THE KING'S SCHOOL

Year 12
Mathematics
3 Unit
Trial Higher School Certificate
1999

Time Allowed: Two Hours
(plus 5 minutes' reading time)

Directions to Candidates:

- Attempt **all** questions.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- A table of standard integrals is provided.
- Approved calculators may be used.
- Each question is to be returned in a separate Writing Booklet clearly labelled, showing your Student Number.

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 1999 Mathematics 3 Unit (Additional) Higher School Certificate Examination.

Use a *separate* Writing Booklet.

Question 1 — Marks

(a) Solve $\frac{x+1}{x} \geq 2$. 3

(b) Find the acute angle between the lines $x + 3y = 4$ and $2x - 5y = 0$.
Give the answer to the nearest degree. 3

(c) Point A is (2,–7) and B is (–6,9). Find the coordinates of point P which divides AB internally in the ratio 3:5. 2

(d) $f(x) = x\sqrt{2 - x^2}, \quad -\sqrt{2} \leq x \leq \sqrt{2}$

For what values of x is $f(x)$ increasing? 4

Use a *separate* Writing Booklet.

Question 2 — Marks

(a) Use the substitution $u = \ln x$ to find $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ — 3

(b) Consider the geometric series $1 - \tan^2 x + \tan^4 x - ...,$ where $0 < x < \frac{\pi}{2}$. — 3

(i) For what values of x does this series have a limiting sum?

(ii) Find the limiting sum in terms of cos$x$.

(c) It is given that $x^2 + x - 2$ is a factor of $x^3 + rx^2 - 4x + s$, where r and s are constants. — 3

(i) Show that $r + s = 3$ and $4r + s = 0$.

(ii) Evaluate r and s.

(d) On a certain railway line, there are eleven stations at which a train can stop. The rail authority wishes to print tickets for travel between every possible pair of stations on the line.

How many different one-way tickets must be printed if the ticket specifies which direction the passenger is travelling? — 1

(e) (i) Five men and five women are arranged in a straight line. How many arrangements are there in which men and women alternate?

(ii) Five men and five women are arranged in a circle. How many arrangements are there in which men and women alternate? — 2

Use a *separate* Writing Booklet.

Question 3 — Marks

(a) (i) By substitution, show that a solution for $3\sin x - x = 0$ lies between $x = 2.2$ and $x = 2.4$.

(ii) Taking $x = 2.3$ as an approximation to a solution of $3\sin x - x = 0$, apply Newton's method once to find a better approximation correct to three decimal places.

(b) (i) Find $\frac{d}{dx}(x\tan^{-1}x)$. 4

(ii) Hence find the exact value of $\int_0^1 \tan^{-1}x\,dx$.

(c) Use mathematical induction to show that 4

$E(n) = n^3 + 5n$ is divisible by 3 for all positive integers n.

Use a *separate* Writing Booklet.

Question 4 — Marks

(a) Assume that Antarctica can be approximately represented as a circle, and that its area is decreasing at the rate of 3000 square kilometres per year. — 3

Find the rate at which the radius of the circle is decreasing when the length of the radius is 2100 kilometres.

(b) — 4

In the diagram, two circles intersect at A and B. Points C and E lie on the circles and C, B and E are collinear. Tangents at C and E meet at D.

Copy the diagram into your booklet

(i) Explain why $\angle DEC = \angle BAE$.

(ii) Prove that quadrilateral AEDC is cyclic.

(c) A particle moves in a straight line and at time t seconds, its velocity, v metres per second, is related to its displacement, x metres, by 5

$$v^2 = 108 + 36x - 9x^2.$$

(i) By deriving an expression for acceleration, show that the motion is simple harmonic, and state the centre of motion.

(ii) Find the period of the motion.

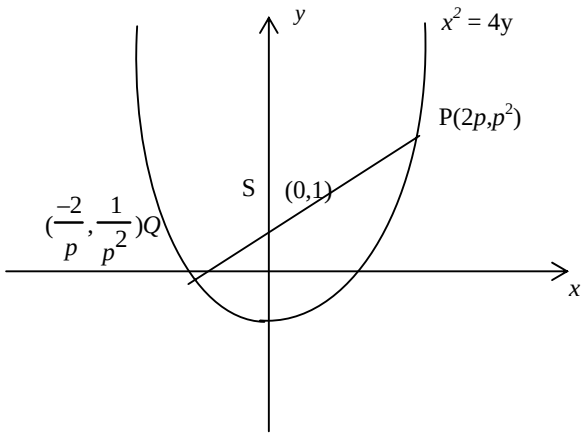
(iii) Find the amplitude of the motion.

Use a *separate* Writing Booklet.

Question 5 — Marks

(a) Solve $|2x-1| < 3x+2$ — 3

(b) — 6



PQ is a focal chord of the parabola $x^2 = 4y$.

$P = (2p, p^2), Q = (\frac{-2}{p}, \frac{1}{p^2})$ and the focus S is (0,1).

(i) State the focus – directrix definition of a parabola.

(ii) Find the length of PQ.

(iii) Find the mid-point, M, of chord PQ.

(iv) Show that the circle drawn with PQ as diameter touches the directrix of the parabola.

Question 5 (continued) — Marks

(c) The region bounded by the curve $y = 2\sin x$ and the x axis from $x = 0$ to $x = 2\pi$ is revolved about the x axis. — 3

Find the volume of the solid of revolution.

Use a *separate* Writing Booklet.

Question 6 — Marks

(a) (i) On the same axes, sketch $y = \cos^{-1} x$ and $y = -\sin^{-1} x$. 4

(ii) Find the area between the two curves and the ordinates $x = -1$ and $x = 1$.

(b) Prove that $(\log_{ab} x)(1 + \log_a b) = \log_a x, \quad x, a, b > 0$ 2

(c) At time t seconds after the parachute has opened, a parachutist falls with speed v metres per second and with acceleration given by 6

$$\frac{dv}{dt} = k(4 - v)$$

where k is constant.

(i) Show that $v = 4 + Ae^{-kt}$, where A is constant, satisfies the above equation.

(ii) At the instant the parachute opens, the parachutist is falling at 25 metres per second. Find the value of A.

(iii) Two seconds after the parachute has opened, the parachutist's speed is 12 metres per second. Evaluate k correct to three significant figures.

(iv) As t increases indefinitely, v approaches a fixed value. Show that after 20 seconds v differs from this fixed value by less than 0.1%.

Use a *separate* Writing Booklet.

Question 7 — Marks

(a) An expression of the form $(1-ax)^n$, where a is constant and n a positive integer, is expanded. The first three terms of the expansion are — 4

$$1-4x+\frac{20}{3}x^2.$$

Find the values of a and n.

(b) A formula for e^x written as an infinite series is — 3

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$$

(i) Use integration to verify that the formula is correct.

(ii) Find $\lim\limits_{x\to 0}\ \dfrac{e^x-1}{x}$.

(c) (i) You are given that $4\sin^2\frac{\pi}{10}+2\sin\frac{\pi}{10}=1$. — 5

Deduce that $\sin\frac{\pi}{10}=\frac{\sqrt{5}-1}{4}$

(ii) Use the result in (i) to prove that

$$\sin\frac{\pi}{5}\cos\frac{\pi}{10}=\frac{\sqrt{5}}{4}.$$

End of Paper

1999 3U Trial HSC: ANSWERS

1 (a) $0 < x \le 1$

(b) 40^0

(c) $(-1,-1)$

(d) $-1 < x < 1$

2 (a) $\sin^{-1}(\ln x) + c$

(b) (i) $0 < x < \frac{\pi}{4}$

(ii) $\cos^2 x$

(c) $r = -1, \quad s = 4$

(d) ${}^{11}P_2$ or $2 \times {}^{11}C_2$ or 110

(e) (i) $5! \times 5! \times 2$ or $28\,800$

(ii) $5! \times 4!$ or $2\,800$

3 (a) 2.279

(b) (i) $\frac{x}{1+x^2} + \tan^{-1} x$

(ii) $\frac{\pi}{4} - \frac{1}{2}\ln 2$

4 (a) $\frac{5}{7\pi}\ km/yr$

(c) (i) $x = 2$

(ii) $\frac{2\pi}{3}$ seconds

(iii) 4m

5 (a) $x > -\frac{1}{5}$

(b) (ii) $p^2 + \frac{1}{p^2} + 2$

(iii) $\left[p - \frac{1}{p}, \quad \frac{1}{2}\left(p^2 + \frac{1}{p^2}\right)\right]$

(c) $4\pi^2 u^3$

6 (a) (ii) $\pi \quad u^2$

(c) (ii) 21

(iii) 0.483

7 (a) $n = 6, \quad a = \frac{2}{3}$

(b) (ii) 1