Year 11 Mathematics Extension 1

Topic: The Binomial Theorem (2 of 2)

Review Questions (Worked Solutions): Mixed one-off exam questions

Term 3, 2026

Mixed one-off exam questions

Q1 2022 Sydney Girls Trial Q10 · 1 mark

Given that x and y are both prime numbers and integers larger than 1, how many terms of the binomial expansion $\left(\sqrt{x}+\sqrt[5]{y}\right)^{301}$ will be rational?

(A) 151

(B) 60

(C) 31

(D) 30

Criteria	Marks
Correct answer (D)	1

Solution.

The general term is $\binom{301}{a}\left(\sqrt{x}\right)^{a}\left(\sqrt[5]{y}\right)^{b}=\binom{301}{a}x^{a/2}\,y^{b/5}$, where $a+b=301$ and $0\leq a,b\leq 301$. There are 302 terms in all.

Since x and y are distinct primes, the term is rational exactly when both exponents are integers, i.e. a is a multiple of 2 **and** b is a multiple of 5. Because a is even and $a+b=301$ is odd, b must be odd. So b is an odd multiple of 5:

$$b=5,15,25,\ldots,295,$$

which gives $\dfrac{295-5}{10}+1=30$ values. Hence there are 30 rational terms: option **(D)**.

Q2 2020 Sydney Girls Prelim Q4(d) · **2 marks**

Suppose that x is a real number and that n is a positive integer. Show that

$$\cos^n 2x = \binom{n}{0}\cos^{2n} x - \binom{n}{1}\cos^{2n-2} x \sin^2 x + \binom{n}{2}\cos^{2n-4} x \sin^4 x - \cdots + (-1)^n \sin^{2n} x.$$

Criteria	Marks
Writes $\cos 2x = \cos^2 x - \sin^2 x$ and raises to the power n	1
Applies the binomial theorem to reach the required expansion	1

Solution.

Using the double-angle identity $\cos 2x = \cos^2 x - \sin^2 x$,

$$\cos^n 2x = \left(\cos^2 x - \sin^2 x\right)^n.$$

Apply the binomial theorem with $A = \cos^2 x$ and $B = -\sin^2 x$:

$$\begin{aligned}\left(\cos^2 x - \sin^2 x\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\cos^2 x\right)^{n-k} \left(-\sin^2 x\right)^k \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos^{2n-2k} x \ \sin^{2k} x \\ &= \binom{n}{0}\cos^{2n} x - \binom{n}{1}\cos^{2n-2} x \sin^2 x + \binom{n}{2}\cos^{2n-4} x \sin^4 x - \cdots + (-1)^n \sin^{2n} x,\end{aligned}$$

as required. ■

Q3 2020 HSC Q14(a) · **8 marks**

(i) Use the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$ to show that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2,$$

where n is a positive integer.

(ii) A club has $2n$ members, with n women and n men. A group consisting of an even number $(0, 2, 4, \ldots, 2n)$ of members is chosen, with the number of men equal to the number of women. Show, giving reasons, that the number of ways to do this is $\binom{2n}{n}$.

(iii) From the group chosen in part (ii), one of the men and one of the women are selected as leaders. Show, giving reasons, that the number of ways to choose the even number of people and then the leaders is

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2.$$

(iv) The process is now reversed so that the leaders, one man and one woman, are chosen first. The rest of the group is then selected, still made up of an equal number of women and men. By considering this reversed process and using part (ii), find a simple expression for the sum in part (iii).

Criteria	**Marks**
(i) Identifies the coefficient of x^n on both sides and equates them	2
(ii) Counts by number of men = number of women and sums using part (i)	2
(iii) Includes the k choices of each leader to get the $k^2\binom{n}{k}^2$ terms	2
(iv) Chooses leaders first and applies part (ii) to $n-1$ of each	2

Solution.

(i) The coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$. Writing $(1+x)^{2n} = (1+x)^n(1+x)^n$, the coefficient of x^n in the product is obtained by pairing an x^k term from the first factor with an x^{n-k} term from the second:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

Since $\binom{n}{n-k} = \binom{n}{k}$, this equals $\sum_{k=0}^{n}\binom{n}{k}^2$. Equating coefficients gives

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

(ii) Choose k men (from n) and k women (from n), for $k = 0, 1, \ldots, n$; this gives an even-sized group of $2k$

with equal numbers of each. The number of ways for a fixed k is $\binom{n}{k}\binom{n}{k} = \binom{n}{k}^2$. Summing over k and using part (i),

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(iii) For a group with k men and k women (chosen in $\binom{n}{k}^2$ ways), the man leader can be chosen in k ways and the woman leader in k ways, contributing a factor $k \times k = k^2$. So the number of ways for that k is $k^2\binom{n}{k}^2$, and the total over $k = 1, \ldots, n$ is

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2.$$

(iv) Reverse the order: first choose the woman leader in n ways and the man leader in n ways (n^2 ways). This leaves $n-1$ women and $n-1$ men, from which we must select an equal number of each (including zero). By part (ii) applied to $n-1$, that can be done in $\binom{2(n-1)}{n-1} = \binom{2n-2}{n-1}$ ways. Hence

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2 = n^2\binom{2n-2}{n-1}.$$

Q4 2023 Normanhurst Boys Trial Q10 · **1 mark**

Consider the expansion of

$$\left(1 + x + x^2 + \cdots + x^n\right)\left(1 + 2x + 3x^2 + \cdots + (n+1)x^n\right).$$

What is the coefficient of x^n when $n = 50$?

(A) 1225

(B) 1300

(C) 1326

(D) 1431

Criteria	**Marks**
Correct answer (C)	1

Solution.

In the first factor the coefficient of x^i is 1; in the second the coefficient of x^j is $j + 1$. The coefficient of x^n in the product is

$$\sum_{i=0}^{n} 1 \cdot \big((n-i)+1\big) = \sum_{i=0}^{n} (n - i + 1) = \sum_{m=1}^{n+1} m = \frac{(n+1)(n+2)}{2}.$$

For $n = 50$ this is $\dfrac{51 \times 52}{2} = 1326$: option **(C)**.

Q5 2025 Sydney Boys Trial Q10 · **1 mark**

What is the coefficient of x^{293} in the expansion of $(x-1)(x^2-2)(x^3-3)\cdots(x^{24}-24)$?

(A) -7

(B) 13

(C) 20

(D) 28

Criteria	**Marks**
Correct answer (B)	1

Solution.

The highest-degree term is $x^1 \cdot x^2 \cdots x^{24} = x^{300}$, since $1+2+\cdots+24 = \dfrac{24\times 25}{2} = 300$. To obtain x^{293} we must lower the degree by 7, which means taking the **constant** $(-k)$ from a set of factors whose exponents k sum to 7 (and the leading x^k from all the others). Each such choice contributes the product of the chosen constants. The subsets of $\{1,2,\ldots,24\}$ summing to 7 are:

$$\{7\},\quad \{1,6\},\quad \{2,5\},\quad \{3,4\},\quad \{1,2,4\},$$

with contributions

$$(-7),\ (-1)(-6),\ (-2)(-5),\ (-3)(-4),\ (-1)(-2)(-4) = -7,\ 6,\ 10,\ 12,\ -8.$$

Summing: $-7+6+10+12-8 = 13$. The coefficient is 13: option **(B)**.