Year 11 Mathematics Advanced

Topic: Working with Functions (5 of 12)

Review Questions (Worked Solutions): Reciprocal Functions

Term 3, 2026

Reciprocal graphs

1. Domain, range and asymptotes of a reciprocal function

Q1 St George Girls 2022 Prelim Q24(a) · **1 mark**

A function is defined by $f(x) = \dfrac{2x-1}{3x+2}$. Find $f(-1)$.

Criteria	Marks
Correct value $f(-1) = 3$	1

Solution.

Substitute $x = -1$ into the rule and simplify:

$$f(-1) = \frac{2(-1)-1}{3(-1)+2} = \frac{-3}{-1} = 3.$$

Q2 Presbyterian 2023 Prelim Q20 · **2 marks**

State the domain and range of $f(x) = \dfrac{1}{2x+1}$.

Criteria	Marks
Correct domain: all real x with $x \neq -\frac{1}{2}$	1
Correct range: all real y with $y \neq 0$	1

Solution.

The function is undefined where the denominator is zero: $2x + 1 = 0$ gives $x = -\frac{1}{2}$, so the domain is all real x except $x = -\frac{1}{2}$. Since $\dfrac{1}{2x+1}$ can never equal 0 (a fraction with numerator 1 is never zero), and every other value is attained, the range is all real y except $y = 0$:

$$\text{Domain: } \{x \in \mathbb{R} : x \neq -\tfrac{1}{2}\}, \qquad \text{Range: } \{y \in \mathbb{R} : y \neq 0\}.$$

Q3 North Sydney Girls 2022 Prelim Q12 · **2 marks**

State the domain and range of $f(x) = 3 - \dfrac{2}{x+4}$.

Criteria	**Marks**
Correct domain: all real x with $x \neq -4$	1
Correct range: all real y with $y \neq 3$	1

Solution.

The function is undefined where $x + 4 = 0$, i.e. $x = -4$, so the domain is all real x except $x = -4$. The term $\dfrac{2}{x+4}$ is never 0, so $f(x) = 3 - \dfrac{2}{x+4}$ never equals 3; every other value is attained, so the range is all real y except $y = 3$:

$$\text{Domain: } \{x \in \mathbb{R} : x \neq -4\}, \qquad \text{Range: } \{y \in \mathbb{R} : y \neq 3\}.$$

Q4 Sydney Grammar 2022 Trial Q27 · **1 mark**

The graph of the function $y = \dfrac{x^2 + 3x}{x^2 + 3}$ has a horizontal asymptote. Find the equation of this asymptote, clearly showing your working.

Criteria	**Marks**
Correct horizontal asymptote $y = 1$ with working	1

Solution.

Divide numerator and denominator by the highest power x^2 and let $x \to \pm\infty$:

$$y = \frac{x^2 + 3x}{x^2 + 3} = \frac{1 + \dfrac{3}{x}}{1 + \dfrac{3}{x^2}} \longrightarrow \frac{1+0}{1+0} = 1.$$

Since the numerator and denominator have equal degree, the horizontal asymptote is the ratio of leading coefficients, $y = 1$.

Q5 North Sydney Boys 2020 Trial Q2 · **1 mark**

If $f(x) = \dfrac{x^2+1}{4-x^2}$ then the vertical and horizontal asymptotes are respectively:

(A) $x = -2,\ x = 2,\ y = \frac{1}{2}$

(B) $x = -2,\ x = 2,\ y = -1$

(C) $x = -2,\ x = 2,\ y = 1$

(D) $x = -2,\ x = 2,\ y = 0$

Criteria	**Marks**
Correct option (B)	1

Solution.

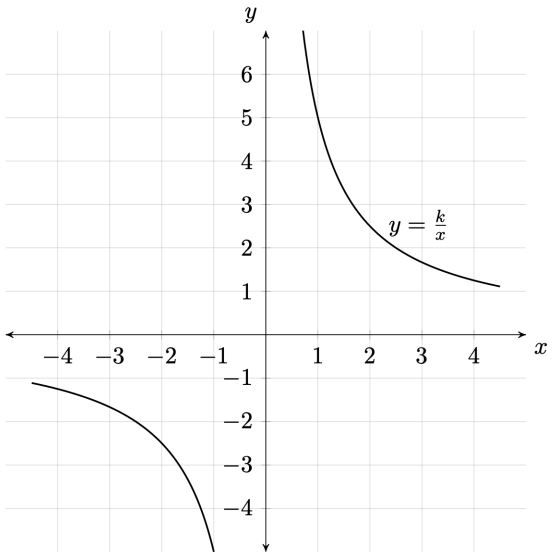
Vertical asymptotes occur where the denominator is zero: $4 - x^2 = 0$ gives $x = \pm 2$. For the horizontal asymptote, the degrees are equal, so take the ratio of leading coefficients:

$$\lim_{x \to \pm\infty} \frac{x^2+1}{4-x^2} = \frac{1}{-1} = -1.$$

Hence $x = -2$, $x = 2$ and $y = -1$. The answer is (B).

Q6 Kambala 2024 Prelim Q22 · **2 marks**

The graph of $y = \dfrac{k}{x}$ is shown on the axes below.



Find the value of k and solve the equation $\dfrac{k}{x} = 4$.

Criteria	**Marks**
Reads a point off the graph to find $k = 5$	1
Solves $\dfrac{5}{x} = 4$ to get $x = \dfrac{5}{4}$	1

Solution.

The curve passes through the point $(1, 5)$ (equivalently $(2, 2.5)$ or $(4, 1.25)$), so $k = 1 \times 5 = 5$ and the rule is $y = \dfrac{5}{x}$. Solving the equation:

$$\frac{5}{x} = 4 \implies 5 = 4x \implies x = \frac{5}{4}.$$

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Q7 Barker 2025 Trial Q7 · **1 mark**

The function $f(x) = \dfrac{1}{x}$ has a restricted domain of $[a, b)$, where a and b are positive real numbers. The range of $f(x) = \dfrac{1}{x}$ is:

(A) $\left(\dfrac{1}{a}, \dfrac{1}{b}\right)$

(B) $\left[\dfrac{1}{a}, \dfrac{1}{b}\right)$

(C) $\left(\dfrac{1}{b}, \dfrac{1}{a}\right)$

(D) $\left(\dfrac{1}{b}, \dfrac{1}{a}\right]$

Criteria	Marks
Correct option (D)	1

Solution.

On $[a, b)$ with $0 < a < b$, the function $f(x) = \dfrac{1}{x}$ is decreasing. The endpoint $x = a$ is included, giving the largest value $\dfrac{1}{a}$ (attained, so closed). As $x \to b^-$ the value approaches $\dfrac{1}{b}$ from above but $x = b$ is excluded, so $\dfrac{1}{b}$ is not attained (open). Hence the range is

$$\left(\frac{1}{b}, \frac{1}{a}\right],$$

which is option (D).

2. Match / describe a hyperbola from its graph (equation and parameters)

Q8 Sydney Grammar 2024 Trial Q2 · **1 mark**

Which equation best represents the graph drawn below?

Diagram to be inserted:
`adv-working-with-functions-d5-q8.png`

(A) $y = 4x,\ x > 0$

(B) $y = 4^x,\ x > 0$

(C) $y = 4x^2,\ x > 0$

(D) $y = \dfrac{4}{x},\ x > 0$

Criteria	Marks
Correct option (D)	1

Solution.

The curve is *decreasing* for $x > 0$ and hugs both axes as asymptotes, which rules out the increasing options: $y = 4x$ (line) and $y = 4^x$ (exponential) both increase, and $y = 4x^2$ (parabola) increases for $x > 0$. Only a reciprocal (hyperbola) decreases towards the axes:

$$\frac{d}{dx}\left(\frac{4}{x}\right) = -\frac{4}{x^2} < 0 \quad \text{for all } x > 0.$$

The answer is (D).

Q9 Moriah 2021 Prelim Q5 · **1 mark**

A curve has a vertical asymptote at $x = -4$ and a horizontal asymptote at $y = 0$. The curve crosses the y-axis at $y = 2$. Which equation could describe the curve?

(A) $y = \dfrac{8}{x+4}$

(B) $y = \dfrac{2}{x+4}$

(C) $y = \dfrac{8}{x-4}$

(D) $y = \dfrac{2}{x-4}$

Criteria	**Marks**
Correct option (A)	1

Solution.

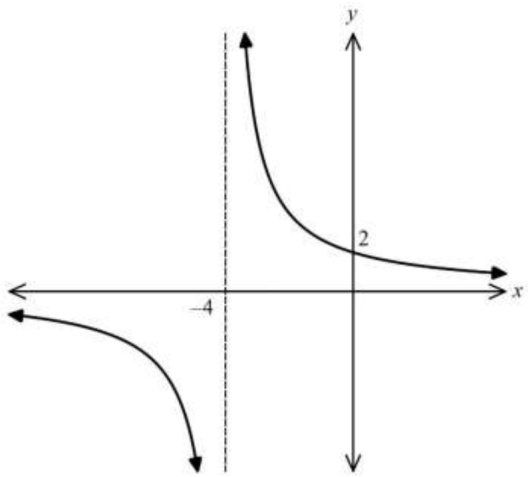
A vertical asymptote at $x = -4$ requires the denominator $(x+4)$, which rules out (C) and (D). The y-intercept is found by setting $x = 0$:

$$\frac{8}{0+4} = 2, \qquad \frac{2}{0+4} = \tfrac{1}{2}.$$

Only $y = \dfrac{8}{x+4}$ gives the required y-intercept of 2 (and its horizontal asymptote is $y = 0$). The answer is (A).

Q10 St George Girls 2020 Trial Q7 · **1 mark**

The diagram below shows the graph of $y = f(x)$, with the point $(-3,\ 2.5)$ marked.



Which of the following statements is *false*?

(A) The horizontal asymptote is $y = 1$.

(B) The curve is continuous.

(C) The curve is concave up for $x < -1$.

(D) The equation of the function is $y = \dfrac{x-2}{x+1}$.

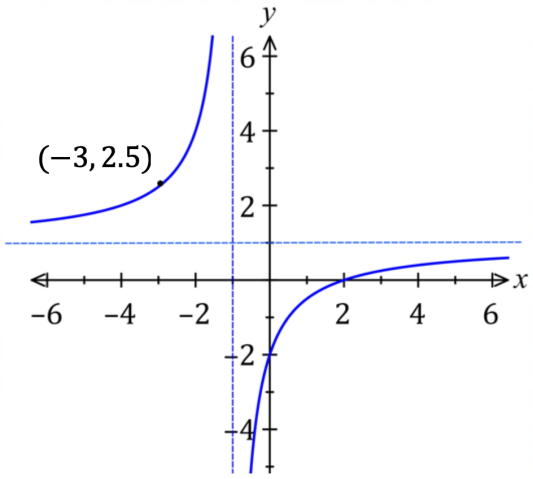
Criteria	Marks
Correct option (B)	1

Solution.

Test option (D): $y = \dfrac{x-2}{x+1}$ has a vertical asymptote at $x = -1$, a horizontal asymptote $y = 1$ (equal degrees, leading ratio 1:1), and passes through the marked point since $f(-3) = \dfrac{-3-2}{-3+1} = \dfrac{-5}{-2} = 2.5$. So (A) and (D) are true. For (C), $f''(x) > 0$ for $x < -1$ (e.g. $f''(-2) = 6 > 0$), so the curve is concave up there: (C) is true. Because of the vertical asymptote at $x = -1$ the curve is broken into two branches, so it is *not* continuous. The false statement is (B).

Q11 Hornsby Girls 2025 Trial Q28(a) · **2 marks**

Consider the graph of a hyperbola $y = f(x)$, as shown below.



Justify that the equation of $y = f(x)$ is $f(x) = \dfrac{x}{x-1}$. Show full working.

Criteria	**Marks**
Reads the asymptotes from the graph: vertical $x = 1$, horizontal $y = 1$, and the curve passes through the origin	1
Confirms $f(x) = \dfrac{x}{x-1}$ matches all three features	1

Solution.

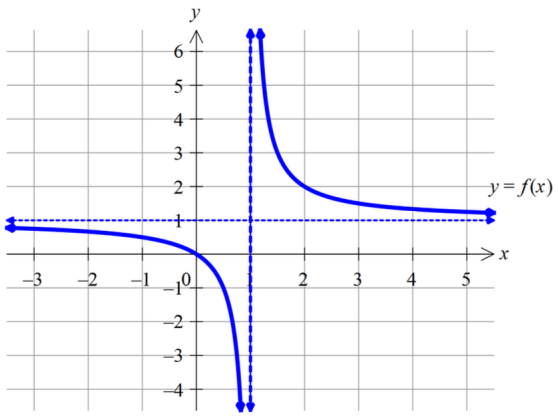
From the graph the curve has a vertical asymptote at $x = 1$, a horizontal asymptote at $y = 1$, and passes through the origin $(0, 0)$. Writing $f(x) = \dfrac{x}{x-1}$ in the form

$$f(x) = \frac{x}{x-1} = \frac{(x-1)+1}{x-1} = 1 + \frac{1}{x-1},$$

we read off a vertical asymptote where the denominator is zero, $x = 1$, and a horizontal asymptote $y = 1$ (the constant term). Finally $f(0) = \dfrac{0}{0-1} = 0$, so the curve passes through the origin. All three features match the graph, so $f(x) = \dfrac{x}{x-1}$.

Q12 Baulkham Hills 2024 Trial Q5 · **1 mark**

Consider the graph of $y = \dfrac{k}{x+b} + c$ below, with y-intercept at $\left(0, -\dfrac{11}{3}\right)$.



Which of the following statements is true?

(A) $k < b < c$

(B) $b < k < c$

(C) $k < c < b$

(D) $b < c < k$

Criteria	Marks
Correct option (D)	1

Solution.

From the graph the standard reciprocal curve $y = \dfrac{k}{x}$ has been translated 3 units right and 2 units down, so the vertical asymptote is $x = 3$ (giving $b = -3$) and the horizontal asymptote is $y = -2$ (giving $c = -2$). Using the y-intercept $\left(0, -\frac{11}{3}\right)$ to find k:

$$\frac{k}{0-3} - 2 = -\frac{11}{3} \implies \frac{k}{-3} = -\frac{5}{3} \implies k = 5.$$

Hence $b = -3$, $c = -2$, $k = 5$, so $b < c < k$. The answer is (D).

Q13 HSC 2022 Q28(b) · **2 marks**

Part of the hyperbola $y = \dfrac{a}{b-x} - 1$, which passes through the points $(0,0)$ and $(1,1)$, is drawn with the circle $x^2 + y^2 = 2$. Show that $a = b = 2$.

Criteria	Marks
Substitutes $(0,0)$ to obtain $a = b$	1
Substitutes $(1,1)$ and solves to obtain $a = b = 2$	1

Solution.

Substitute the point $(0,0)$ into $y = \dfrac{a}{b-x} - 1$:

$$0 = \frac{a}{b-0} - 1 \implies \frac{a}{b} = 1 \implies a = b.$$

Substitute the point $(1,1)$:

$$1 = \frac{a}{b-1} - 1 \implies \frac{a}{b-1} = 2 \implies a = 2(b-1) = 2b - 2.$$

Since $a = b$, replace a with b: $b = 2b - 2$, so $b = 2$, and therefore $a = b = 2$.

3. Sketch a transformed hyperbola k/(x-a)+c from its equation

Q14 Blacktown Boys 2022 Prelim Q13 · **4 marks**

(a) Show $f(x)$ neither even nor odd. (b) State the domain. (c) Sketch $y = \dfrac{3}{x+2}$.

Criteria	Marks
Shows $f(x)$ is neither even nor odd	1
States the domain correctly	1
Correct graph with all key features (asymptotes, intercept)	2

Solution.

(a) $f(-x) = \dfrac{3}{-x+2} = \dfrac{3}{2-x}$. Since $f(-x) \neq f(x)$, f is not even. Also $-f(-x) = -\dfrac{3}{2-x} = \dfrac{3}{x-2} \neq f(x)$, so f is not odd. $\therefore$ $f(x)$ is neither even nor odd.

(b) The function is undefined when $x + 2 = 0$, i.e. $x = -2$. $\therefore$ domain is $x \neq -2$, that is $(-\infty, -2) \cup (-2, \infty)$.

(c) $y = \dfrac{3}{x+2}$ is a rectangular hyperbola: vertical asymptote $x = -2$, horizontal asymptote $y = 0$, y-intercept $\left(0, \frac{3}{2}\right)$, in the top-right and bottom-left branches relative to the asymptotes.

Q15 Sydney Girls 2021 Prelim Q3(c)(ii) · **2 marks**

Sketch the graph of $y = 2 - \dfrac{1}{x+1}$, showing all the key features.

Criteria	Marks
Correct asymptotes ($x = -1$, $y = 2$)	1
Correct intercepts $\left(-\frac{1}{2}, 0\right)$, $(0, 1)$ and shape	1

Solution.

Write $y = 2 - \dfrac{1}{x+1}$: this is $y = -\dfrac{1}{x}$ shifted 1 left and 2 up. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 2$.

y-intercept: $y = 2 - \dfrac{1}{1} = 1$, i.e. $(0, 1)$. x-intercept: $0 = 2 - \dfrac{1}{x+1} \Rightarrow \dfrac{1}{x+1} = 2 \Rightarrow x + 1 = \frac{1}{2} \Rightarrow x = -\frac{1}{2}$, i.e. $\left(-\frac{1}{2}, 0\right)$.

Because of the $-\dfrac{1}{x+1}$ term the branches sit in the "lower-left/upper-right" pair about $(-1, 2)$.

Q16 St George Girls 2022 Trial Q17 · **3 marks**

Without using calculus, sketch the graph of $y = 2 - \dfrac{1}{x+4}$, showing any intercepts and asymptotes.

Criteria	Marks
Correct asymptotes ($x = -4$, $y = 2$)	1
Correct intercepts $\left(-\frac{7}{2}, 0\right)$ and $\left(0, \frac{7}{4}\right)$	1
Correct shape of both branches	1

Solution.

This is $y = -\dfrac{1}{x}$ shifted 4 left and 2 up. The vertical asymptote is $x = -4$ and the horizontal asymptote is $y = 2$.

y-intercept: $y = 2 - \dfrac{1}{4} = \dfrac{7}{4}$, i.e. $\left(0, \frac{7}{4}\right)$. x-intercept: $0 = 2 - \dfrac{1}{x+4} \Rightarrow x + 4 = \frac{1}{2} \Rightarrow x = -\frac{7}{2}$, i.e. $\left(-\frac{7}{2}, 0\right)$.

The $-\dfrac{1}{x+4}$ term places the branches in the "lower-left/upper-right" pair about $(-4, 2)$.

Q17 Blacktown Boys 2025 Trial Q15(b) · **2 marks**

Given the function $f(x) = \dfrac{3}{2-x} + 1$, sketch the graph of $y = f(x)$, showing all key features.

Criteria	**Marks**
Correct asymptotes ($x = 2$, $y = 1$)	1
Correct intercepts $(5, 0)$, $\left(0, \frac{5}{2}\right)$ and shape	1

Solution.

The vertical asymptote is where $2 - x = 0$, i.e. $x = 2$, and the horizontal asymptote is $y = 1$.

y-intercept: $f(0) = \dfrac{3}{2} + 1 = \dfrac{5}{2}$, i.e. $\left(0, \frac{5}{2}\right)$. x-intercept: $0 = \dfrac{3}{2-x} + 1 \Rightarrow \dfrac{3}{2-x} = -1 \Rightarrow 2 - x = -3 \Rightarrow x = 5$, i.e. $(5, 0)$.

Writing $f(x) = 1 - \dfrac{3}{x-2}$ shows the branches sit in the "lower-left/upper-right" pair about $(2, 1)$: for $x < 2$ the curve is below $y = 1$ rising toward the asymptote, and for $x > 2$ it is above $y = 1$.

Q18 Knox 2022 Prelim Q11(g) · **5 marks**

(i) Asymptotes and axis intercepts of $y = 2 + \dfrac{2}{1-x}$. (ii) Sketch of the curve.

Criteria	**Marks**
(i) Vertical asymptote $x = 1$	1
(i) Horizontal asymptote $y = 2$	1
(i) Intercepts $(2, 0)$ and $(0, 4)$	1
(ii) Correct shape of both branches about $(1, 2)$	1
(ii) Asymptotes and intercepts labelled on the sketch	1

Solution.

(i) Write $y = 2 + \dfrac{2}{1-x}$. The vertical asymptote is where the denominator is zero, $1 - x = 0$, i.e. $x = 1$. As $x \to \pm\infty$ the fraction $\dfrac{2}{1-x} \to 0$, so the horizontal asymptote is $y = 2$.

y-intercept ($x = 0$): $y = 2 + \dfrac{2}{1} = 4$, i.e. $(0, 4)$. x-intercept ($y = 0$): $2 + \dfrac{2}{1-x} = 0 \Rightarrow \dfrac{2}{1-x} = -2 \Rightarrow 1 - x = -1 \Rightarrow x = 2$, i.e. $(2, 0)$.

(ii) The curve is a hyperbola with centre $(1, 2)$ (the intersection of the asymptotes $x = 1$ and $y = 2$). Draw the two branches: one through $(0, 4)$ and one through $(2, 0)$, each approaching the asymptotes. Label $x = 1$, $y = 2$ and the intercepts $(0, 4)$ and $(2, 0)$.

Q19 Glenwood 2020 Trial Q15 · 3 marks

(a) Show that $\dfrac{4x-3}{x} = 4 - \dfrac{3}{x}$. (b) Hence sketch $y = \dfrac{4x-3}{x}$, showing asymptotes and the x-intercept.

Criteria	Marks
(a) Splits the fraction correctly to obtain $4 - \dfrac{3}{x}$	1
(b) Identifies both asymptotes $x = 0$ and $y = 4$	1
(b) Correct sketch with the x-intercept at $x = \frac{3}{4}$	1

Solution.

(a) Split the fraction term by term:

$$\frac{4x-3}{x} = \frac{4x}{x} - \frac{3}{x} = 4 - \frac{3}{x}.$$

(b) Written as $y = 4 - \dfrac{3}{x}$, the graph is the reciprocal curve $-\dfrac{3}{x}$ shifted up 4 units. Vertical asymptote: $x = 0$. Horizontal asymptote: $y = 4$. x-intercept: $4x - 3 = 0 \Rightarrow x = \dfrac{3}{4}$. Sketch a hyperbola with these asymptotes, crossing the x-axis at $\left(\frac{3}{4}, 0\right)$, lying below $y = 4$ for $x > 0$ and above $y = 4$ for $x < 0$.

Q20 Hornsby Girls 2023 Trial Q29(a) · 2 marks

Sketch $f(x) = \dfrac{1-x}{x-2}$, clearly labelling the asymptotes and the x- and y-intercepts.

Criteria	Marks
Correct asymptotes ($x = 2$, $y = -1$)	1
Correct intercepts $(1, 0)$, $\left(0, -\frac{1}{2}\right)$ and shape	1

Solution.

Rewrite by polynomial division: $f(x) = \dfrac{1-x}{x-2} = \dfrac{-(x-2)-1}{x-2} = -1 - \dfrac{1}{x-2}$. Hence the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = -1$.

y-intercept: $f(0) = \dfrac{1}{-2} = -\dfrac{1}{2}$, i.e. $\left(0, -\frac{1}{2}\right)$. x-intercept: numerator zero, $1 - x = 0 \Rightarrow x = 1$, i.e. $(1, 0)$.

The $-\dfrac{1}{x-2}$ term places the branches in the "lower-left/upper-right" pair about $(2, -1)$.

Q21 Girraween 2021 Trial Q22(i) · **3 marks**

Find the vertical and horizontal asymptotes of the hyperbola $y = \dfrac{x-2}{x-4}$, and hence sketch the graph of $y = \dfrac{x-2}{x-4}$, showing all intercepts with the coordinate axes.

Criteria	**Marks**
Vertical asymptote $x = 4$	1
Horizontal asymptote $y = 1$	1
Correct intercepts $(2,0)$, $\left(0, \frac{1}{2}\right)$ and shape	1

Solution.

Divide: $y = \dfrac{x-2}{x-4} = \dfrac{(x-4)+2}{x-4} = 1 + \dfrac{2}{x-4}$. The vertical asymptote is where $x - 4 = 0$, i.e. $x = 4$, and the horizontal asymptote is $y = 1$.

y-intercept: $y = \dfrac{-2}{-4} = \dfrac{1}{2}$, i.e. $\left(0, \frac{1}{2}\right)$. x-intercept: numerator zero, $x - 2 = 0 \Rightarrow x = 2$, i.e. $(2,0)$.

Since the reciprocal term $+\dfrac{2}{x-4}$ is positive, the branches lie in the "upper-right/lower-left" pair about $(4,1)$.

Q22 Barker 2021 Prelim Q12(f) · **3 marks**

Sketch the function $f(x) = 1 - \dfrac{5}{x-2}$, showing all intercepts and other important features.

Criteria	**Marks**
Correct asymptotes ($x = 2$, $y = 1$)	1
Correct intercepts $(7,0)$ and $\left(0, \frac{7}{2}\right)$	1
Correct shape of both branches	1

Solution.

This is $y = -\dfrac{5}{x}$ shifted 2 right and 1 up. The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 1$.

y-intercept: $f(0) = 1 - \dfrac{5}{-2} = 1 + \dfrac{5}{2} = \dfrac{7}{2}$, i.e. $\left(0, \frac{7}{2}\right)$. x-intercept: $0 = 1 - \dfrac{5}{x-2} \Rightarrow \dfrac{5}{x-2} = 1 \Rightarrow x - 2 = 5 \Rightarrow x = 7$, i.e. $(7,0)$.

The $-\dfrac{5}{x-2}$ term places the branches in the "lower-left/upper-right" pair about $(2,1)$.

Solving and combining

4. Intersections of a hyperbola with a line and solving rational equations / inequalities

Q23 Blacktown Boys 2020 Trial Q29(a) · **1 mark**

The diagram shows the graphs of $y = 5 - x$ and $y = \dfrac{4}{2x-1}$, meeting at the points A and B. Show that the x-coordinate values of A and B are 1 and 4.5 respectively.

Criteria	**Marks**
Equates the two curves, clears the denominator and solves for $x = 1$ and $x = 4.5$	1

Solution.

At the points of intersection the two y-values are equal:

$$5 - x = \frac{4}{2x-1}.$$

Multiplying both sides by $(2x-1)$,

$$(5-x)(2x-1) = 4 \quad \Rightarrow \quad 10x - 5 - 2x^2 + x = 4 \quad \Rightarrow \quad -2x^2 + 11x - 9 = 0.$$

Multiplying by -1 gives $2x^2 - 11x + 9 = 0$, i.e. $(x-1)(2x-9) = 0$. Hence $x = 1$ or $x = \frac{9}{2} = 4.5$, which are the x-coordinates of A and B.

Q24 Caringbah 2020 Trial Q20(a) · **2 marks**

The curves $y = 2 - \dfrac{3}{x}$ and $y = x - 2$ (for $x \geq 0$) intersect at two points P and Q. Find the coordinates of P and Q.

Criteria	Marks
Sets the curves equal and reduces to $x^2 - 4x + 3 = 0$, obtaining $x = 1,\ 3$	1
States both points $(1, -1)$ and $(3, 1)$	1

Solution.

Equating the two expressions for y:

$$2 - \frac{3}{x} = x - 2.$$

Multiplying through by x gives $2x - 3 = x^2 - 2x$, so

$$x^2 - 4x + 3 = 0 \quad \Rightarrow \quad (x-1)(x-3) = 0 \quad \Rightarrow \quad x = 1 \text{ or } x = 3.$$

Substituting into $y = x - 2$: when $x = 1$, $y = -1$; when $x = 3$, $y = 1$. Hence the intersection points are $(1, -1)$ and $(3, 1)$.

Q25 Blacktown Boys 2022 Trial Q9 · **1 mark**

The graph of $y = \dfrac{x-2}{x-4}$ is shown. The values of x for which $\dfrac{x-2}{x-4} \leq 3$ are: (A) $x < 4,\ x \geq 5$ (B) $x \leq 3$ (C) $2 \leq x < 4$ (D) $x < 1$.

Criteria	Marks
Correct option (A)	1

Solution.

The line $y = 3$ meets the curve where $\dfrac{x-2}{x-4} = 3$, i.e. $x - 2 = 3(x-4) = 3x - 12$, giving $2x = 10$, $x = 5$. There is a vertical asymptote at $x = 4$. From the graph the curve lies on or below $y = 3$ for $x < 4$ (where the branch is below 3) and for $x \geq 5$ (where the right branch has come back down to and below 3). Hence $x < 4$ or $x \geq 5$, which is option (A).

Q26 Gosford 2024 Trial Q27(a) · **4 marks**

Sketch the graphs of $y = \dfrac{4}{x}$ and $y = |x+1|-4$ on the same number plane, showing their point(s) of intersection and any intercepts on the axes.

Criteria	Marks
Correct rectangular hyperbola $y = \dfrac{4}{x}$ (both branches, asymptotes the axes)	1
Correct V-shape $y = \|x+1\|-4$ with vertex $(-1,-4)$ and x-intercepts $(-5,0)$, $(3,0)$, y-intercept $(0,-3)$	1
Marks the intersection points $(4,1)$ and $(-4,-1)$	1
Marks the third intersection $(-1,-4)$ (the vertex lies on the hyperbola)	1

Solution.

Hyperbola. $y = \dfrac{4}{x}$ is a rectangular hyperbola with the x- and y-axes as asymptotes; it passes through $(1,4)$, $(2,2)$, $(4,1)$ and $(-1,-4)$, $(-2,-2)$, $(-4,-1)$, occupying the first and third quadrants.

Absolute-value graph. $y = |x+1|-4$ is a V with vertex at $(-1,-4)$. Its x-intercepts satisfy $|x+1| = 4$, so $x = 3$ or $x = -5$, giving $(3,0)$ and $(-5,0)$; the y-intercept is $(0, |1|-4) = (0,-3)$.

Intersections. Solve $\dfrac{4}{x} = |x+1|-4$. For $x \geq -1$: $\dfrac{4}{x} = x-3 \Rightarrow x^2-3x-4 = 0 \Rightarrow (x-4)(x+1) = 0$, giving $x = 4$ (point $(4,1)$) and $x = -1$ (point $(-1,-4)$). For $x < -1$: $\dfrac{4}{x} = -x-5 \Rightarrow x^2+5x+4 = 0 \Rightarrow (x+4)(x+1) = 0$, giving $x = -4$ (point $(-4,-1)$). The three intersection points are $(4,1)$, $(-4,-1)$ and $(-1,-4)$.

Q27 Girraween 2021 Trial Q22(ii) · **2 marks**

The graph of $y = \dfrac{x-2}{x-4}$ (vertical asymptote $x = 4$, horizontal asymptote $y = 1$) has been sketched. Hence, or otherwise, find the values of x for which $\dfrac{x-2}{x-4} \leq 3$.

Criteria	Marks
Finds the boundary $x = 5$ from $\dfrac{x-2}{x-4} = 3$ and notes the asymptote $x = 4$	1
States the full solution $x < 4$ or $x \geq 5$	1

Solution.

Solve the boundary equation $\dfrac{x-2}{x-4} = 3$: $x-2 = 3(x-4) = 3x-12$, so $2x = 10$, $x = 5$. The expression is undefined at the vertical asymptote $x = 4$. Testing the three regions determined by $x = 4$ and $x = 5$: for $x < 4$ (e.g. $x = 0$) the value is $\frac{1}{2} \leq 3$ (true); for $4 < x < 5$ (e.g. $x = 4.5$) the value is $5 > 3$ (false); for $x \geq 5$ (e.g. $x = 6$) the value is $2 \leq 3$ (true), with equality at $x = 5$. Hence

$$\frac{x-2}{x-4} \leq 3 \quad \Longleftrightarrow \quad x < 4 \ \text{ or } \ x \geq 5.$$

5. Reciprocal graphs via composition and combined curves

Q28 Caringbah 2020 Trial Q2 · 1 mark

What are the asymptotes of the graph of $y = \dfrac{1}{x^2 - 9}$?

(A) $x = \pm 3$

(B) $x = \pm 9$

(C) $y = \pm 3$

(D) $y = \pm 9$

Criteria	Marks
Correct option (A)	1

Solution.

Vertical asymptotes occur where the denominator is zero and the numerator is not. Setting $x^2 - 9 = 0$ gives $x^2 = 9$, so $x = \pm 3$. (The horizontal asymptote is $y = 0$, since $\frac{1}{x^2-9} \to 0$ as $x \to \pm\infty$, so options C and D are wrong.) The asymptotes asked for are the vertical ones,

$$x = \pm 3.$$

The answer is (A).

Q29 Blacktown Boys 2023 Trial Q14 · **5 marks**

(a) Find $f\big(g(x)\big)$ in simplest form. **(b)** Find the domain of $f\big(g(x)\big)$. **(c)** Sketch $y = f\big(g(x)\big) + 1$, showing all intercepts and asymptotes.

Criteria	Marks
(a) Substitutes $g(x)$ into f and simplifies to $f\big(g(x)\big) = \dfrac{1}{6x+3}$	2
(b) States the domain $x \neq -\frac{1}{2}$ (all real x except $-\frac{1}{2}$)	1
(c) Finds the asymptotes ($x = -\frac{1}{2}$, $y = 1$) and intercepts ($(-\frac{2}{3}, 0)$, $(0, \frac{4}{3})$)	1
(c) Correct hyperbola shape drawn on both sides of $x = -\frac{1}{2}$, consistent with the asymptotes and intercepts	1

Solution.

(a) Substitute $g(x) = 3x + 2$ into f: $f\big(g(x)\big) = \dfrac{1}{2(3x+2)-1} = \dfrac{1}{6x+4-1} = \dfrac{1}{6x+3}$.

(b) The expression is undefined where the denominator is zero: $6x + 3 = 0$ gives $x = -\frac{1}{2}$. Hence the domain is all real x with $x \neq -\frac{1}{2}$, i.e. $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

(c) The curve is $y = \dfrac{1}{6x+3} + 1$, a reciprocal (hyperbola) shifted up by 1. *Asymptotes:* the denominator vanishes at $x = -\frac{1}{2}$ (vertical asymptote); since $\dfrac{1}{6x+3} \to 0$ as $x \to \pm\infty$, $y \to 1$ (horizontal asymptote $y = 1$). *Intercepts:* for the x-intercept put $y = 0$: $\dfrac{1}{6x+3} = -1$, so $6x + 3 = -1$, $6x = -4$, $x = -\frac{2}{3}$, giving $\left(-\frac{2}{3}, 0\right)$; for the y-intercept put $x = 0$: $y = \dfrac{1}{3} + 1 = \dfrac{4}{3}$, giving $\left(0, \frac{4}{3}\right)$. *Shape:* a rectangular hyperbola with asymptotes $x = -\frac{1}{2}$ and $y = 1$; the left branch passes through $\left(-\frac{2}{3}, 0\right)$ and falls to $-\infty$ as $x \to -\frac{1}{2}^{-}$, and the right branch comes down from $+\infty$, passes through $\left(0, \frac{4}{3}\right)$, and approaches $y = 1$ from above.

Q30 Killara 2023 Prelim Q29 · **5 marks**

(a) Find an expression for $g\big(f(x)\big)$ and state its domain. (b) Hence sketch $g\big(f(x)\big)$ over its natural domain.

Criteria	**Marks**
(a) Correct expression $g\big(f(x)\big) = -\dfrac{1}{x+2}$	1
(a) Correct natural domain $[-4,-2)\cup(-2,\infty)$	1
(b) Asymptotes $x=-2$, $y=0$ and the included endpoint $\left(-4,\frac{1}{2}\right)$	1
(b) y-intercept $\left(0,-\frac{1}{2}\right)$ and no x-intercept	1
(b) Correct hyperbola shape over the restricted domain	1

Solution.

(a) Substitute $f(x)=\sqrt{x+4}$ into g:

$$g\big(f(x)\big) = -\frac{1}{\left(\sqrt{x+4}\right)^2-2} = -\frac{1}{(x+4)-2} = -\frac{1}{x+2}.$$

For $f(x)=\sqrt{x+4}$ to be defined we need $x+4\geq 0$, i.e. $x\geq -4$; and g is undefined where the denominator is zero, i.e. $x+2=0\Rightarrow x=-2$. Hence the natural domain is $x\geq -4$ with $x\neq -2$, i.e. $[-4,-2)\cup(-2,\infty)$.
(b) The curve is $y=-\dfrac{1}{x+2}$, drawn only over $[-4,-2)\cup(-2,\infty)$. The vertical asymptote is $x=-2$ and the horizontal asymptote is $y=0$. The domain begins at $x=-4$, where $y=-\dfrac{1}{-4+2}=\dfrac{1}{2}$, so the graph starts at the included point $\left(-4,\frac{1}{2}\right)$ (closed dot). The y-intercept is $y=-\dfrac{1}{2}$, at $\left(0,-\frac{1}{2}\right)$; setting $y=0$ has no solution, so there is no x-intercept. On $[-4,-2)$ the branch rises from $\left(-4,\frac{1}{2}\right)$ to $+\infty$ as $x\to -2^-$; on $(-2,\infty)$ the branch comes up from $-\infty$ as $x\to -2^+$, passes through $\left(0,-\frac{1}{2}\right)$ and approaches $y=0$ from below.