SYDNEY BOYS HIGH SCHOOL

2023

YEAR 9 STAGE 5.3
TERM 2
HALF YEARLY EXAMINATION

Mathematics

General Instructions: **Time Allowed:** 90 minutes

- All questions may be attempted.
- **Marks may be deducted for careless or badly arranged work.**
- Show ALL relevant mathematical reasoning and/or calculations.
- Write using a non-erasable black pen.
- If you wish to rewrite an answer, draw a line through your faulty answer and rewrite your answer on the back page of the booklet.
  Show the number and part of the answer being rewritten.
- Leave your answers in the **simplest exact form**, unless otherwise stated.
- NESA approved calculators may be used
- Clearly indicate your class by placing a cross **X** next to your class.

Examiner: AF

Name: _____

Class	Teacher	
9Ma**A**	AYW & JJ	
9Ma**B**	MZ	
9Ma**C**	JT	
9Ma**P**	DT	
9Ma**L**	RW	
9Ma**U**	AW	
9Ma**S**	JC	

Section	Marks
A	/20
B	/17
C	/16
D	/12
E	/14
Total	/79

Section A

Question 1 (4 marks)
State whether the following a rational or irrational:

(a) $\sqrt{3}$ **1**

……

(b) $1\frac{2}{3}$ **1**

……

(c) $\frac{\pi}{4}$ **1**

……

(d) $0.3\dot{4}$ **1**

……

Question 2 (4 marks)

Simplify the following surds:

(a) $\sqrt{147}$ **1**

……

……

(b) $7\sqrt{2} - \sqrt{2}$ **1**

……

(c) $\sqrt{4a} + 5\sqrt{a}$ **1**

……

……

(d) $\sqrt{10} \times \sqrt{15}$ **1**

……

……

Question 3 (2 marks)

Evaluate the following given that $a = -2$, $b = \frac{3}{4}$ and $c = 5$.

(a) ac^2 **1**

……

……

(b) $\frac{a+b}{bc}$ **1**

……

……

Question 4 (1 marks)

Write in ascending order: **1**

$5.421,\ 1.1 \times 10^5,\ 9.654 \times 10^{-10},\ 1.235 \times 10^2$

……

Question 5 (3 marks)

Find the value of the pronumerals in the following:

(a) $a^3 = 125$ **1**

……

(b) $10^b = 1\ 000\ 000\ 000\ 000$ **1**

……

(c) $400 = 2^c \times 5^d$ **1**

……

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Question 6 (6 marks)

Simplify the following:

(a) $3^a \times 3^b$ **1**

……

……

(b) $a^{11} \div a^7$ **1**

……

……

(c) $(a^3)^5$ **1**

……

……

(d) $2^a \times 5^a$ **1**

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……

(e) $3^a + 3^a + 3^a$ **1**

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……

(f) $k^k \div \underbrace{(1+1+\ldots+1)}_{k \text{ times}}$ **1**

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Extra Working Page for Section A only

Section B

Name: _____

Class	Teacher	
9**A**	AYW & JJ	
9**B**	MZ	
9**C**	JT	
9**P**	DT	
9**L**	RW	
9**U**	AW	
9**S**	JC	

Mark	**/17**

Question 1 (1 mark)

From a 3.6 m piece of timber, 5 pieces of equal length are cut, leaving 28 cm. **1**

What length were the timbers cut to?

……

……

……

……

Question 2 (5 marks)

Express the following in the form $\sqrt{\triangle}$:

(a) $\sqrt{3} \times \sqrt{7}$ **1**

……

(b) $\sqrt{18} + \sqrt{32}$ **2**

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(c) $\dfrac{x-y}{x+y}\sqrt{\dfrac{x+y}{x-y}}$ **2**

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Question 3 (2 marks)

$Q = \{1, 2, 3, 4\}$.
Write down a possible set P where:

(a) $P \subset Q$ **1**

……

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……

(b) $P \cup Q = \{1, 2, 3, 4, 5, 6\}$ **1**

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Question 4 (2 marks)

Write the following without negative or fractional indices in simplest form:

$$27a^{\frac{1}{3}}b^{-4}$$ **2**

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Question 5 (5 marks)

Simplify the following:

(a) $\frac{4}{3a}+\frac{5}{3a}$ **1**

……

(b) $\frac{7x}{12y}-\frac{5x}{8y}$ **2**

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(c) $\frac{5+x}{8y}-\frac{7+x}{16y}$ **2**

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Question 6 (2 marks)

For the following, <u>circle</u> the larger of the two numbers:

(a) 300^{900}, 900^{300} **1**

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(b) 2^{450}, 3^{300} **1**

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Extra Working Page for Section B only

Section C

Name: _____

Class	Teacher	
9**A**	AYW & JJ	
9**B**	MZ	
9**C**	JT	
9**P**	DT	
9**L**	RW	
9**U**	AW	
9**S**	JC	

Mark	**/16**

Question 1 (3 marks)

If $2^a = 5$. Evaluate the following:

(a) 2^{a+2} **1**

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(b) 4^a **1**

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(c) 2^{-a} **1**

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Question 2 (2 marks)

What is half of:

(a) 2^{2n} **1**

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(b) 1.3×10^{57} (write your answer in scientific notation) **1**

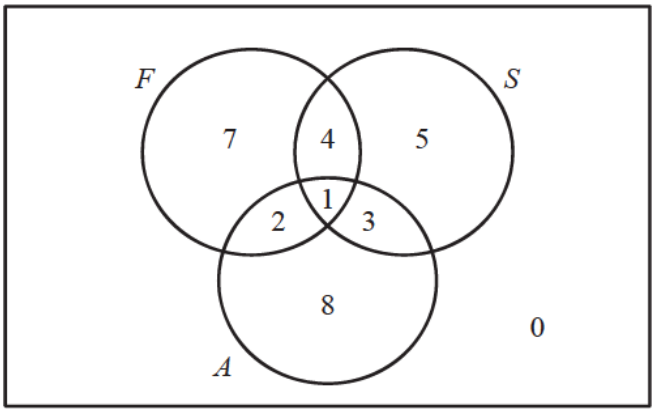
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Question 3 (2 marks)

The Venn diagram shows the number of students who study French (F), Spanish (S) and Arabic (A).



(a) Find $n(A \cup (F \cap S))$ **1**

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(b) On the Venn diagram, shade the region $\tilde{F} \cap S$, where $\tilde{F}$ is the complement of F. **1**

Question 4 (4 marks)

Expand and simplify the following:

(a) $2\sqrt{3}\left(\sqrt{5}+3\right)$ **1**

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(b) $\left(2-\sqrt{3}\right)\left(3\sqrt{5}+3\right)$ **1**

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(c) $\left(\sqrt{2}+\sqrt{5}+\sqrt{7}\right)\left(\sqrt{2}+\sqrt{5}-\sqrt{7}\right)$ **2**

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Question 5 (5 marks)

The time is 2 pm.

(a) Express this time in 24-hour time. **1**

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(b) What is the angle between the hour hand and the minute hand on an analogue clock at this time? **1**

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(c) Calculate the least time which passes until the angle between the hour hand and the minute hand is the same as what was found in part (b). **3**

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Extra Working Page for Section C only

Section D

Name: _____

Class	Teacher	
9**A**	AYW & JJ	
9**B**	MZ	
9**C**	JT	
9**P**	DT	
9**L**	RW	
9**U**	AW	
9**S**	JC	

Mark	**/12**

Question 1 (4 marks)

Mickey lists the positive integers, in order, leaving out the integers that are multiples of 5.

The resulting list is: 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, …

(a) How many integers has Mickey listed just before he leaves out the 6^{th} multiple of 5? **1**

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(b) If the kth integer in Mickey's list is 2023, determine the value of k. **3**

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Question 2 (8 marks)

The positive integers are arranged in rows and columns, as shown below.

	A	B	C	D	E	F	G	H	I
Row 1		1	2	3	4	5	6	7	8
Row 2	16	15	14	13	12	11	10	9	
Row 3		17	18	19	20	21	22	23	24
Row 4	32	31	30	29	28	27	26	25	

(a) Determine the largest integer in row 20 **1**

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(b) Determine the sum of the eight integers in row 2023 **2**

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(c) Determine the row and column in which the integer 2023 appears. **2**

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(d) For how many rows is the sum of the eight integers greater than 10 000 and less than 20 000? **3**

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Extra Working Page for Section D only

Section E

Name: _____

Class	Teacher	
9**A**	AYW & JJ	
9**B**	MZ	
9**C**	JT	
9**P**	DT	
9**L**	RW	
9**U**	AW	
9**S**	JC	

Mark	**/14**

Question 1 (8 marks)

In a Dubla sequence, the first term is a positive integer and each term after the first is calculated by adding 1 to the previous term in the sequence, then doubling the result.
For example, the first seven terms of a Dubla sequence with first term 4 are: 4, 10, 22, 46, 94, 190, 382

(a) The 5^{th} term of a Dubla sequence is 142. What are the 4^{th} and 6^{th} terms in the sequence? **2**

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(b) Determine all possible first terms which give a Dubla sequence that includes 702. **2**

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(c) Which possible first terms from 10 to 19 inclusive produce a Dubla sequence in which all the terms after the first have the same ones (unit) digits? **2**

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(d) Determine the number of positive integers between 1 and 2023, inclusive, that can be the third term in a Dubla sequence. **2**

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Question 2 (6 marks)

$$a^n - b^n = (a - b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}\right)$$

You may use this result to answer the following questions.

(a) Factorise $32 - x^5$. **1**

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(b) Prove that the difference between the squares of consecutive numbers is equal to twice the smaller number plus one. **2**

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Question 2 continues over the page

Question 2 (continued)

(c) You may use this result to answer the following question.

$$a^n - b^n = (a - b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}\right)$$

Prove that **3**

$$\underbrace{(11111\ldots111)}_{2n\ 1\text{s}} - \underbrace{(22222\ldots222)}_{n\ 2\text{s}}$$

is divisible by 9.

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End of paper

Extra Working Page for Section E only

SYDNEY BOYS HIGH SCHOOL

2023

YEAR 9 STAGE 5
TERM 2
HALF YEARLY

Sample Solutions

NOTE: Some of you may be disappointed with your mark.
This process of checking your mark is NOT the opportunity to improve your marks. Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done. Writing something down does not justify that your working can be linked to a mark.

Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

Section A

Question 1 (4 marks)
State whether the following a rational or irrational:

(a) $\sqrt{3}$ IRRATIONAL **1**

.....

(b) $1\frac{2}{3}$ RATIONAL **1**

.....

(c) $\frac{\pi}{4}$ IRRATIONAL **1**

.....

(d) $0.3\dot{4}$ RATIONAL $0.3\dot{4} = \frac{31}{90}$ **1**

.....

Marker's comment: Mostly well done. Some students have difficulty with (c) and (d). Although (c) was written in fraction form, pi is irrational; and (d) can be written as a fraction, hence rational.

Question 2 (4 marks)

Simplify the following surds:

(a) $\sqrt{147}$ $= 7\sqrt{3}$ Use the FACT operation on the calculator: **1**

$\sqrt{147} = \sqrt{49 \times 3} = \sqrt{49} \times \sqrt{3} = 7\sqrt{3}$ ..... [1] [4] [7] [=] [SHIFT] [$\sqrt{\square}$] .....

(b) $7\sqrt{2} - \sqrt{2}$ $= 6\sqrt{2}$ **1**

.....

(c) $\sqrt{4a} + 5\sqrt{a}$ $2\sqrt{a} + 5\sqrt{a}$ $= 7\sqrt{a}$ **1**

.....

(d) $\sqrt{10} \times \sqrt{15}$ $= \sqrt{150}$ $= \sqrt{25 \times 6}$ $= 5\sqrt{6}$ **1**

Marker's comment: Mostly well done. Some students struggle with (c) and simplifying or operating with surds.

Question 3 (2 marks)

Evaluate the following given that $a = -2$, $b = \frac{3}{4}$ and $c = 5$.

(a) ac^2 $-2 \times 5^2 = -50$ **1**

(b) $\frac{a+b}{bc}$ **1**

$$\frac{-2+\frac{3}{4}}{\frac{3}{4}\times 5} = \frac{-\frac{5}{4}}{\frac{15}{4}} = -\frac{1}{3}$$

Half mark if student shows the working out of $\frac{-\frac{5}{4}}{\frac{15}{4}}$

Or if student answered $\frac{1}{3}$

Question 4 (1 marks)

Write in ascending order: **1**

$5.421,\ 1.1 \times 10^5,\ 9.654 \times 10^{-10},\ 1.235 \times 10^2$

$9.654 \times 10^{-10},\ 5.421,\ 1.235 \times 10^2,\ 1.1 \times 10^5$

Marker's comment: Half mark for arranging the numbers in descending order or if there are only two correctly ordered.

Question 5 (3 marks)

Find the value of the pronumerals in the following:

(a) $a^3 = 125$ $a = 5$ Half mark for -5 **1**

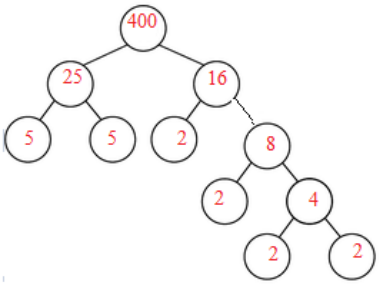
(b) $10^b = 1\ 000\ 000\ 000\ 000$ $b = 12$ **1**

Some students had difficulty with this question, it is not the same as 1.0×10^{11}

(c) $400 = 2^c \times 5^d$ $c = 4$ and $d = 2$ **1**

Use the FACT operation on the calculator:

4 0 0 = SHIFT °'''



Question 6 (6 marks)

Simplify the following:

(a) $3^a \times 3^b$ $= 3^{a+b}$ **1**

.....

(b) $a^{11} \div a^7$ $= a^4$ **1**

.....

(c) $(a^3)^5$ $= a^{15}$ **1**

.....

(d) $2^a \times 5^a$ $= 10^a$ Half mark for $(2 \times 5)^a$ **1**

.....

(e) $3^a + 3^a + 3^a$ $= 3$ lots of 3^a Half mark for $3(3^a)$ **1**

$= 3^1 \times 3^a$

$= 3^{a+1}$

.....

(f) $k^k \div \underbrace{(1 + 1 + \ldots + 1)}_{k \text{ times}}$ $k^k \div k$ lots of 1 **1**

$= \dfrac{k^k}{k(1)}$

$= \dfrac{k^k}{k^1}$

$= k^{k-1}$

.....

Marker's comment: Fairly well done. Parts (d) and (f) was hard to comprehend for some students.

2023 YR9 HYE Section B Solutions and Comments

Answers in pencil and/or with white out will **NOT** have their marks changed.

Question 1

From a 3.6 m piece of timber, 5 pieces of equal length are cut, leaving 28 cm. 1

What lengths were the timbers cut to?

Solution	Comment(s)
Let x be the pieces of timber of equal length. Then: $$5x + 28 = 360$$ $$x = \frac{360 - 28}{5}$$ $$= 66.4$$ Hence, each piece is 66.4 cm long.	No half marks. Students generally performed well on this question.

Question 2

Express the following in the form $\sqrt{\Delta}$:

A. $\sqrt{3} \times \sqrt{7}$ 1

Solution	Comment(s)
$$\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7}$$ $$= \sqrt{21}$$	No half marks. Students generally performed well on this question.

B. $\sqrt{18} + \sqrt{32}$ 2

Solution	Comment(s)
$$\sqrt{18} + \sqrt{32} = \sqrt{9 \times 2} + \sqrt{16 \times 2}$$ $$= (\sqrt{9} \times \sqrt{2}) + (\sqrt{16} \times \sqrt{2})$$ $$= 3\sqrt{2} + 4\sqrt{2}$$ $$= 7\sqrt{2}$$ $$= \sqrt{49 \times 2}$$ $$= \sqrt{98}$$	Common error(s): • Writing the answer as $7\sqrt{2}$ instead of $\sqrt{98}$, as required by the question. • Incorrectly writing $\sqrt{18} + \sqrt{32} = \sqrt{50}$.

C. $\frac{x-y}{x+y}\sqrt{\frac{x+y}{x-y}}$ 2

Solution	Comment(s)
$$\frac{x-y}{x+y}\sqrt{\frac{x+y}{x-y}} = \sqrt{\left(\frac{x-y}{x+y}\right)^2} \times \sqrt{\frac{x+y}{x-y}}$$ $$= \sqrt{\frac{(x-y)^2}{(x+y)^2} \times \frac{x+y}{x-y}}$$ $$= \sqrt{\frac{x-y}{x+y}}$$	Students who wrote $\frac{x-y}{x+y} = \sqrt{\frac{x^2-y^2}{x^2+y^2}}$ instead of $\frac{x-y}{x+y} = \sqrt{\frac{(x-y)^2}{(x+y)^2}}$ could not receive any marks.

2023 YR9 HYE Section B Solutions and Comments

Answers in pencil and/or with white out will **NOT** have their marks changed.

Question 3

$Q = \{1, 2, 3, 4\}$.

Write down a possible set P where:

A. $P \subset Q$ 1

Solution	Comment(s)
One of the simplest solutions is $P = \{1\}$.	No half marks. Various alternatives such as $P = \{2\}$, $P = \{1, 2\}$ etc. were accepted. Students who wrote $P = \{1, 2, 3, 4\}$ risked receiving no marks.

B. $P \cup Q = \{1, 2, 3, 4, 5, 6\}$ 1

Solution	Comment(s)
The simplest solution is $P = \{5, 6\}$.	No half marks. Various alternatives such as $P = \{4, 5, 6\}$ etc. were accepted.

Question 4

Write the following without negative or fractional indices in simplest form: 2

$$27a^{\frac{1}{3}}b^{-4}$$

Solution	Comment(s)
$27a^{\frac{1}{3}}b^{-4} = \dfrac{27\sqrt[3]{a}}{b^4}$	Common error(s): • Incorrectly writing $a^{\frac{1}{3}} = \dfrac{1}{a^3}$. • Misinterpreting $27a^{\frac{1}{3}}$ as $(27a)^{\frac{1}{3}}$ and hence incorrectly writing that $27a^{\frac{1}{3}} = 3\sqrt[3]{a}$ or variants.

2023 YR9 HYE Section B Solutions and Comments

Answers in pencil and/or with white out will **NOT** have their marks changed.

Question 5

Simplify the following:

A. $\frac{4}{3a}+\frac{5}{3a}$ 1

Solution	Comment(s)
$\frac{4}{3a}+\frac{5}{3a}=\frac{9}{3a}$ $=\frac{3}{a}$	Students who didn't simplify $\frac{9}{3a}$ into $\frac{3}{a}$ could not receive any marks.

B. $\frac{7x}{12y}-\frac{5x}{8y}$ 2

Solution	Comment(s)
$\frac{7x}{12y}-\frac{5x}{8y}=\frac{2\times 7x}{2\times 12y}-\frac{3\times 5x}{3\times 8y}$ $=\frac{14x}{24y}-\frac{15x}{24y}$ $=\frac{-x}{24y}$	Some students took the less efficient approach of taking a common denominator of $96y^2$ instead of $24y$. Of those who did, some were subsequently penalised for not simplifying their answer.

C. $\frac{5+x}{8y}-\frac{7+x}{16y}$ 2

Solution	Comment(s)
$\frac{5+x}{8y}-\frac{7+x}{16y}=\frac{2(5+x)}{16y}-\frac{7+x}{16y}$ $=\frac{10+2x-7-x}{16y}$ $=\frac{x+3}{16y}$	Common error(s): • Incorrectly writing $-\frac{7+x}{16y}=\frac{-7+x}{16y}$. • Incorrectly writing $2x-x=3x$.

2023 YR9 HYE Section B Solutions and Comments

Answers in pencil and/or with white out will **NOT** have their marks changed.

Question 6

For the following, circle the larger of the two numbers:

A. $300^{900}, 900^{300}$ 1

Solution	Comment(s)
$300^{900} = 300^{3\times300}$ $= (300^3)^{300}$ $= (2.7 \times 10^7)^{300}$ $> 900^{300}$	No half marks. Students generally performed well on this question.

B. $2^{450}, 3^{300}$ 1

Solution	Comment(s)
$2^{450} = 2^{3\times150}$ $= (2^3)^{150}$ $= 8^{150}$ $3^{300} = 3^{2\times150}$ $= (3^2)^{150}$ $= 9^{150}$ $9 > 8$ $\therefore 9^{150} > 8^{150}$ $3^{300} > 2^{450}$	No half marks. Students generally performed well on this question.

Section C

SOLUTIONS

Question 1 (3 marks)

If $2^a = 5$. Evaluate the following:

(a) 2^{a+2} **1**

$2^{a+2} = 2^a \times 2^2 = 5 \times 4 = 20$

(b) 4^a **1**

$4^a = (2^2)^a = (2^a)^2 = 5^2 = 25$

(c) 2^{-a}

$2^{-a} = \frac{1}{2^a} = \frac{1}{5}$ or 0.2 **1**

In general, this question was poorly done.
Students could not correctly apply index laws.

*Common **mistakes**:*

Part (a)
1. $2^{a+2} = 2^a + 2^2 = 5 + 4 = 9$
2. $2^{a+2} = (2^a)^2 = 5^2 = 25$

Part (b)
1. $4^a = 2 \times 2^a = 2 \times 5 = 10$
2. $4^a = 2 \times 2^a = 4 \times 5 = 20$

Part (c)
1. $2^{-a} = -2^a = -5$
2. $2^{-a} = \frac{1}{2^5} = \frac{1}{32}$

Question 2 (2 marks)

What is half of:

(a) 2^{2n} **1**

$\frac{1}{2} \times 2^{2n} = 2^{2n-1}$

Poorly done. No half marks.

*Common **mistakes**:*

- *Some students halved the index*: 2^{n}
- *Some students halved the base:* 1^{2n}
- *Some students did not evaluate their answers*: $\frac{2^{2n}}{2}, \frac{4^{n}}{2}$

(b) 1.3×10^{57} (write your answer in scientific notation) **1**

$\frac{1}{2} \times 1.3 \times 10^{57} = 0.65 \times 10^{57} = 6.5 \times 10^{56}$

Generally well done. No half marks.

*Common **mistakes**:*

- *Some students did not correctly write their answers in scientific notation: e.g.* 0.65×10^{57}, 65×10^{55}
- *Some students halved the index*: 1.3×10^{28}
- *Some students halved both* 1.3 *and* 56*:* 6.5×10^{28}

Question 3 (2 marks)

The Venn diagram shows the number of students who study French (F), Spanish (S) and Arabic (A).

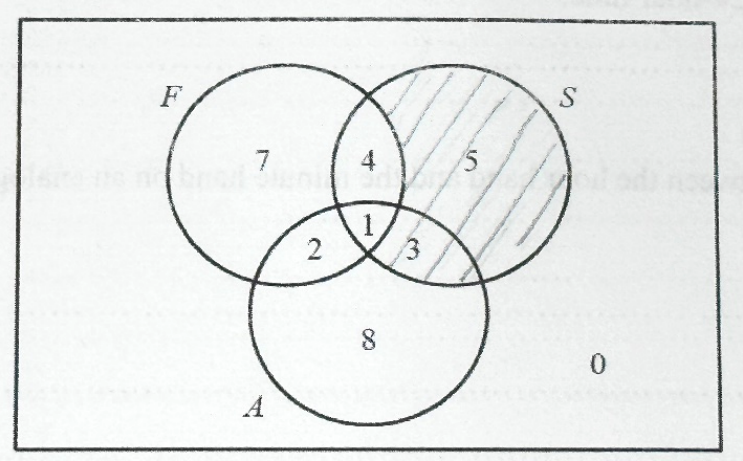
(a) Find $n(A \cup (F \cap S))$ **1**

$n(A \cup (F \cap S)) = n(A) + n(F \cap S) - n(A \cap (F \cap S)) = 14 + 5 - 1 = 18$

Generally well done.

Common mistake: 1

(b) On the Venn diagram, shade the region $\tilde{F} \cap S$, where $\tilde{F}$ is the complement of F. **1**



Generally well done.
Some students did not see the question, so they did not answer it !

Common mistake: *students shaded the region $F \cap S$.*

Question 4 (4 marks)

Expand and simplify the following:

(a) $2\sqrt{3}\left(\sqrt{5}+3\right)$ **1**

$2\sqrt{3}(\sqrt{5}+3) = 2\sqrt{15}+6\sqrt{3}$

Generally well done.

(b) $\left(2-\sqrt{3}\right)\left(3\sqrt{5}+3\right)$ **1**

$(2-\sqrt{3})(3\sqrt{5}+3) = 6\sqrt{5}+6-3\sqrt{15}-3\sqrt{3}$

Generally well done.

(c) $\left(\sqrt{2}+\sqrt{5}+\sqrt{7}\right)\left(\sqrt{2}+\sqrt{5}-\sqrt{7}\right)$ **2**

$(\sqrt{2}+\sqrt{5}+\sqrt{7})(\sqrt{2}+\sqrt{5}-\sqrt{7})$

$= (\sqrt{2}+\sqrt{5})^2 - (\sqrt{7})^2$ (difference of squares)

$= 2+2\sqrt{10}+5-7$ (perfect square)

$= 2\sqrt{10}$

Generally well done. No half marks.
Most students directly expanded brackets.
There were many careless mistakes.

Question 5 (5 marks)

The time is 2 pm.

(a) Express this time in 24-hour time. **1**

14.00

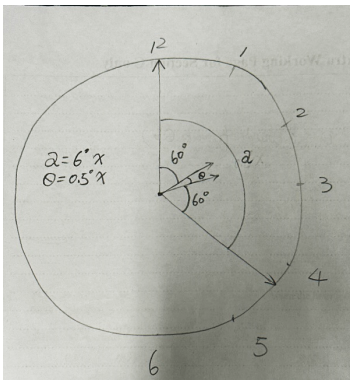
Every student got this question right. 😊

(b) What is the angle between the hour hand and the minute hand on an analogue clock at this time? **1**

$$\frac{2}{12} \times 360° = 60°$$

Extremely well done.
Few mistakes: 30°, 75°

(c) Calculate the least time which passes until the angle between the hour hand and the minute hand is the same as what was found in part (b). **3**



Every minute, hour hand moves $\frac{1}{60} \times \frac{360°}{12} = 0.5°$,

minute hand moves $\frac{1}{60} \times 360° = 6°$

Method 1:

Let x be the least minute which passes until the angle between the hour hand and the minute hand is 60°

$$6°x - (60° + 0.5°x) = 60°$$

$\therefore x = \frac{120}{5.5} = 21\frac{9}{11}$ minutes or 21 minutes 49 seconds

Method 2:

$$\frac{60° + 60°}{6° - 0.5°} = \frac{120°}{5.5°} = 21\frac{9}{11} \text{ minutes}$$

Poorly done.

In order to get full marks, students needed to:

- *correctly identify the degrees that hour hand and minute hand move every minute, or something equivalent.*
- *correctly translate the question into an equation, or* ***thoroughly*** *explain how they got the time.*
- *include sufficient working out*
- *have the correct time.*

Marks were awarded for:

- *correctly identifying the angle between the hour hand and the minute hand at* 2:10, 2:15, 2:20, 2:25 *etc.*
- *correctly identifying the ratio of speed:* *Hour hand : Minute hand* = 1 : 12
- *identifying that the minute hand catches up* 5.5° *per minute.*

Question 1 (Section D)

Mickey lists the positive integers in order, leaving out multiples of 5: 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, ...

a) How many integers has Mickey listed before he leaves out the 6th multiple of 5?

The 6th multiple of 5 is $6 \times 5 = 30$.

So $30 - 6 = 24$.

∴ He has listed 24 integers.

Some students saw groups of 4, so divided 30 by 4, forgetting the missing integers.

If you have time, you could test your method on a smaller list.

b) If the k^{th} integer in Mickey's list is 2023, determine the value of k.

Find how many multiples of 5 first:

$2023 \div 5 = 404 \text{ r } 3$

Subtract this from 2023:

$$\begin{array}{r} 2023 - \\ 404 \\ \hline 1619 \end{array}$$

∴ There are 1619 integers in the list up to 2023.

Some students mistook k from the number of digits. Read carefully.

Question 2

a) The largest integer in row n is $8n$.

∴ In row 20 the largest is $8 \times 20 = 160$.

Come up with a formula and check it on each of the 4 given rows.

Section 1 (cont'd)

Question 2

d) The first row: $64n - 28 > 10\,000$ (from (b))

$64n > 10\,028$

$n > \frac{10\,028}{64} = 156 \text{ r } 4$

So first row is 157.

The last row: $64n - 28 < 20\,000$

$64n < 20\,028$

$n < \frac{20\,028}{64} = 312 \text{ r } 60$

So the last row is 312 (since less than).

The number of rows $= 312 - 157 + 1$ (+1 to count first row)

$= 156.$

These questions can be tricky when you need to round up, round down and add 1. Some students ended up off by 1.

Think carefully.

Also read carefully. Some students missed the word "sum", some focused on the largest numbers in the row and some just divided by 8.

Overall, Section was done reasonably well, but some falling over on the small details. Read carefully, find a pattern or formula, test it, see if your answer seems reasonable.

Section D (cont'd)

Question 2

b) Each row (n^{th} row) has sum:

$$\begin{array}{r} 8n + \\ 8n-1 \\ 8n-2 \\ 8n-3 \\ 8n-4 \\ 8n-5 \\ 8n-6 \\ 8n-7 \\ \hline 64n-28 \end{array}$$

For row 2023:

$\text{sum} = 64 \times 2023 - 28$

$= 129\,444$

Be careful to add the full value of each number. Some students used $8n-28$.

Again, come up with a formula, and test it if you have time.

c) Find the row:

$2023 \div 8 = 252 \text{ r } 7$

So this is row 253. (Think that $7 \div 8 = 0 \text{r} 7$ and is in row 1, not row 0.)

The column is for the 7^{th} value.

2023 is odd, so the 7^{th} column is H.

∴ row 253, column N.

Section E

SOLUTIONS

Question 1 (8 marks)

In a Dubla sequence, the first term is a positive integer and each term after the first is calculated by adding 1 to the previous term in the sequence, then doubling the result.
For example, the first seven terms of a Dubla sequence with first term 4 are: 4, 10, 22, 46, 94, 190, 382

(a) The 5th term of a Dubla sequence is 142. What are the 4th and 6th terms in the sequence? **2**

$T_4 = \frac{142}{2} - 1 = 70$

$T_6 = (142+1) \times 2 = 286$

Generally well done. Students should clearly indicate $T_4 = 70$ and $T_6 = 286$, though no marks were penalised.

(b) Determine all possible first terms which give a Dubla sequence that includes 702. **2**

$\frac{702}{2} - 1 = 350$

$\frac{350}{2} - 1 = 174$

$\frac{174}{2} - 1 = 86$

$\frac{86}{2} - 1 = 42$

$\frac{42}{2} - 1 = 20$

$\frac{20}{2} - 1 = 9$

$\frac{9}{2} - 1 = 3.5$ which is not an integer

$\therefore$ All possible first terms: 702, 350, 174, 86, 42, 20, 9

Some students forgot to include 702 as a solution.

(c) Which possible first terms from 10 to 19 inclusive produce a Dubla sequence in which all the terms after the first have the same ones (unit) digits? **2**

$10 \to 22 \to 46$

$11 \to 24 \to 50$

$12 \to 26 \to 54$

$13 \to 28 \to 58$

$14 \to 30 \to 62$

$15 \to 32 \to 66$

$16 \to 34 \to 70$

$17 \to 36 \to 74$

$18 \to 38 \to 78$

$19 \to 40 \to 82$

$\therefore$ 13 and 18.

'after the first' means you need to calculate T_2 and T_3, and compare their ones digit.

Many students did not include 13 in their solution.

Note: If the digit ends in 8, so will the next number: $(8+1)2 = 18$

(d) Determine the number of positive integers between 1 and 2023, inclusive, that can be the third term in a Dubla sequence. **2**

Let the first term be $x \Rightarrow T_1 = x$

$T_2 = 2(T_1+1) = 2(x+1)$

$T_3 = 2(T_2+1) = 2[2(x+1)+1]$

$= 2(2x+2+1)$

$= 4x + 6$

$4x + 6 < 2023$

$4x < 2017$

$x < \frac{2017}{4} = 504.25$

$\therefore$ 504 positive integers.

Some students took $T_1 = 2(x+1)$ and $T_3 = 8x+14$, hence they were unsuccessful in obtaining the correct solution.

Question 2 (6 marks)

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}\right)$$

You may use this result to answer the following questions.

(a) Factorise $32 - x^5$. **1**

$32 - x^5 = 2^5 - x^5$

$= (2-x)(2^{5-1} + 2^{5-2}x + 2^{5-3}x^2 + 2^{5-4}x^3 + 2^{5-5}x^4)$

$= (2-x)(16 + 8x + 4x^2 + 2x^3 + x^4)$

Note: powers of 2 are decreasing, powers of x are increasing.

(b) Prove that the difference between the squares of consecutive numbers is equal to twice the smaller number plus one. **2**

Let the consecutive numbers be x and y. If x is the smaller number, then $y = x+1$.

We want to prove: $(x+1)^2 - x^2 = 2x+1$.

LHS $= (x+1)^2 - x^2$

$= x^2 + 2x + 1 - x^2$

$= 2x + 1$

$= 2($smaller number $+ 1) =$ RHS

- Many students proved using specific numbers. Students need to prove the question generally i.e. for all numbers.
- Some students recognised they needed to prove $y^2 - x^2 = 2x+1$, but failed to make the connection between x and y, i.e. $y = x+1$.

Question 2 continues over the page

- Poor setting out. Many students omitted LHS and RHS.

e.g. LHS $= (x+1)^2 - x^2 = 2x+1$

$= x^2 + 2x + 1 - x^2 = 2x+1$

$= 2x + 1 = 2x+1$

$=$ RHS

Question 2 (continued)

(c) You may use this result to answer the following question.

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}\right)$$

Prove that **3**

$$\underbrace{(11111\ldots111)}_{2n\ 1\text{s}} - \underbrace{(22222\ldots222)}_{n\ 2\text{s}}$$

is divisible by 9.

Method 1

$\underbrace{11111\ldots111}_{2n\ 1\text{s}} - \underbrace{22222\ldots222}_{n\ 2\text{s}}$

$= \underbrace{11111\ldots111}_{2n\ 1\text{s}} - 2\underbrace{(11111\ldots111)}_{n\ 1\text{s}}$

$= \underbrace{11111\ldots111}_{n\ 1\text{s}}(10^n + 1) - \underbrace{11111\ldots111}_{n\ 1\text{s}} - \underbrace{11111\ldots111}_{n\ 1\text{s}}$

$= \underbrace{11111\ldots111}_{n\ 1\text{s}} \times 10^n + \underbrace{11111\ldots111}_{n\ 1\text{s}} - \underbrace{11111\ldots111}_{n\ 1\text{s}} - \underbrace{11111\ldots111}_{n\ 1\text{s}}$

$= \underbrace{11111\ldots111}_{n\ 1\text{s}} \times 10^n - \underbrace{11111\ldots111}_{n\ 1\text{s}}$

$= \underbrace{11111\ldots111}_{n\ 1\text{s}}(10^n - 1)$

$= \underbrace{11111\ldots111}_{n\ 1\text{s}}(10-1)(10^{n-1} + 10^{n-2} + 10^{n-3} + \ldots + 1)$ using given result

$= 9\left[\underbrace{11111\ldots111}_{n\ 1\text{s}}(10^{n-1} + 10^{n-2} + 10^{n-3} + \ldots + 1)\right]$

$= 9N$ where $N = \underbrace{11111\ldots111}_{n\ 1\text{s}}(10^{n-1} + 10^{n-2} + 10^{n-3} + \ldots + 1) \in \mathbb{Z}$

$\therefore \underbrace{11111\ldots111}_{2n\ 1\text{s}} - \underbrace{22222\ldots222}_{n\ 2\text{s}}$ is divisible by 9.

End of solutions

Extra Working Page for Section E only

Method 2

$\underbrace{11111\ldots111}_{\text{'2n' 1s}} - \underbrace{22222\ldots222}_{\text{n 2s}}$

$n=1: 11-2=9$

$n=2: 1111-22=1089$ 'one 1' 'zero' 'one 8' 'nine'

$n=3: 111111-222=110889$ 'two 1' 'zero' 'two 8' 'nine'

$n=4: 11111111-2222=11108889$ 'three 1' 'zero' 'three 8' 'nine'

$n=5: 1111111111-22222=1111088889$ 'four 1' 'zero' 'four 8' 'nine'

⋮

Looking for a pattern:

$n=n: \underbrace{11111\ldots111}_{\text{'2n' 1s}} - \underbrace{22222\ldots222}_{\text{n 2s}} = \underbrace{11111\ldots111}_{\text{(n-1) 1s}}0\underbrace{88888\ldots888}_{\text{(n-1) 8s}}9$

Divisibility test for 9: sum of all the digits is a multiple of 9.

For the following number: $\underbrace{11111\ldots111}_{\text{(n-1) 1s}}0\underbrace{88888\ldots888}_{\text{(n-1) 8s}}9$

Adding up all the digits: $1(n-1)+0+8(n-1)+9$

$=n-1+8n-8+9$

$=9n$ which is divisible by 9

Since the sum of the digits is divisible by 9.

$\therefore \underbrace{11111\ldots111}_{\text{(n-1) 1s}}0\underbrace{88888\ldots888}_{\text{(n-1) 8s}}9$ is divisible by 9.

$\therefore \underbrace{11111\ldots111}_{\text{'2n' 1s}} - \underbrace{22222\ldots222}_{\text{n 2s}}$ is divisible by 9.

- Some students proved using specific numbers. Students need to prove the question generally i.e. for all numbers.

Method 3

Question 2 (continued)

(c) You may use this result to answer the following question.

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$

3

Prove that

$\underbrace{(11111\ldots111)}_{2n\ 1s} - \underbrace{(22222\ldots222)}_{n\ 2s}$

is divisible by 9.

~~Recall~~

Notation: $a \mid b \iff b$ divisible by $a \iff b \equiv 0 \pmod{a}$

~~Recall $9 \mid a \iff 9 \mid$ (digit sum of a)~~

Let $a = \underbrace{(11111\ldots111)}_{2n\ 1s}$

Let $b = \underbrace{(22222\ldots222)}_{n\ 2s}$

RTP $9 \mid (a-b) \iff a \equiv b \pmod{9}$

$a \equiv$ digit sum of $a \equiv \underbrace{1+1+1+1+\cdots+1+1+1}_{2n\ 1s} \equiv (1) \times 2n = 2n \pmod{9}$

$b \equiv$ digit sum of $b \equiv \underbrace{2+2+2+2+2+\cdots+2+2+2}_{n\ 2s} \equiv (2) \times n \equiv 2n \pmod{9}$

$\therefore a \equiv 2n \equiv b \pmod{9}$

$\therefore 9 \mid (a-b)$

Proven.

End of paper