$, show that

$$\binom{2n}{1} + 2\binom{2n}{2} + 3\binom{2n}{3} + \cdots + 2n\binom{2n}{2n} = n\,2^{2n}.$$

Criteria	Marks
Writes the expansion of $(1+x)^{2n}$	1
Differentiates both sides with respect to x	1
Substitutes $x = 1$	1
Simplifies $2n \cdot 2^{2n-1} = n\,2^{2n}$ to reach the identity	1

Solution.

The binomial expansion is

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n}x^{2n}.$$

Differentiating both sides with respect to x:

$$2n(1+x)^{2n-1} = \binom{2n}{1} + 2\binom{2n}{2}x + 3\binom{2n}{3}x^2 + \cdots + 2n\binom{2n}{2n}x^{2n-1}.$$

Substituting $x = 1$:

$$2n\,2^{2n-1} = \binom{2n}{1} + 2\binom{2n}{2} + 3\binom{2n}{3} + \cdots + 2n\binom{2n}{2n}.$$

Since $2n \cdot 2^{2n-1} = n \cdot 2 \cdot 2^{2n-1} = n\,2^{2n}$, we obtain

$$\binom{2n}{1} + 2\binom{2n}{2} + 3\binom{2n}{3} + \cdots + 2n\binom{2n}{2n} = n\,2^{2n},$$

as required.

Q59 2022 Baulkham Hills Prelim Q14(c) · 4 marks

By considering the expansion of $(1+x)^n$ and differentiating, show the following identity, given that n is even.

$$n\,2^{n-1} = 2\left[\binom{n}{1} + 3\binom{n}{3} + 5\binom{n}{5} + \cdots + (n-1)\binom{n}{n-1}\right].$$

Criteria	Marks
Differentiates $(1+x)^n$ correctly	1
Substitutes $x=1$ to get $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$	1
Substitutes $x=-1$ to get $0 = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots$	1
Adds the two results and factors to reach the identity	1

Solution.

Start from

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

and differentiate both sides with respect to x:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

Substituting $x = 1$:

$$n\,2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}. \tag{A}$$

Substituting $x = -1$ (with n even, so $n(1+(-1))^{n-1} = 0$):

$$0 = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots - n\binom{n}{n}. \tag{B}$$

Adding (A) and (B), the even-position terms cancel and each odd-position term doubles:

$$n\,2^{n-1} = 2\binom{n}{1} + 6\binom{n}{3} + 10\binom{n}{5} + \cdots + 2(n-1)\binom{n}{n-1}.$$

Taking out the common factor of 2:

$$n\,2^{n-1} = 2\left[\binom{n}{1} + 3\binom{n}{3} + 5\binom{n}{5} + \cdots + (n-1)\binom{n}{n-1}\right],$$

as required.

Note. The paper's printed target line reads $n\,2^n = \binom{n}{1} + 3\binom{n}{3} + 5\binom{n}{5} + \cdots + (n-1)\binom{n}{n-1}$, which drops the factor of 2 while keeping the left side and is off by a factor of 4. Checking the bracket at $n=2$ gives $\binom{2}{1} = 2$, whereas $n\,2^n = 2\cdot 2^2 = 8$; the correct value is $n\,2^{n-2} = 2\cdot 2^0 = 2$. Hence the corrected identity is $n\,2^{n-1} = 2\left[\,\cdots\right]$ as proved above (equivalently $\binom{n}{1} + 3\binom{n}{3} + \cdots + (n-1)\binom{n}{n-1} = n\,2^{n-2}$).

Q60 ★ 2021 Cheltenham Girls Trial Q7(c) · 4 marks

★ *Part (ii) uses integration, a Year 12 topic.*

Consider the binomial expansion

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n = (1+x)^n.$$

(i) Show that

$$1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0.$$

(ii) Hence, by integrating both sides with respect to x, show that

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}.$$

Criteria	Marks
(i) Substitutes $x = -1$ into the expansion to show the alternating sum is 0	1
(ii) Integrates both sides to get $x + \frac{1}{2}\binom{n}{1}x^2 + \cdots = \frac{(1+x)^{n+1}}{n+1} + C$	1
(ii) Finds $C = -\frac{1}{n+1}$ by substituting $x = 0$	1
(ii) Substitutes $x = -1$ and simplifies to reach the required identity	1

Solution.

(i) Substitute $x = -1$ into

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (1+x)^n:$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = (1-1)^n = 0.$$

Since $\binom{n}{0} = 1$, this is

$$1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0,$$

as required.

(ii) Integrate both sides of the original expansion with respect to x:

$$x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \cdots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{(1+x)^{n+1}}{n+1} + C.$$

To find C, put $x = 0$: the left side is 0, so

$$0 = \frac{(1+0)^{n+1}}{n+1} + C = \frac{1}{n+1} + C \implies C = -\frac{1}{n+1}.$$

Hence

$$x + \frac{1}{2}\binom{n}{1}x^2 + \cdots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1}.$$

Now put $x = -1$. The right side becomes $\dfrac{(1-1)^{n+1}}{n+1} - \dfrac{1}{n+1} = -\dfrac{1}{n+1}$, and the left side is

$$-1 + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \cdots + \frac{(-1)^{n+1}}{n+1}\binom{n}{n} = -\frac{1}{n+1}.$$

Multiplying through by -1:

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + \frac{(-1)^{n}}{n+1}\binom{n}{n} = \frac{1}{n+1},$$

as required.

Q61 2021 James Ruse Prelim Q42(b) · **2 marks**

Use the binomial expansion of $(1+x)^n$ to prove that

$$\binom{n}{1} + 4\binom{n}{2} + 12\binom{n}{3} + \cdots + r\binom{n}{r}2^{r-1} + \cdots + n\binom{n}{n}2^{n-1} = n\,3^{n-1}.$$

Criteria	Marks
Differentiates $(1+x)^n$ to get $n(1+x)^{n-1} = \sum r\binom{n}{r}x^{r-1}$	1
Substitutes $x = 2$ and simplifies to $n\,3^{n-1}$	1

Solution.

Differentiate both sides of

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

with respect to x:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}x^{r-1}.$$

Substitute $x = 2$:

$$n(1+2)^{n-1} = \binom{n}{1} + 2\binom{n}{2}\cdot 2 + 3\binom{n}{3}\cdot 2^2 + \cdots + n\binom{n}{n}\cdot 2^{n-1}.$$

The general term on the right is $r\binom{n}{r}2^{r-1}$; the first few coefficients are $1,\ 2\cdot 2 = 4,\ 3\cdot 2^2 = 12, \ldots$ Hence

$$\binom{n}{1} + 4\binom{n}{2} + 12\binom{n}{3} + \cdots + n\binom{n}{n}2^{n-1} = n\,3^{n-1},$$

as required.

10. Identities by comparing coefficients

Q62 2023 Cheltenham Girls Trial Q12(c) · **3 marks**

(i) Find the value of the coefficient of x^4 in the expansion of $(1+x)^3(1+x)^6$. (1 mark)

(ii) Using the expansion of $(1+x)^3(1+x)^6$, show that

$$\binom{3}{0}\binom{6}{6} + \binom{3}{1}\binom{6}{5} + \binom{3}{2}\binom{6}{4} + \binom{3}{3}\binom{6}{3} = \binom{9}{6}.$$ (2 marks)

Criteria	Marks
(i) Coefficient of x^4 is $\binom{9}{4} = 126$	1
(ii) Collects the x^3 coefficient of the product as a sum of $\binom{3}{k}\binom{6}{6-k}$	1
(ii) Equates it to the x^3 coefficient $\binom{9}{6}$ of $(1+x)^9$	1

Solution.

(i) Since $(1+x)^3(1+x)^6 = (1+x)^9$, the coefficient of x^4 is

$$\binom{9}{4} = 126.$$

(ii) Note $\binom{6}{6} = \binom{6}{0}$, $\binom{6}{5} = \binom{6}{1}$, $\binom{6}{4} = \binom{6}{2}$, so the left-hand side is

$$\binom{3}{0}\binom{6}{0} + \binom{3}{1}\binom{6}{1} + \binom{3}{2}\binom{6}{2} + \binom{3}{3}\binom{6}{3}.$$

Consider the coefficient of x^3 on both sides of $(1+x)^3(1+x)^6 = (1+x)^9$.

On the right, the coefficient of x^3 in $(1+x)^9$ is $\binom{9}{3} = \binom{9}{6}$ (using $\binom{9}{3} = \binom{9}{6}$).

On the left, a term in x^3 is formed by taking x^k from $(1+x)^3$ and x^{3-k} from $(1+x)^6$, for $k = 0, 1, 2, 3$. Its coefficient is

$$\sum_{k=0}^{3} \binom{3}{k}\binom{6}{3-k} = \binom{3}{0}\binom{6}{3} + \binom{3}{1}\binom{6}{2} + \binom{3}{2}\binom{6}{1} + \binom{3}{3}\binom{6}{0}.$$

Equating the two coefficients of x^3 and reversing the order of the sum gives

$$\binom{3}{0}\binom{6}{6} + \binom{3}{1}\binom{6}{5} + \binom{3}{2}\binom{6}{4} + \binom{3}{3}\binom{6}{3} = \binom{9}{6},$$

as required. (Numerically both sides equal $1 \cdot 1 + 3 \cdot 6 + 3 \cdot 15 + 1 \cdot 20 = 84$.)

Q63 2023 Glenwood Prelim Q10(c) · 3 marks

By expanding both sides of the identity $(1+x)^{11} = (1+x)^8(1+x)^3$, prove that

$$\binom{11}{3} = \binom{8}{0}\binom{3}{3} + \binom{8}{1}\binom{3}{2} + \binom{8}{2}\binom{3}{1} + \binom{8}{3}\binom{3}{0}.$$

Criteria	Marks
States the coefficient of x^3 in $(1+x)^{11}$ is $\binom{11}{3}$	1
Forms the coefficient of x^3 in $(1+x)^8(1+x)^3$ as $\sum_k \binom{8}{3-k}\binom{3}{k}$	1
Equates the two coefficients to obtain the identity	1

Solution.

Compare the coefficient of x^3 on each side of $(1+x)^{11} = (1+x)^8(1+x)^3$.

Right-hand product. A term in x^3 comes from taking x^{3-k} from $(1+x)^8$ and x^k from $(1+x)^3$, for $k = 0, 1, 2, 3$. Its coefficient is

$$\sum_{k=0}^{3} \binom{8}{3-k}\binom{3}{k} = \binom{8}{3}\binom{3}{0} + \binom{8}{2}\binom{3}{1} + \binom{8}{1}\binom{3}{2} + \binom{8}{0}\binom{3}{3}.$$

Left-hand side. The coefficient of x^3 in $(1+x)^{11}$ is $\binom{11}{3}$.

Equating the coefficients of x^3 gives

$$\binom{11}{3} = \binom{8}{0}\binom{3}{3} + \binom{8}{1}\binom{3}{2} + \binom{8}{2}\binom{3}{1} + \binom{8}{3}\binom{3}{0},$$

as required. (Check: $165 = 1 \cdot 1 + 8 \cdot 3 + 28 \cdot 3 + 56 \cdot 1 = 1 + 24 + 84 + 56 = 165$.)

Q64 2021 Scots Trial Q13(b) · 3 marks

(i) Given the identity $(1+x)^{m+n} = (1+x)^m(1+x)^n$, prove that

$$\binom{m+n}{4} = \binom{n}{4} + \binom{n}{3}\binom{m}{1} + \binom{n}{2}\binom{m}{2} + \binom{n}{1}\binom{m}{3} + \binom{m}{4}.\text{(2 marks)}$$

(ii) Hence, by showing working, find the value of

$$\binom{16}{3}\binom{4}{1} + \binom{16}{2}\binom{4}{2} + \binom{16}{1}\binom{4}{3}.\text{(1 mark)}$$

Criteria	Marks
(i) Forms the coefficient of x^4 in $(1+x)^m(1+x)^n$ as $\sum_k \binom{m}{k}\binom{n}{4-k}$ and equates to $\binom{m+n}{4}$	1
(i) Writes out all five terms to complete the proof	1
(ii) Uses $m = 16$, $n = 4$ and isolates the three middle terms to get 3024	1

Solution.

(i) Compare the coefficient of x^4 on both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.

On the left, the coefficient of x^4 in $(1+x)^{m+n}$ is $\binom{m+n}{4}$.

On the right, a term in x^4 comes from taking x^k from $(1+x)^m$ and x^{4-k} from $(1+x)^n$, for $k = 0, 1, 2, 3, 4$. Its coefficient is

$$\sum_{k=0}^{4} \binom{m}{k}\binom{n}{4-k} = \binom{m}{0}\binom{n}{4} + \binom{m}{1}\binom{n}{3} + \binom{m}{2}\binom{n}{2} + \binom{m}{3}\binom{n}{1} + \binom{m}{4}\binom{n}{0}.$$

Since $\binom{m}{0} = \binom{n}{0} = 1$, equating the two coefficients of x^4 gives

$$\binom{m+n}{4} = \binom{n}{4} + \binom{n}{3}\binom{m}{1} + \binom{n}{2}\binom{m}{2} + \binom{n}{1}\binom{m}{3} + \binom{m}{4},$$

as required.

(ii) Put $m = 16$ and $n = 4$ in the identity from (i). Then $m + n = 20$ and

$$\binom{20}{4} = \binom{4}{4} + \binom{4}{3}\binom{16}{1} + \binom{4}{2}\binom{16}{2} + \binom{4}{1}\binom{16}{3} + \binom{16}{4}.$$

The required sum is exactly the three middle terms, so

$$\binom{16}{3}\binom{4}{1} + \binom{16}{2}\binom{4}{2} + \binom{16}{1}\binom{4}{3} = \binom{20}{4} - \binom{4}{4} - \binom{16}{4}.$$

Now $\binom{20}{4} = 4845$, $\binom{4}{4} = 1$ and $\binom{16}{4} = 1820$, so the value is

$$4845 - 1 - 1820 = 3024.$$

Q65 2023 Girraween Prelim Q11(b) · **3 marks**

By considering $(1+x)^{11} = (1+x)^3(1+x)^8$, show that

$$\binom{11}{5} = \binom{8}{5} + \binom{3}{1}\binom{8}{4} + \binom{3}{2}\binom{8}{3} + \binom{8}{2}.$$

Criteria	Marks
States the coefficient of x^5 in $(1+x)^{11}$ is $\binom{11}{5}$	1
Forms the coefficient of x^5 in $(1+x)^3(1+x)^8$ as $\sum_k \binom{3}{k}\binom{8}{5-k}$	1
Uses $\binom{3}{0} = \binom{3}{3} = 1$, $\binom{3}{1} = \binom{3}{2} = 3$ and equates coefficients	1

Solution.

Compare the coefficient of x^5 on each side of $(1+x)^{11} = (1+x)^3(1+x)^8$.

Left-hand side. The coefficient of x^5 in $(1+x)^{11}$ is $\binom{11}{5}$.

Right-hand product. A term in x^5 comes from taking x^k from $(1+x)^3$ and x^{5-k} from $(1+x)^8$, for $k = 0, 1, 2, 3$. Its coefficient is

$$\sum_{k=0}^{3} \binom{3}{k}\binom{8}{5-k} = \binom{3}{0}\binom{8}{5} + \binom{3}{1}\binom{8}{4} + \binom{3}{2}\binom{8}{3} + \binom{3}{3}\binom{8}{2}.$$

Since $\binom{3}{0} = \binom{3}{3} = 1$ and $\binom{3}{1} = \binom{3}{2} = 3$, this equals $\binom{8}{5} + \binom{3}{1}\binom{8}{4} + \binom{3}{2}\binom{8}{3} + \binom{8}{2}$.

Equating the coefficients of x^5 gives

$$\binom{11}{5} = \binom{8}{5} + \binom{3}{1}\binom{8}{4} + \binom{3}{2}\binom{8}{3} + \binom{8}{2},$$

as required. (Check: $462 = 56 + 3 \cdot 70 + 3 \cdot 56 + 28 = 56 + 210 + 168 + 28 = 462$.)

Q66 2025 Baulkham Hills Prelim Q14(c) · **5 marks**

(i) Show that $x^n(1+x)^n\left(1+\dfrac{1}{x}\right)^n=(1+x)^{2n}$. (2 marks)

(ii) Hence show that

$$1+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}.$$ (3 marks)

Criteria	Marks
(i) Writes $x^n\left(1+\frac{1}{x}\right)^n=(x+1)^n$	1
(i) Combines with $(1+x)^n$ to give $(1+x)^{2n}$	1
(ii) Finds the coefficient of x^n in $(1+x)^n(1+x^{-1})^n\cdot x^n$ as $\sum_k \binom{n}{k}^2$	1
(ii) Finds the coefficient of x^n on the right as $\binom{2n}{n}$	1
(ii) Equates the coefficients to obtain the identity	1

Solution.

(i) Distribute the factor x^n into the third bracket:

$$x^n\left(1+\frac{1}{x}\right)^n=\left[x\left(1+\frac{1}{x}\right)\right]^n=(x+1)^n.$$

Therefore

$$x^n(1+x)^n\left(1+\frac{1}{x}\right)^n=(1+x)^n(1+x)^n=(1+x)^{2n}.$$

(ii) Compare the coefficient of x^n on both sides of the identity in (i).

Right-hand side. The coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$.

Left-hand side. Write $x^n\left(1+\frac{1}{x}\right)^n=(1+x)^n$ (from part (i)), so the left side is $(1+x)^n(1+x)^n$. We seek the coefficient of x^n in $\left[(1+x)^n\right]\left[(1+x)^n\right]$. A term in x^n is formed by taking x^k from the first factor and x^{n-k} from the second, giving coefficient

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}=\sum_{k=0}^{n}\binom{n}{k}^2,$$

using $\binom{n}{n-k}=\binom{n}{k}$. Written out this is $\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2=1+\binom{n}{1}^2+\cdots+\binom{n}{n}^2$.

Equating the coefficients of x^n gives

$$1+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n},$$

as required.

Q67 2020 James Ruse Prelim Q7(e) · **4 marks**

(i) Show that $(1+x)^m\left(1-\frac{1}{x}\right)^m=\left(x-\frac{1}{x}\right)^m$. (2 marks)

(ii) By considering the term(s) independent of x in the expansion of (i), show that

$$\binom{2n}{0}^2-\binom{2n}{1}^2+\binom{2n}{2}^2-\cdots+\binom{2n}{2n}^2=(-1)^n\binom{2n}{n}.$$ (2 marks)

Criteria	Marks
(i) Rewrites $\left(1-\frac{1}{x}\right)^m=\left(\frac{x-1}{x}\right)^m$ and combines with $(1+x)^m$	1
(i) Simplifies $\frac{(1+x)(x-1)}{x}$ to $x-\frac{1}{x}$ to complete the proof	1
(ii) With $m=2n$, finds the constant term of the left side as $\sum_k(-1)^k\binom{2n}{k}^2$	1
(ii) Finds the constant term of $\left(x-\frac{1}{x}\right)^{2n}$ as $(-1)^n\binom{2n}{n}$ and equates	1

Solution.

(i) Write the second bracket over a common denominator:

$$\left(1-\frac{1}{x}\right)^m=\left(\frac{x-1}{x}\right)^m=\frac{(x-1)^m}{x^m}.$$

Then

$$(1+x)^m\left(1-\frac{1}{x}\right)^m=\frac{(1+x)^m(x-1)^m}{x^m}=\frac{\left[(1+x)(x-1)\right]^m}{x^m}=\frac{(x^2-1)^m}{x^m}.$$

Since $\frac{x^2-1}{x}=x-\frac{1}{x}$, this is

$$\left(\frac{x^2-1}{x}\right)^m=\left(x-\frac{1}{x}\right)^m,$$

as required.

(ii) Take $m=2n$ in the identity from (i):

$$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n}=\left(x-\frac{1}{x}\right)^{2n}.$$

Now find the term independent of x (the constant term) on each side.

Left-hand side. Expand each factor:

$$(1+x)^{2n}=\sum_{i=0}^{2n}\binom{2n}{i}x^i,\qquad\left(1-\frac{1}{x}\right)^{2n}=\sum_{j=0}^{2n}\binom{2n}{j}(-1)^jx^{-j}.$$

In the product, a term is independent of x when $i - j = 0$, i.e. $i = j$. Summing those terms gives the constant

$$\sum_{i=0}^{2n} \binom{2n}{i}\binom{2n}{i}(-1)^i = \sum_{i=0}^{2n}(-1)^i\binom{2n}{i}^2 = \binom{2n}{0}^2 - \binom{2n}{1}^2 + \binom{2n}{2}^2 - \cdots + \binom{2n}{2n}^2.$$

Right-hand side. The general term of $\left(x - \frac{1}{x}\right)^{2n}$ is

$$\binom{2n}{k}x^{2n-k}\left(-\frac{1}{x}\right)^k = \binom{2n}{k}(-1)^k x^{2n-2k}.$$

This is independent of x when $2n - 2k = 0$, i.e. $k = n$, giving the constant $(-1)^n\binom{2n}{n}$.

Equating the two constant terms gives

$$\binom{2n}{0}^2 - \binom{2n}{1}^2 + \binom{2n}{2}^2 - \cdots + \binom{2n}{2n}^2 = (-1)^n\binom{2n}{n},$$

as required.