Year 11 Mathematics Advanced

Topic: Exponential and Logarithmic Functions (3 of 4)

Review Questions (Worked Solutions): Graphing exponential and logarithmic functions

Term 3, 2026

Sketching and identifying graphs

1. Sketching exponential graphs

Q1 North Sydney Boys 2024 Trial Q20(a) · **2 marks**

Sketch the curve $y = 4e^{-2x}$.

Criteria	Marks
Correct shape (decreasing exponential) with the horizontal asymptote $y = 0$	1
Correct y-intercept $(0, 4)$ labelled	1

Solution.

$y = 4e^{-2x}$ is a decreasing exponential (the coefficient -2 of x is negative).

- **y-intercept:** $x = 0 \Rightarrow y = 4e^{0} = 4$, so $(0, 4)$.
- **x-intercept:** $4e^{-2x} = 0$ has no solution (an exponential is always positive), so there is no x-intercept.
- **Asymptote:** as $x \to \infty$, $y \to 0^{+}$, so the horizontal asymptote is $y = 0$ (the x-axis). As $x \to -\infty$, $y \to \infty$.

The curve lies entirely above the x-axis, passes through $(0, 4)$, is decreasing, and flattens onto $y = 0$ for large x (e.g. $\left(1, \dfrac{4}{e^{2}}\right) \approx (1, 0.5)$).

Q2 Hornsby Girls 2022 Prelim Q16(c) · **2 marks**

On the given number plane (which shows $y = \log_2 x$ dotted), sketch $y = 2^x$, labelling any intercepts and asymptotes.

Criteria	Marks
Correct increasing exponential shape with horizontal asymptote $y = 0$	1
Correct y-intercept $(0, 1)$ labelled	1

Solution.

$y = 2^x$ is an increasing exponential (it is the reflection of the given $y = \log_2 x$ in the line $y = x$).

- **y-intercept:** $x = 0 \Rightarrow y = 2^0 = 1$, so $(0, 1)$.
- **x-intercept:** $2^x = 0$ has no solution, so there is no x-intercept.
- **Asymptote:** as $x \to -\infty$, $y \to 0^+$, so the horizontal asymptote is $y = 0$. As $x \to \infty$, $y \to \infty$.

The curve passes through $(0, 1)$ and $(1, 2)$, is increasing, lies above the x-axis, and flattens onto $y = 0$ for large negative x.

Q3 Sydney Grammar 2021 Prelim Q9(e)(i) · **2 marks**

Sketch the curve $y = 2^{-x}$, clearly showing all important features.

Criteria	Marks
Correct decreasing exponential shape with horizontal asymptote $y = 0$	1
Correct y-intercept $(0, 1)$ labelled	1

Solution.

$y = 2^{-x} = \left(\frac{1}{2}\right)^x$ is a decreasing exponential (the reflection of $y = 2^x$ in the y-axis).

- **y-intercept:** $x = 0 \Rightarrow y = 2^0 = 1$, so $(0, 1)$.
- **x-intercept:** $2^{-x} = 0$ has no solution, so there is no x-intercept.
- **Asymptote:** as $x \to \infty$, $y \to 0^+$, so the horizontal asymptote is $y = 0$. As $x \to -\infty$, $y \to \infty$.

The curve passes through $(0, 1)$ and $\left(1, \frac{1}{2}\right)$, is decreasing, lies above the x-axis, and flattens onto $y = 0$ for large x.

Q4 Shore 2023 Prelim Q15(a) · **2 marks**

For $f(x) = 3^{-x} - 1$, sketch $y = f(x)$, clearly showing any asymptote(s), intercept(s), and one other point.

Criteria	Marks
Correct decreasing exponential shape with horizontal asymptote $y = -1$	1
Intercept at the origin $(0, 0)$ and one further point (e.g. $(-1, 2)$) shown	1

Solution.

$y = 3^{-x} - 1$ is $y = 3^{-x}$ (a decreasing exponential) shifted down by 1.

- **y-intercept:** $x = 0 \Rightarrow y = 3^0 - 1 = 0$, so $(0, 0)$.
- **x-intercept:** $3^{-x} - 1 = 0 \Rightarrow 3^{-x} = 1 \Rightarrow -x = 0 \Rightarrow x = 0$, so the only x-intercept is also $(0, 0)$ (the curve passes through the origin).
- **Asymptote:** as $x \to \infty$, $3^{-x} \to 0$ so $y \to -1$; the horizontal asymptote is $y = -1$. As $x \to -\infty$, $y \to \infty$.
- **One other point:** $x = -1 \Rightarrow y = 3^1 - 1 = 2$, so $(-1, 2)$.

The curve is decreasing, passes through $(0, 0)$ and $(-1, 2)$, and flattens onto $y = -1$ from above for large x.

Q5 Abbotsleigh 2024 Prelim Q16 · **2 marks**

On the given set of axes, sketch $y = e^{-x} - 2$, showing all intercepts and asymptotes.

Criteria	Marks
Correct decreasing exponential shape with horizontal asymptote $y = -2$	1
Both intercepts $(0, -1)$ and $(-\ln 2,\ 0)$ labelled	1

Solution.

$y = e^{-x} - 2$ is $y = e^{-x}$ (a decreasing exponential) shifted down by 2.

- **y-intercept:** $x = 0 \Rightarrow y = e^0 - 2 = -1$, so $(0, -1)$.
- **x-intercept:** $e^{-x} - 2 = 0 \Rightarrow e^{-x} = 2 \Rightarrow -x = \ln 2 \Rightarrow x = -\ln 2 \approx -0.69$, so $(-\ln 2,\ 0)$.
- **Asymptote:** as $x \to \infty$, $e^{-x} \to 0$ so $y \to -2$; the horizontal asymptote is $y = -2$. As $x \to -\infty$, $y \to \infty$.

The curve is decreasing, crosses the y-axis at $(0, -1)$ and the x-axis at $(-\ln 2,\ 0)$, and flattens onto $y = -2$ from above for large x.

Q6 Hurlstone 2024 Trial Q15 · **2 marks**

Sketch $y = e^{(1-x)} + 2$ by transforming $y = e^x$, showing any intercepts, asymptotes and an additional point.

Criteria	Marks
Correct decreasing exponential shape with horizontal asymptote $y = 2$	1
y-intercept $(0,\ e + 2)$ and an additional point (e.g. $(1, 3)$) labelled	1

Solution.

$y = e^{(1-x)} + 2 = e^{-(x-1)} + 2$. Starting from $y = e^x$: reflect in the y-axis ($y = e^{-x}$), shift right 1 unit ($y = e^{-(x-1)} = e^{1-x}$), then shift up 2 units. The result is a decreasing exponential.

- **y-intercept:** $x = 0 \Rightarrow y = e^1 + 2 = e + 2 \approx 4.72$, so $(0,\ e + 2)$.
- **x-intercept:** $e^{1-x} + 2 = 0 \Rightarrow e^{1-x} = -2$ has no solution, so there is no x-intercept.
- **Asymptote:** as $x \to \infty$, $e^{1-x} \to 0$ so $y \to 2$; the horizontal asymptote is $y = 2$. As $x \to -\infty$, $y \to \infty$.
- **Additional point:** $x = 1 \Rightarrow y = e^0 + 2 = 3$, so $(1, 3)$.

The curve is decreasing, passes through $(0,\ e + 2)$ and $(1, 3)$, lies above $y = 2$, and flattens onto $y = 2$ from above for large x.

2. Sketching logarithmic graphs

Q7 Barker 2021 Trial Q14(a) · **1 mark**

Sketch the curve of $y = \ln(x + 1)$ indicating intercepts and asymptotes.

Criteria	Marks
Correct log curve with vertical asymptote $x = -1$ and intercept at the origin	1

Solution.

The graph is $y = \ln x$ shifted 1 unit to the left.

- **Domain:** $x + 1 > 0$, i.e. $x > -1$, so there is a **vertical asymptote at** $x = -1$.
- **x-intercept:** set $y = 0$: $\ln(x + 1) = 0 \Rightarrow x + 1 = 1 \Rightarrow x = 0$. So $(0, 0)$.
- **y-intercept:** set $x = 0$: $y = \ln(1) = 0$. So $(0, 0)$.

The curve passes through the origin, increases for all $x > -1$, and falls away steeply towards $-\infty$ as $x \to -1^+$. Sketch an increasing logarithmic curve through $(0, 0)$ with a dashed vertical asymptote at $x = -1$.

Q8 Sydney Grammar 2021 Prelim Q11(a)(ii) · **2 marks**

Sketch $y = \log_e x + 1$ on a number plane, showing all intercepts with the axes and all asymptotes, and at least two specific points.

Criteria	Marks
Correct shape with vertical asymptote $x = 0$ and x-intercept at $\left(\frac{1}{e}, 0\right)$	1
Labels at least two specific points (e.g. $(1, 1)$ and $(e, 2)$)	1

Solution.

This is $y = \ln x$ shifted **up** 1 unit (note $\log_e x + 1$ means $(\ln x) + 1$, not $\ln(x + 1)$).

- **Domain:** $x > 0$, so there is a **vertical asymptote at** $x = 0$ (the y-axis). There is *no* y-intercept.
- **x-intercept:** set $y = 0$: $\ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \dfrac{1}{e} \approx 0.37$. So $\left(\frac{1}{e}, 0\right)$.
- **Specific points:** at $x = 1$, $y = \ln 1 + 1 = 1$, giving $(1, 1)$; at $x = e$, $y = \ln e + 1 = 2$, giving $(e, 2)$.

Sketch an increasing logarithmic curve for $x > 0$, crossing the x-axis at $\left(\frac{1}{e}, 0\right)$, passing through $(1, 1)$ and $(e, 2)$, with a dashed vertical asymptote along the y-axis.

Q9 Presbyterian 2022 Prelim Q34(a) · **2 marks**

Given the sketch of $y = \log_e x$, sketch $y = \log_e(x + 2) - 1$ on the same axes, showing any asymptotes and the y intercept.

Criteria	Marks
Correct horizontal shift left 2 giving vertical asymptote $x = -2$	1
Correct vertical shift down 1 with y-intercept $(0, \log_e 2 - 1)$	1

Solution.

Transform $y = \log_e x$ by shifting 2 units **left** (replace x with $x + 2$) and 1 unit **down**.

- **Vertical asymptote:** $x + 2 > 0 \Rightarrow x > -2$, so the asymptote moves from $x = 0$ to $x = -2$.
- **y-intercept:** set $x = 0$: $y = \log_e(0 + 2) - 1 = \log_e 2 - 1 \approx -0.31$. So $\left(0,\ \log_e 2 - 1\right)$.
- **x-intercept:** set $y = 0$: $\log_e(x + 2) = 1 \Rightarrow x + 2 = e \Rightarrow x = e - 2 \approx 0.72$.

Sketch an increasing logarithmic curve with a dashed vertical asymptote at $x = -2$, passing through $\left(0, \log_e 2 - 1\right)$ on the y-axis and cutting the x-axis at $x = e - 2$.

Q10 Caringbah 2023 Trial Q16 · **3 marks**

Sketch $y = \ln(x+4) - 2$ showing all the key features and clearly labelling any intercepts in exact form.

Criteria	Marks
Vertical asymptote at $x = -4$	1
x-intercept in exact form $(e^2 - 4,\ 0)$	1
y-intercept in exact form $(0,\ 2\ln 2 - 2)$ and correct increasing shape	1

Solution.

This is $y = \ln x$ shifted 4 units **left** and 2 units **down**.

- **Vertical asymptote:** $x + 4 > 0 \Rightarrow x > -4$, so $x = -4$.
- **x-intercept:** set $y = 0$: $\ln(x+4) = 2 \Rightarrow x + 4 = e^2 \Rightarrow x = e^2 - 4$. So $(e^2 - 4,\ 0) \approx (3.39, 0)$.
- **y-intercept:** set $x = 0$: $y = \ln 4 - 2 = 2\ln 2 - 2$. So $(0,\ 2\ln 2 - 2) \approx (0, -0.61)$.

Sketch an increasing logarithmic curve with a dashed vertical asymptote at $x = -4$, cutting the y-axis at $(0,\ 2\ln 2 - 2)$ and the x-axis at $(e^2 - 4,\ 0)$.

Q11 St George Girls 2022 Prelim Q31 · **3 marks**

Sketch the graph of $y = \ln x - 2$, labelling the x-intercept and vertical asymptote.

Criteria	Marks
Correct increasing logarithmic shape	1
Vertical asymptote labelled $x = 0$ (the y-axis)	1
x-intercept labelled $(e^2, 0)$	1

Solution.

This is $y = \ln x$ shifted **down** 2 units.

- **Vertical asymptote:** $x > 0$, so the asymptote is $x = 0$ (the y-axis). There is no y-intercept.
- **x-intercept:** set $y = 0$: $\ln x = 2 \Rightarrow x = e^2 \approx 7.39$. So $(e^2, 0)$.

Sketch an increasing logarithmic curve for $x > 0$, hugging the y-axis (dashed asymptote $x = 0$) as $x \to 0^+$ and cutting the x-axis at $(e^2, 0)$.

Q12 James Ruse 2023 Trial Q12 · **3 marks**

Sketch $y = 3\log_{10}(x-2)$ showing all important features.

Criteria	Marks
Vertical asymptote at $x = 2$	1
x-intercept at $(3, 0)$	1
Correct increasing shape with the vertical stretch (e.g. through $(12, 3)$)	1

Solution.

This is $y = \log_{10} x$ shifted 2 units **right** and vertically stretched by a factor of 3.

- **Vertical asymptote:** $x - 2 > 0 \Rightarrow x > 2$, so $x = 2$. There is no y-intercept ($x = 0$ is outside the domain).
- **x-intercept:** set $y = 0$: $\log_{10}(x-2) = 0 \Rightarrow x - 2 = 1 \Rightarrow x = 3$. So $(3, 0)$.
- **Extra point:** at $x = 12$, $x - 2 = 10$, so $y = 3\log_{10} 10 = 3$, giving $(12, 3)$.

Sketch an increasing logarithmic curve with a dashed vertical asymptote at $x = 2$, cutting the x-axis at $(3, 0)$ and rising through $(12, 3)$; the factor of 3 makes it rise three times as fast as the basic $\log_{10}$ curve.

3. Identifying a graph or finding its equation

Q13 Hurlstone 2022 Trial Q9 · **1 mark**

Identify which of the four graphs (A)–(D) is $y = \ln(x+1)$.

Criteria	Marks
Correct option (B)	1

Solution.

The graph of $y = \ln(x+1)$ is the graph of $y = \ln x$ shifted 1 unit left. So it has a vertical asymptote at $x = -1$, passes through $(0, 0)$ since $\ln(0+1) = \ln 1 = 0$, and is increasing and concave down. At $x = 3$, $y = \ln 4 \approx 1.39$.

Only graph (B) has the vertical asymptote at $x = -1$, passes through the origin, and rises to about 1.4 near $x = 3$. (Graph C has its asymptote at $x = 0$, i.e. it is $y = \ln x$.)

The answer is **(B)**.

Q14 Moriah 2021 Prelim Q4 · **1 mark**

Identify which of the four graphs (A)–(D) is $y = \log_2(x)$.

Criteria	Marks
Correct option (C)	1

Solution.

The graph of $y = \log_2 x$ has domain $x > 0$, a vertical asymptote at $x = 0$, passes through $(1, 0)$ since $\log_2 1 = 0$, and is *increasing* (base $2 > 1$).

Only graph (C) lies in $x > 0$, is increasing, cuts the x-axis at $x = 1$, and drops down its asymptote at $x = 0$.

The answer is **(C)**.

Q15 2021 HSC Q5 · **1 mark**

Identify which of the four graphs (A)–(D) best represents $y = 10(0.8)^x$.

Criteria	Marks
Correct option (A)	1

Solution.

Setting $x = 0$ gives the y-intercept $y = 10(0.8)^0 = 10$. Since the base 0.8 satisfies $0 < 0.8 < 1$, the curve is *decreasing* (exponential decay) and stays above the x-axis.

Only graph (A) has y-intercept 10 and decreases towards the x-axis as x increases.

The answer is **(A)**.

Q16 SCEGGS 2022 Prelim Q7 · **1 mark**

The graph of $y = ka^x$ passes through $(1, 15)$ and $(2, 90)$. Find a and k.

Criteria	Marks
Correct option (D)	1

Solution.

Substituting the two points:

$$ka^1 = 15 \qquad \text{and} \qquad ka^2 = 90.$$

Dividing the second by the first:

$$\frac{ka^2}{ka^1} = \frac{90}{15} \implies a = 6.$$

Then $k = \dfrac{15}{a} = \dfrac{15}{6} = 2.5$.

So $a = 6$, $k = 2.5$. The answer is **(D)**.

Q17 Barker 2021 Prelim Q10 · **1 mark**

The graph of $y = a \ln x + b$ passes through $(e^2, 0)$ and is decreasing. Find a and b.

Criteria	Marks
Correct option (C)	1

Solution.

The curve is *decreasing*, so the coefficient of $\ln x$ is negative: $a < 0$. For a standard log-type curve reflected this way, $a = -1$.

Using the point $(e^2, 0)$:

$$a \ln(e^2) + b = 0 \implies 2a + b = 0 \implies b = -2a.$$

With $a = -1$, $b = -2(-1) = 2$.

So $a = -1$, $b = 2$. The answer is **(C)**.

Q18 North Sydney Boys 2024 Prelim Q2 · **1 mark**

From the graph of $y = a^x + b$ (a decreasing curve with a horizontal asymptote above the x-axis), state the constraints on a and b.

Criteria	Marks
Correct option (A)	1

Solution.

The curve is *decreasing*, so the base satisfies $0 < a < 1$.

As $x \to \infty$, $a^x \to 0$, so $y \to b$: the horizontal asymptote is $y = b$. In the figure this asymptote lies *above* the x-axis, so $b > 0$.

Hence $0 < a < 1$ and $b > 0$. The answer is **(A)**.

Q19 Baulkham Hills 2022 Prelim Q6 · **1 mark**

The graph of $y = 2^{x+a} + b$ has y-intercept -2 and x-intercept 1. Find a and b.

Criteria	Marks
Correct option (D)	1

Solution.

Read two points off the graph: it cuts the y-axis at $(0, -2)$ and the x-axis at $(1, 0)$.

Test option (D), $a = 1$, $b = -4$, i.e. $y = 2^{x+1} - 4$:

$$y(0) = 2^{0+1} - 4 = 2 - 4 = -2 \quad \checkmark \qquad y(1) = 2^{1+1} - 4 = 4 - 4 = 0 \quad \checkmark$$

Both intercepts match. (The horizontal asymptote is $y = b = -4$, consistent with the curve flattening below -2.)

The answer is **(D)**.

Q20 St George Girls 2023 Prelim Q5 · **1 mark**

$f(x) = a^x$ is a reflection of $g(x) = b^x$ in the y-axis, with f decreasing and g increasing. Decide which of I: $0 < a < 1$, II: $b > 1$, III: $ab = 1$ hold.

Criteria	**Marks**
Correct option (D)	1

Solution.

From the diagram f is decreasing, so its base satisfies $0 < a < 1$ (statement **I** is true).

g is increasing, so its base satisfies $b > 1$ (statement **II** is true).

A reflection in the y-axis replaces x by $-x$: $f(x) = g(-x)$, so

$$a^x = b^{-x} = \left(\tfrac{1}{b}\right)^x \implies a = \frac{1}{b} \implies ab = 1$$

(statement **III** is true).

All three are true. The answer is **(D)**.

Domain, range and intersections

4. Domain and range

Q21 Sydney Tech 2022 Trial Q2 · **1 mark**

What is the domain of $y = \ln(x - 1)$? (A) $x > 0$; (B) $x \geq 1$; (C) $x \geq -1$; (D) $x > 1$.

Criteria	Marks
Correct option (D)	1

Solution.

The logarithm $\ln(x - 1)$ is defined only when its argument is strictly positive:

$$x - 1 > 0 \implies x > 1.$$

So the domain is $x > 1$, which is option **(D)**.

Q22 Barker 2023 Trial Q2 · **1 mark**

What is the range of the function $y = 3^{-x} + 2$? (A) $(3, \infty)$; (B) $(-\infty, 2)$; (C) $(2, \infty)$; (D) $(-\infty, 1)$.

Criteria	Marks
Correct option (C)	1

Solution.

For all real x, $3^{-x} > 0$ (an exponential is always positive), and 3^{-x} takes every positive value. Hence

$$y = 3^{-x} + 2 > 2,$$

with $y \to 2^+$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$. The range is $(2, \infty)$, which is option **(C)**.

Q23 Riverview 2022 Trial Q1 · **1 mark**

The domain of $y = \log_e(x+2)$ is: (A) $x > 0$; (B) $x > 1$; (C) $x > -3$; (D) $x > -2$.

Criteria	Marks
Correct option (D)	1

Solution.

The logarithm is defined only when its argument is strictly positive:

$$x + 2 > 0 \implies x > -2.$$

So the domain is $x > -2$, which is option **(D)**.

Q24 Girraween 2023 Prelim Q5 · **1 mark**

The domain of the graph of $y = \log_2(2x-3)$ is: (A) all real x, $x > \frac{3}{2}$; (B) all real x, $x \neq \frac{3}{2}$; (C) $x > 0$; (D) all real x.

Criteria	Marks
Correct option (A)	1

Solution.

The logarithm requires its argument to be strictly positive:

$$2x - 3 > 0 \implies 2x > 3 \implies x > \tfrac{3}{2}.$$

So the domain is all real x with $x > \frac{3}{2}$, which is option **(A)**.

Q25 Blacktown Boys 2022 Prelim Q6 · **1 mark**

The range of $y = e^{-x} + 1$ is: (A) $(-\infty, 1)$; (B) $(-\infty, \infty)$; (C) $(-1, \infty)$; (D) $(1, \infty)$.

Criteria	Marks
Correct option (D)	1

Solution.

For all real x, $e^{-x} > 0$ (an exponential is always positive), taking every positive value. Hence

$$y = e^{-x} + 1 > 1,$$

with $y \to 1^+$ as $x \to \infty$ and $y \to \infty$ as $x \to -\infty$. The range is $(1, \infty)$, which is option **(D)**.

Q26 Fort St 2025 Trial Q6 · **1 mark**

What is the domain of $f(x) = \log_{10}\big[(x-3)(x-5)\big]$? (A) $\{3 \le x \le 5\}$; (B) $\{3 < x < 5\}$; (C) $\{x \le 3 \text{ or } x \ge 5\}$; (D) $\{x < 3 \text{ or } x > 5\}$.

Criteria	Marks
Correct option (D)	1

Solution.

The logarithm requires its argument to be strictly positive:

$$(x-3)(x-5) > 0.$$

This concave-up parabola is positive outside its roots $x = 3$ and $x = 5$. Hence

$$x < 3 \quad \text{or} \quad x > 5.$$

(The endpoints are excluded because the argument would be 0, where log is undefined.) The domain is $\{x < 3 \text{ or } x > 5\}$, which is option **(D)**.

5. Intersections of curves

Q27 Fort St 2020 Trial Q14 · **3 marks**

The graphs of $y = 4e^{-x}$ and $y = e^{x} - 3$ meet at a point P. (a) Show that the curves intersect when $e^{2x} - 3e^{x} - 4 = 0$. (b) Hence, show the x coordinate of P is $x = \ln 4$.

Criteria	Marks
(a) Equates the curves and clears e^{-x} to obtain $e^{2x} - 3e^{x} - 4 = 0$	1
(b) Factorises the quadratic in $u = e^{x}$ and rejects $u = -1$	1
(b) Concludes $e^{x} = 4$, hence $x = \ln 4$	1

Solution.

(a) At an intersection the y-values are equal:

$$4e^{-x} = e^{x} - 3.$$

Multiply both sides by e^{x} (which is positive, so the direction is preserved):

$$4 = e^{2x} - 3e^{x} \quad \Longrightarrow \quad e^{2x} - 3e^{x} - 4 = 0,$$

as required.

(b) Let $u = e^{x}$, so $u > 0$. The equation becomes

$$u^{2} - 3u - 4 = 0 \quad \Longrightarrow \quad (u - 4)(u + 1) = 0,$$

so $u = 4$ or $u = -1$. Since $u = e^{x} > 0$, reject $u = -1$. Hence

$$e^{x} = 4 \quad \Longrightarrow \quad x = \ln 4.$$

So the x coordinate of P is $\boxed{x = \ln 4}$ (and $y = e^{\ln 4} - 3 = 1$).

Q28 Barker 2022 Trial Q25 · **2 marks**

Find the point(s) of intersection of the graphs of $y = 9^x + 4$ and $y = 13 - 8 \times 3^x$.

Criteria	**Marks**
Forms a quadratic in $u = 3^x$ and factorises (or solves) it	1
Rejects $u = -9$, solves $3^x = 1$ and states the point $(0, 5)$	1

Solution.

At an intersection the y-values are equal:

$$9^x + 4 = 13 - 8 \times 3^x.$$

Since $9^x = (3^x)^2$, let $u = 3^x$ (so $u > 0$):

$$u^2 + 4 = 13 - 8u \quad \Longrightarrow \quad u^2 + 8u - 9 = 0 \quad \Longrightarrow \quad (u - 1)(u + 9) = 0.$$

So $u = 1$ or $u = -9$. Since $u = 3^x > 0$, reject $u = -9$. Hence

$$3^x = 1 \quad \Longrightarrow \quad x = 0.$$

Then $y = 9^0 + 4 = 5$. The only point of intersection is $\boxed{(0, 5)}$.

Q29 Cherrybrook Tech 2021 Trial Q28(a) · **3 marks**

Two curves $y = 2e^{-x}$ and $y = e^{x} - 1$ intersect at point P. Show that the coordinates of P are $(\ln 2,\ 1)$.

Criteria	**Marks**
Equates the curves and clears e^{-x} to obtain $e^{2x} - e^{x} - 2 = 0$	1
Factorises in $u = e^{x}$, rejects $u = -1$, and finds $x = \ln 2$	1
Computes $y = 1$ and states $P = (\ln 2, 1)$	1

Solution.

At P the y-values are equal:

$$2e^{-x} = e^{x} - 1.$$

Multiply both sides by e^{x}:

$$2 = e^{2x} - e^{x} \quad \Longrightarrow \quad e^{2x} - e^{x} - 2 = 0.$$

Let $u = e^{x}$ $(u > 0)$:

$$u^{2} - u - 2 = 0 \quad \Longrightarrow \quad (u - 2)(u + 1) = 0,$$

so $u = 2$ or $u = -1$. Reject $u = -1$ since $e^{x} > 0$. Hence

$$e^{x} = 2 \quad \Longrightarrow \quad x = \ln 2.$$

Then $y = e^{\ln 2} - 1 = 2 - 1 = 1$. Therefore $P = \boxed{(\ln 2,\ 1)}$, as required.

Q30 Normanhurst Boys 2020 Trial Q29 · **3 marks**

The functions $f(x) = 2 - \dfrac{1}{e^x}$ and $g(x) = 4e^x - 3$ intersect at points P and Q. (a) Show that the curves intersect when $4e^{2x} - 5e^x + 1 = 0$. (b) Show that the points of intersection occur at $x = 0$ and $x = \ln \frac{1}{4}$.

Criteria	**Marks**
(a) Equates f and g and clears e^{-x} to obtain $4e^{2x} - 5e^x + 1 = 0$	1
(b) Factorises the quadratic in $u = e^x$	1
(b) Solves $e^x = 1$ and $e^x = \frac{1}{4}$ to give $x = 0$ and $x = \ln \frac{1}{4}$	1

Solution.

(a) At an intersection $f(x) = g(x)$:

$$2 - e^{-x} = 4e^x - 3.$$

Multiply both sides by e^x:

$$2e^x - 1 = 4e^{2x} - 3e^x \quad \Longrightarrow \quad 4e^{2x} - 5e^x + 1 = 0,$$

as required.

(b) Let $u = e^x$ $(u > 0)$:

$$4u^2 - 5u + 1 = 0 \quad \Longrightarrow \quad (4u - 1)(u - 1) = 0,$$

so $u = \frac{1}{4}$ or $u = 1$. Both are positive, so both are valid. Hence

$$e^x = 1 \implies x = 0, \qquad e^x = \tfrac{1}{4} \implies x = \ln \tfrac{1}{4}.$$

So the points of intersection occur at $\boxed{x = 0}$ and $\boxed{x = \ln \frac{1}{4}}$ (i.e. $x = -\ln 4$).

Q31 Sydney Girls 2022 Trial Q29(a) · **4 marks**

The diagram shows the graph of $y = 6 - 8e^{-x}$. On the same diagram $y = e^x$ is sketched (passing through $(1, e)$). Show that the x coordinates of the two points of intersection (x_1 and x_2, where $x_1 < x_2$) are the solutions of $e^{2x} - 6e^x + 8 = 0$, and solve to find the exact values of x_1 and x_2.

Criteria	**Marks**
Sketches $y = e^x$ (increasing, through $(0, 1)$ and $(1, e)$) cutting the given curve twice	1
Equates $e^x = 6 - 8e^{-x}$ and clears e^{-x} to obtain $e^{2x} - 6e^x + 8 = 0$	1
Factorises the quadratic in $u = e^x$	1
States $x_1 = \ln 2$ and $x_2 = \ln 4$ (correctly ordered)	1

Solution.

The curve $y = e^x$ is an increasing exponential through $(0, 1)$ and $(1, e)$, with the x-axis as asymptote; it crosses $y = 6 - 8e^{-x}$ at two points.

At an intersection the y-values are equal:

$$e^x = 6 - 8e^{-x}.$$

Multiply both sides by e^x:

$$e^{2x} = 6e^x - 8 \quad \Longrightarrow \quad e^{2x} - 6e^x + 8 = 0,$$

as required. Let $u = e^x$ ($u > 0$):

$$u^2 - 6u + 8 = 0 \quad \Longrightarrow \quad (u - 2)(u - 4) = 0,$$

so $u = 2$ or $u = 4$ (both positive, both valid). Hence

$$e^x = 2 \implies x = \ln 2, \qquad e^x = 4 \implies x = \ln 4.$$

Since $\ln 2 < \ln 4$, the exact values are $\boxed{x_1 = \ln 2}$ and $\boxed{x_2 = \ln 4}$.

Q32 Barker 2021 Trial Q24(a) · **3 marks**

The graph shows the functions $f(x) = 2^x - 3$ and $g(x) = 4 \times 2^{-x}$, meeting at A. By creating a quadratic equation, show that the coordinates of A are $(2, 1)$.

Criteria	**Marks**
Equates f and g and clears 2^{-x} to form $(2^x)^2 - 3(2^x) - 4 = 0$	1
Factorises in $u = 2^x$, rejects $u = -1$, and finds $x = 2$	1
Computes $y = 1$ and states $A = (2, 1)$	1

Solution.

At A the y-values are equal:

$$2^x - 3 = 4 \times 2^{-x}.$$

Multiply both sides by 2^x:

$$(2^x)^2 - 3(2^x) = 4 \quad \Longrightarrow \quad (2^x)^2 - 3(2^x) - 4 = 0.$$

Let $u = 2^x$ $(u > 0)$:

$$u^2 - 3u - 4 = 0 \quad \Longrightarrow \quad (u - 4)(u + 1) = 0,$$

so $u = 4$ or $u = -1$. Reject $u = -1$ since $2^x > 0$. Hence

$$2^x = 4 \quad \Longrightarrow \quad x = 2.$$

Then $y = 2^2 - 3 = 1$. Therefore $A = \boxed{(2, 1)}$, as required.

Graphing applied models

6. Sketching an applied exponential model over its domain

Q33 ★ Blacktown Boys 2022 Prelim Q20 · **6 marks**

The amount of vaccine, M millilitres, in a rat's bloodstream t hours later is $M = e^{-\frac{t}{4}} + 2$. (a) State the initial amount. (b) Find the rate of decrease at $t = 3$. (c) Show more than 2 mL is always present. (d) Sketch M against t.

Criteria	Marks
(a) Correct initial value $M = 3$ mL	1
(b) [Year 12] correct rate ≈ -0.118 mL/h	2
(c) Shows $M \to 2$ from above, so $M > 2$ always	1
(d) Starts at $(0, 3)$, decreasing	1
(d) Horizontal asymptote $M = 2$ shown	1

Solution.

(a) Initial value. At $t = 0$: $M = e^0 + 2 = 1 + 2 = 3$. Initially 3 mL of vaccine was injected.

(b) Rate at $t = 3$ (Year-12 method). $\dfrac{dM}{dt} = -\dfrac{1}{4}e^{-\frac{t}{4}}$. At $t = 3$: $\dfrac{dM}{dt} = -\dfrac{1}{4}e^{-\frac{3}{4}} \approx -0.118$. The amount is decreasing at about 0.118 mL per hour at the end of 3 hours.

(c) Always more than 2 mL. For all t, $e^{-t/4} > 0$, so $M = e^{-t/4} + 2 > 2$. As $t \to \infty$, $e^{-t/4} \to 0$ and $M \to 2^+$, but M never reaches or falls below 2. Hence there is always more than 2 mL present. ■

(d) Sketch. This is exponential *decay* shifted up by 2. The curve starts at $(0, 3)$, is strictly decreasing and concave up for $t \geq 0$, and flattens toward the horizontal asymptote $M = 2$ (never reaching it). Sample point: at $t = 4$, $M = e^{-1} + 2 \approx 2.37$. Draw a smooth decreasing curve from $(0, 3)$ levelling off just above $M = 2$.

Q34 Gosford 2023 Prelim Q25 · **6 marks**

A heated metal bar cools so that $T = 15 + 125e^{-0.3t}$ (t in minutes). (a) Find the initial temperature. (b) Find T after 10 minutes. (c) Graph T for $0 \le t \le 20$. (d) Find when $T = 30°\text{C}$.

Criteria	Marks
(a) Correct initial temperature $T = 140°\text{C}$	2
(b) Correct value $T \approx 21.2°\text{C}$ at $t = 10$	1
(c) Decreasing curve from $(0, 140)$ toward asymptote $T = 15$	2
(d) Correct time $t \approx 7.1$ minutes	1

Solution.

(a) Initial temperature. At $t = 0$: $T = 15 + 125e^{0} = 15 + 125 = 140$. The initial temperature is 140°C.

(b) After 10 minutes. $T = 15 + 125e^{-0.3(10)} = 15 + 125e^{-3} \approx 15 + 6.22 = 21.2°\text{C}$ (to 1 d.p.).

(c) Graph. As $t \to \infty$, $e^{-0.3t} \to 0$, so $T \to 15$: the line $T = 15$ is the horizontal asymptote (ambient temperature). The curve starts at $(0, 140)$, is strictly decreasing and concave up, dropping steeply then flattening. Plotted points: $t = 5 \Rightarrow T \approx 42.9$; $t = 10 \Rightarrow T \approx 21.2$; $t = 20 \Rightarrow T \approx 15.3$. Draw a smooth decreasing curve on $0 \le t \le 20$ from $(0, 140)$ flattening just above $T = 15$.

(d) Time to reach 30°C. Solve $15 + 125e^{-0.3t} = 30$: $125e^{-0.3t} = 15 \Rightarrow e^{-0.3t} = \dfrac{15}{125} = 0.12$. Taking logs: $-0.3t = \ln 0.12$, so $t = \dfrac{\ln 0.12}{-0.3} \approx 7.07$ minutes.

Q35 Hurlstone 2024 Prelim Q30 · **5 marks**

The energy stored in a battery is $E(t) = 100 - Ae^{-0.1t}$, with $E(0) = 10$. (a) Show $A = 90$. (b) Fill the missing table value ($t = 10$). (c) Draw the graph. (d) State the value E approaches as t increases.

Criteria	Marks
(a) Substitutes $E(0) = 10$ to show $A = 90$	1
(b) Missing value $E(10) = 66.9$	1
(c) Plots the table points and joins with a smooth increasing curve toward $E = 100$	2
(d) Limiting value $E = 100$	1

Solution.

(a) Show $A = 90$. At $t = 0$: $E(0) = 100 - Ae^0 = 100 - A$. Given $E(0) = 10$: $100 - A = 10$, so $A = 90$. ■

With $A = 90$, $E(t) = 100 - 90e^{-0.1t}$.

(b) Missing value. At $t = 10$: $E = 100 - 90e^{-0.1(10)} = 100 - 90e^{-1} \approx 100 - 33.1 = 66.9$ (to 1 d.p.). Full table: $(0, 10)$, $(10, 66.9)$, $(20, 87.8)$, $(30, 95.5)$, $(40, 98.4)$.

(c) Graph. As $t \to \infty$, $e^{-0.1t} \to 0$, so $E \to 100$ (horizontal asymptote). E is strictly increasing and concave down. Plot the five table points and join them with a smooth curve rising steeply from $(0, 10)$ then flattening toward $E = 100$.

(d) Limiting value. As t increases, $90e^{-0.1t} \to 0$, so $E \to 100$. The electrical energy approaches 100 units.

Note: the printed part (c) states the interval as "$0 \le t \le 4$", but the supplied table and the printed grid both run to $t = 40$ (the grid axis is marked to 70); the intended interval is $0 \le t \le 40$.

Q36 Barker 2021 Trial Q26 · 5 marks

A bird population is modelled by $P = 150 + 300e^{-0.05t}$, where t is the number of years after the start of 2020. (a) Find the population in 2020. (b) Sketch P against t for $t \geq 0$, showing the limiting value. (c) Find the year the population first falls below 200.

Criteria	Marks
(a) Correct initial population $P = 450$	1
(b) Decreasing curve from $(0, 450)$ with limiting value $P = 150$ indicated	2
(c) Solves $P = 200$ to get $t \approx 35.8$	1
(c) States the year (2055/2056)	1

Solution.

(a) Population in 2020. At $t = 0$: $P = 150 + 300e^{0} = 150 + 300 = 450$ birds.

(b) Sketch. This is exponential *decay* toward a lower limit. As $t \to \infty$, $e^{-0.05t} \to 0$, so $P \to 150$: the line $P = 150$ is the horizontal asymptote and the limiting value that must be indicated. The curve starts at $(0, 450)$, is strictly decreasing and concave up, flattening toward $P = 150$. Sample points: $t = 20 \Rightarrow P \approx 260$; $t = 50 \Rightarrow P \approx 175$.

(c) Year population falls below 200**.** Solve $150+300e^{-0.05t} = 200$: $300e^{-0.05t} = 50 \Rightarrow e^{-0.05t} = \dfrac{50}{300} = \dfrac{1}{6}$. Taking logs: $-0.05t = \ln\dfrac{1}{6}$, so $t = \dfrac{\ln(1/6)}{-0.05} \approx 35.8$ years. Since t is measured from the start of 2020, the population drops below 200 about 35.8 years later, i.e. during 2055 (so it is below 200 by 2056). The model predicts the species becomes eligible in 2055.

Q37 ★ 2024 HSC Q17 · 6 marks

The voltage across a capacitor is $V(t) = 6.5\left(1 - e^{-kt}\right)$, $k > 0$, $t \geq 0$ (seconds). (a) Sketch $V(t)$, showing its behaviour as t increases. (b) Given $V(1) = 2.6$, find k to 3 d.p. (c) Find the rate the voltage is increasing at $t = 2$, to 3 d.p.

Criteria	Marks
(a) Curve starts at origin $(0, 0)$ and is increasing	1
(a) Approaches the asymptote $V = 6.5$ as $t \to \infty$	1
(b) Sets up $6.5(1 - e^{-k}) = 2.6$ and takes logs	1
(b) $k \approx 0.511$	1
(c) [Year 12] correct rate ≈ 1.195 V/s	2

Solution.

(a) Sketch. This is a *saturating charging* curve. At $t = 0$: $V = 6.5(1 - e^0) = 6.5(1 - 1) = 0$, so the curve starts at the origin $(0, 0)$. As $t \to \infty$, $e^{-kt} \to 0$ (since $k > 0$), so $V \to 6.5(1 - 0) = 6.5$: the line $V = 6.5$ is the horizontal asymptote (the maximum voltage). V is strictly increasing and concave down: it rises steeply from the origin then flattens toward $V = 6.5$.

(b) Find k. $V(1) = 6.5(1 - e^{-k}) = 2.6 \Rightarrow 1 - e^{-k} = \dfrac{2.6}{6.5} = 0.4$, so $e^{-k} = 0.6$. Taking logs: $-k = \ln 0.6$, giving $k = -\ln 0.6 \approx 0.511$ (to 3 d.p.).

(c) Rate at $t = 2$ (Year-12 method). $\dfrac{dV}{dt} = 6.5k\, e^{-kt}$. With $k = -\ln 0.6 \approx 0.510826$: at $t = 2$, $\dfrac{dV}{dt} = 6.5(0.510826)e^{-0.510826(2)} \approx 1.195$. The voltage is increasing at about 1.195 volts per second when $t = 2$.