2014 HSC Examination

Mathematics Extension 2

SOLUTIONS

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Multiple Choice

1. D
2. A
3. B
4. C
5. C
6. D
7. B
8. B
9. A
10. D

Brief Explanations

Question 1 Standard partial fractions decomposition.

Question 2 Use the formula $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\text{Re}(\alpha)z + |\alpha|^2$.

Question 3 Express in standard form $\frac{x^2}{25/9} + \frac{y^2}{25/16} = 1$ then use the formula $e^2 = \frac{a^2 - b^2}{a^2}$.

Question 4 Conjugation then reciprocation preserves the argument, but modulus is flipped.

Question 5 The section $0 < x < 2$ gets 'ignored', so it must be (C).

Question 6 Standard volumes by discs.

Question 7 Differentiate the available options.

Question 8 Either (A) or (B) because the argument increases. Must be (B) for *u* to be flat.

Question 9 At first, (A) and (B) satisfy the velocity, but only (A) satisfies the acceleration condition.

Question 10 Split the integrand and let $u = a - x$.

Written Response

Question 11 (a) (i)

$$z+w=1-i=\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

Question 11 (a) (ii)

$$\frac{z}{w}=\frac{-2-2i}{3+i}=\frac{(-2-2i)(3-i)}{10}=\frac{-8-4i}{10}=-\frac{4}{5}-\frac{2}{5}i$$

Question 11 (b)

Use integration by parts.

$$\int_0^{\frac{1}{2}}(3x-1)\cos(\pi x)\,dx = \frac{1}{\pi}(3x-1)\sin(\pi x)\Big|_0^{\frac{1}{2}}-\frac{3}{\pi}\int_0^{\frac{1}{2}}\sin(\pi x)\,dx$$

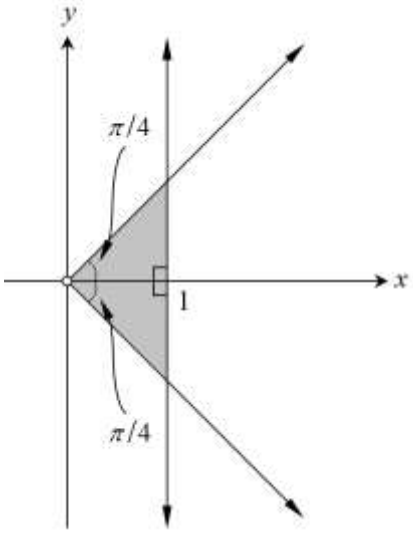
$$=\frac{1}{\pi}\left(\frac{3}{2}-1\right)\sin\left(\frac{\pi}{2}\right)+\frac{3}{\pi}\times\frac{1}{\pi}\cos(\pi x)\Big|_0^{\frac{1}{2}}$$

$$=\frac{1}{2\pi}-\frac{3}{\pi^2}$$

$$=\frac{\pi-6}{2\pi^2}$$

Question 11 (c)

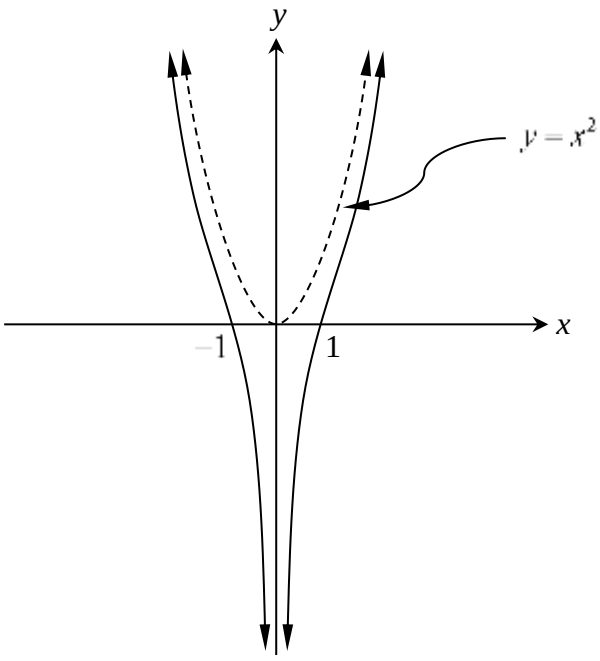
The locus is the set of all points in between two rays with arguments $\pm\frac{\pi}{4}$ and closer to the origin than to the point $z=2$. In other words, the left side of the line $x=1$.



Question 11 (d)

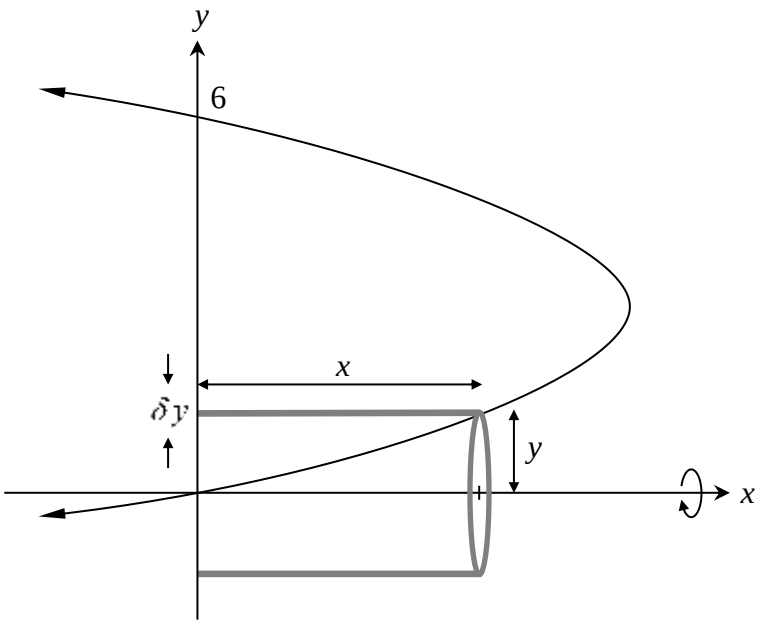
We know the following features.

- The function is even, so we focus on the right half only, then flip it afterwards.
- As $x\to\infty$, the curve tends to approach $y=x^2$ from below (curvillinear asymptote).
- There is a vertical asymptote at $x=0$ and as $x\to 0^+$, we have $y\to-\infty$.
- There is an x intercept at $x=1$.



Question 11 (e)

We will use the method of cylindrical shells since that will allow for integrating with respect to *y*, and our function is primarily in terms of *y*.



From the diagram above, we can see that the volume of an arbitrary shell is $\delta V = 2\pi xy\,\delta y$.

Hence, we can compute the volume.

$$V = \lim_{\delta y \to 0} \sum_{0 \le y \le 6} 2\pi xy\,\delta y$$

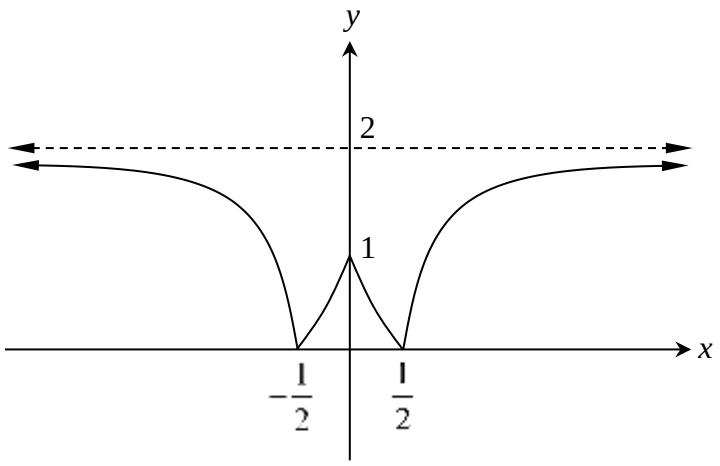
$$= 2\pi \int_0^6 xy\,dy$$

$$= 2\pi \int_0^6 y^2 (6 - y)\,dy$$

$$= 216\pi$$

Question 12 (a) (i)

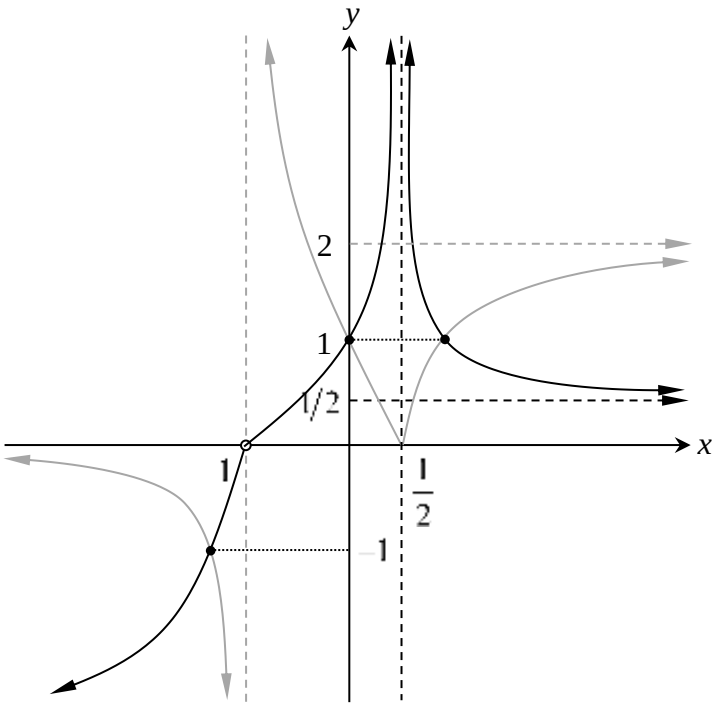
The graph is the right half, reflected onto the left half.



Question 12 (a) (i)

We know the following features about $y = \dfrac{1}{f(x)}$.

- There is a vertical asymptote at $x = 1/2$.
- As $x \to \infty$, we have $y \to 1/2^{+}$.
- The curve also passes through $(0,1)$.
- The curve will have an 'x intercept' at $(-1,0)$, except there is an open circle there.
- As $x \to -\infty$, we have $y \to -\infty$.
- We cannot guarantee that the two branches at $x = -1$ connect 'smoothly', so it has been drawn in a disconnected manner.



Question 12 (b) (i)

Substitute $x = 2\cos\theta$ into the cubic.

$$8\cos^3\theta - 6\cos\theta = \sqrt{3}$$
$$4\cos^3\theta - 3\cos\theta = \frac{\sqrt{3}}{2}$$
$$\cos 3\theta = \frac{\sqrt{3}}{2}$$

Question 12 (b) (ii)

We solve $\cos 3\theta = \frac{\sqrt{3}}{2}$ for three distinct values of θ that solve the cubic $x^3 - 3x = \sqrt{3}$, since we have the substitution $x = 2\cos\theta$.

$$\cos 3\theta = \frac{\sqrt{3}}{2}$$
$$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$$
$$\theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}$$

Hence, the solutions to $x^3 - 3x = \sqrt{3}$ are $x = 2\cos\left(\frac{\pi}{18}\right), 2\cos\left(\frac{11\pi}{18}\right), 2\cos\left(\frac{13\pi}{18}\right)$.

Question 12 (c)

We use implicit differentiation on both curves.

The implicit derivative of the $x^2 - y^2 = 5$ is $2x - 2yy' = 0 \Rightarrow y' = \frac{x}{y}$.

By using the product rule, the implicit derivative of $xy = 6$ is $y + xy' = 0 \Rightarrow y' = -\frac{y}{x}$.

At their point of intersection $P(x_0, y_0)$, the gradients are $y_1' = \frac{x_0}{y_0}$ and $y_2' = -\frac{y_0}{x_0}$.

Multiplying them yields $y_1' \times y_2' = \frac{x_0}{y_0} \times -\frac{y_0}{x_0} = -1$, hence the two curve intersect at right angles. In other words, the tangents are perpendicular to one another at *P*.

Question 12 (d) (i)

$$I_0 = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 = \frac{\pi}{4}$$

Question 12 (d) (ii)

$$I_n + I_{n-1} = \int_0^1 \frac{x^{2n} + x^{2n-2}}{1+x^2}\,dx$$

$$= \int_0^1 \frac{x^{2n-2}\left(x^2+1\right)}{1+x^2}\,dx$$

$$= \int_0^1 x^{2n-2}\,dx$$

$$= \frac{1}{2n-1} x^{2n-1}\Big|_0^1$$

$$= \frac{1}{2n-1}$$

Question 12 (d) (iii)

From part (i), we have $I_0 = \frac{\pi}{4}$. From part (ii), we have $I_n = \frac{1}{2n-1} - I_{n-1}$

So we have $I_1 = 1 - \frac{\pi}{4}$ and thus

$$I_2 = \frac{1}{3} - I_1$$

$$= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

$$= \frac{3\pi - 8}{12}$$

Question 13 (a)

From the substitution $t = \tan\left(\frac{x}{2}\right)$, we have $dx = \frac{2}{1+t^2}dt$.

For the limits, we have $x = \frac{\pi}{2} \Rightarrow t = 1$ and $x = \frac{\pi}{3} \Rightarrow t = \frac{1}{\sqrt{3}}$.

So substituting everything in, we have

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{3\sin x - 4\cos x + 5} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{\frac{2}{1+t^2}}{\frac{6t}{1+t^2} - \frac{4-4t^2}{1+t^2} + 5}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{6t - 4 + 4t^2 + 5 + 5t^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{9t^2 + 6t + 1}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{(3t+1)^2}\,dt$$

$$= -\frac{2}{3} \times \frac{1}{3t+1}\Bigg|_{\frac{1}{\sqrt{3}}}^{1}$$

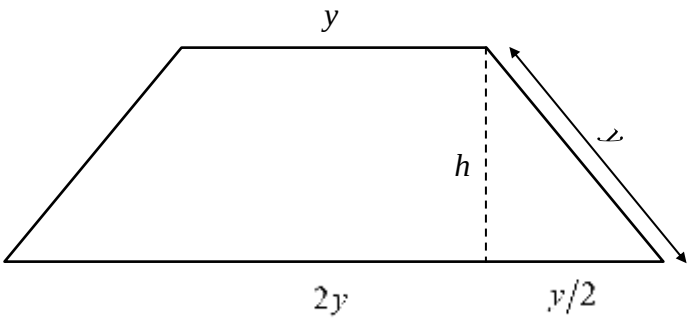
$$= -\frac{2}{3}\left[\frac{1}{4} - \frac{1}{\sqrt{3}+1}\right]$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{2}$$

Question 13 (b)

If we let an arbitrary point on the curve $y = x^2$ be $P(x, y)$, then the base of the trapezium would be $2y$. But since the base is twice the length of any of the other sides (given), we can deduce that the length of the top side is just *y*.

We now need to find the height of the trapezium to find the area of the cross section.



By Pythagoras' Theorem,

$$h^2 = y^2 - \left(\frac{y}{2}\right)^2$$

$$h = \sqrt{y^2 - \frac{y^2}{4}}$$

$$= \frac{\sqrt{3}}{2} y$$

Noting that the thickness of the slices are in terms of *x*, the volume of the cross section is

$$\delta V = \frac{h}{2}(y + 2y)\delta x$$

$$= \frac{y\sqrt{3}}{4} \times 3y\,\delta x$$

$$= \frac{3\sqrt{3}}{4} y^2\,\delta x$$

$$= \frac{3\sqrt{3}}{4} x^4\,\delta x \quad \ldots(\text{since } y = x^2)$$

And so the volume of the solid is

$$V = \lim_{\delta x \to 0} \sum_{0 \le x \le 2} \frac{3\sqrt{3}}{4} x^4\,\delta x$$

$$= \int_0^2 \frac{3\sqrt{3}}{4} x^4 dx$$

$$= \frac{3\sqrt{3}}{20} \times x^5 \Big|_0^2$$

$$= \frac{24\sqrt{3}}{5}$$

Question 13 (c) (i)

Substitute *M* into the left side of the equation of the hyperbola.

$$\frac{\left(\frac{a\left(t^2+1\right)}{2t}\right)^2}{a^2}-\frac{\left(\frac{b\left(t^2-1\right)}{2t}\right)^2}{b^2}=\frac{\frac{a^2\left(t^2+1\right)^2}{4t^2}}{a^2}-\frac{\frac{b^2\left(t^2-1\right)^2}{4t^2}}{b^2}$$

$$=\frac{\left(t^2+1\right)^2}{4t^2}-\frac{\left(t^2-1\right)^2}{4t^2}$$

$$=\frac{\left(t^2+1\right)^2-\left(t^2-1\right)^2}{4t^2}$$

$$=\frac{4t^2}{4t^2}$$

$$=1$$

Question 13 (c) (ii)

The Cartesian Equation of the tangent to the hyperbola at $P\left(x_0,y_0\right)$ is $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$.

Replace x_0 with $\frac{a\left(t^2+1\right)}{2t}$ and y_0 with $\frac{b\left(t^2-1\right)}{2t}$, so the equation of the tangent at *M* is

$$\frac{x}{a^2}\times\frac{a\left(t^2+1\right)}{2t}-\frac{y}{b^2}\times\frac{b\left(t^2-1\right)}{2t}=1$$

$$x\left(\frac{t^2+1}{2at}\right)-y\left(\frac{t^2-1}{2bt}\right)=1$$

To show that the line through *P* and *Q* is a tangent to the hyperbola at *M*, we simply show that they both lie on the tangent equation derived above.

For *P*, $(at)\left(\frac{t^2+1}{2at}\right)-(bt)\left(\frac{t^2-1}{2bt}\right)=\frac{t^2+1}{2}-\frac{t^2-1}{2}=1$

For *Q*, $\left(\frac{a}{t}\right)\left(\frac{t^2+1}{2at}\right)-\left(-\frac{b}{t}\right)\left(\frac{t^2-1}{2bt}\right)=\frac{t^2+1}{2t^2}+\frac{t^2-1}{2t^2}=1$

Hence, they both lie on the tangent at *M*.

Question 13 (c) (iii)

First, note that $OS^2 = a^2e^2 = a^2 + b^2$, since $b^2 = a^2\left(e^2 - 1\right)$ yields $e^2 = \dfrac{a^2 + b^2}{a^2}$.

$$\begin{aligned} OP \times OQ &= \sqrt{a^2t^2 + b^2t^2} \times \sqrt{\frac{a^2}{t^2} + \frac{b^2}{t^2}} \\ &= t\sqrt{a^2 + b^2} \times \frac{1}{t}\sqrt{a^2 + b^2} \\ &= a^2 + b^2 \\ &= OS^2 \end{aligned}$$

Question 13 (c) (iv)

Since *P* is vertically above *S*, we have $at = ae$ by equating their *x* coordinates. Hence $t = e$.

We now compute the gradient of *MS* and show that it's equal to $-\dfrac{b}{a}$.

$$m_{MS} = \frac{\dfrac{b}{2e}\left(e^2 - 1\right)}{\dfrac{a\left(e^2 + 1\right)}{2e} - ae}$$

$$= \frac{b\left(e^2 - 1\right)}{a\left(e^2 + 1\right) - 2ae^2}$$

$$= \frac{b\left(e^2 - 1\right)}{a\left(1 - e^2\right)}$$

$$= -\frac{b}{a}$$

By symmetry of the hyperbola, if we have *M* being say on the other branch, we will have the same result of *MS* being parallel to one of the asymptotes.

Question 14 (a) (i)

$$P'(x) = 5x^4 - 20x + 15 \qquad \Rightarrow P'(1) = 5 - 20 + 15 = 0$$
$$P''(x) = 20x^3 - 20 \qquad \Rightarrow P''(1) = 20 - 20 = 0$$

Since $P''(1) = P'(1) = 0$, we can deduce that $x = 1$ is a root of multiplicity 3.

Question 14 (a) (ii)

From (i), we have $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ being a factor. Hence, we can deduce (either by observation or by polynomial long division) that the other quadratic factor is $(x^2 + 3x + 6)$, which has roots $x = -\frac{3}{2} \pm i\frac{\sqrt{15}}{2}$.

Question 14 (b) (i)

We use the angle between two lines formula, with one gradient being the gradient of the normal and the other being the gradient of *OP*.

Using parametric differentiation, the gradient of the normal is $-\frac{1}{dy/dx} = -\frac{dx/d\theta}{dy/d\theta} = \frac{a\sin\theta}{b\cos\theta}$.

The gradient of *OP* can simply be found using $\frac{\text{Rise}}{\text{Run}} = \frac{b\sin\theta}{a\cos\theta}$.

$$\begin{aligned}
\tan\phi &= \left| \frac{\frac{b\sin\theta}{a\cos\theta} - \frac{a\sin\theta}{b\cos\theta}}{1 + \frac{b\sin\theta}{a\cos\theta} \times \frac{a\sin\theta}{b\cos\theta}} \right| \\
&= \left| \frac{\left(\frac{b}{a} - \frac{a}{b}\right)\frac{\sin\theta}{\cos\theta}}{1 + \tan^2\theta} \right| \\
&= \left| \frac{b^2 - a^2}{ab} \right| \left| \frac{\tan\theta}{\sec^2\theta} \right| \\
&= \left(\frac{a^2 - b^2}{ab} \right) |\sin\theta\cos\theta| \qquad \ldots(\text{since } a > b)
\end{aligned}$$

Question 14 (b) (ii)

To maximise ϕ, we want to maximise the value of $\sin\theta\cos\theta$ since the value of $\dfrac{a^2-b^2}{ab}$ is fixed. But notice that $\sin\theta\cos\theta=\dfrac{1}{2}\sin 2\theta$, which has a maximum point when $\theta=\dfrac{\pi}{4}$.

Question 14 (c) (i)

The following diagram below is shows the forces acting upon the train. Notice that the vector for *F* is longer than the resistive force vector Kv^2. This must be the case in order for the train to be moving forwards.



So the equation of motion is $m\ddot{x}=F-Kv^2$, and we now must work out *K*.

We are given that the terminal velocity of the train is 300 km/hr, which occurs when $\ddot{x}=0$.

In other words, we have $0=F-K\left(300^2\right)$ and so $K=\dfrac{F}{300^2}$.

Hence, the equation of motion is $m\ddot{x}=F-\dfrac{F}{300^2}\times v^2=F\left[1-\left(\dfrac{v}{300}\right)^2\right]$.

Question 14 (c) (i)

We use $\ddot{x} = \dfrac{dv}{dt}$ for this problem, as we are finding a time value.

$$m \times \frac{dv}{dt} = F\left[1 - \left(\frac{v}{300}\right)^2\right]$$

$$\frac{dv}{dt} = \frac{F}{m}\left[1 - \left(\frac{v}{300}\right)^2\right]$$

$$= \frac{F}{m}\left(1 + \frac{v}{300}\right)\left(1 - \frac{v}{300}\right)$$

$$= \frac{F}{m}\left(\frac{300 + v}{300}\right)\left(\frac{300 - v}{300}\right)$$

$$\int_0^T dt = \frac{m}{F}\int_0^{200} \frac{300}{300 + v} \times \frac{300}{300 - v}\, dv$$

$$= 150 \times \frac{m}{F}\int_0^{200} \frac{1}{300 + v} + \frac{1}{300 - v}\, dv$$

$$= 150 \times \frac{m}{F}\ln\left(\frac{300 + v}{300 - v}\right)\Bigg|_0^{200}$$

$$= 150 \times \frac{m}{F}\ln 5$$

Question 15 (a)

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac) = 1$, since we are given that $a+b+c=1$.

Since $a \le b \le c$, we have $2ab \ge 2a^2$ and $2ac \ge 2a^2$, then add them to get $2ab+2ac \ge 4a^2$.

Similarly, we have $c > b$ and so $2bc \ge 2b^2$.

Adding these, we get $2ab+2bc+2ac \ge 4a^2+2b^2$

Substitute these into our equation above

$$\begin{aligned} a^2+b^2+c^2+2ab+2bc+2ac &= 1 \\ &\ge a^2+b^2+c^2+\left(4a^2+2b^2\right) \\ &= 5a^2+3b^2+c^2 \end{aligned}$$

And thus the result.

Question 15 (b) (i)

By de Moivre's theorem, we have $(1+i)^n = \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^n = \left(\sqrt{2}\right)^n \text{cis}\left(\frac{n\pi}{4}\right)$.

Recall the conjugate property $\overline{z}^n = \overline{z^n}$. Using the property, where $z = 1+i$, we have

$$(1+i)^n + (1-i)^n = z^n + \overline{z}^n = z^n + \overline{z^n} = 2\,\text{Re}\left(z^n\right) = 2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right)$$

Question 15 (b) (ii)

$$(1+x)^n + (1-x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k + \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k$$

$$= \sum_{k=0}^{n}\left[\binom{n}{k}x^k + (-1)^k\binom{n}{k}x^k\right]$$

$$= \sum_{k=0}^{n}\left[\binom{n}{k}x^k\left(1+(-1)^k\right)\right]$$

$$= \sum_{\text{even } k}^{n} 2\binom{n}{k}x^k \qquad \ldots(\text{since when } k \text{ is odd, the term becomes zero})$$

We now let $x = i$.

$$(1+i)^n + (1-i)^n = \sum_{\text{even } k}^{n} 2\binom{n}{k}i^k$$

$$2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) = 2\left[\binom{n}{0} + i^2\binom{n}{2} + i^4\binom{n}{4} + i^6\binom{n}{6} + \ldots + i^n\binom{n}{n}\right] \qquad \ldots(\text{from part (i)})$$

$$\left(\sqrt{2}\right)^n (-1)^{\frac{n}{4}} = \left[\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots + \binom{n}{n}\right] \qquad \ldots(\text{since } n \text{ is a multiple of 4, so } i^n = 1)$$

Question 15 (c) (i)

Vertical: $kv^2 - T\sin\phi = mg$

Horizontal: $T\cos\phi = \dfrac{mv^2}{r}$, so $T = \dfrac{mv^2}{r\cos\phi}$.

Combine them

$$kv^2 - \frac{mv^2\sin\phi}{r\cos\phi} = mg$$

But notice that $r = l\cos\phi$, from the triangle drawn for us. Substitute this in too.

$$kv^2 - \frac{mv^2\sin\phi}{l\cos^2\phi} = mg$$

$$\frac{mv^2\sin\phi}{l\cos^2\phi} = kv^2 - mg$$

$$\frac{\sin\phi}{\cos^2\phi} = \frac{l}{mv^2}\left(kv^2 - mg\right)$$

$$= \frac{lk}{m} - \frac{\lg}{v^2}$$

Question 15 (c) (ii)

We are given that $\dfrac{\sin\phi}{\cos^2\phi} < \dfrac{lk}{m}$, so $\dfrac{\sin\phi}{1-\sin^2\phi} < \dfrac{lk}{m}$.

Re-arrange as a quadratic in terms of $\sin\phi$ to get $lk\sin^2\phi + m\sin\phi - lk < 0$.

Use the quadratic formula. The required result is the right hand branch.

$$\frac{-m - \sqrt{m^2 + 4l^2k^2}}{2lk} < \sin\phi < \frac{-m + \sqrt{m^2 + 4l^2k^2}}{2lk}$$

Question 15 (c) (iii)

$$\frac{d}{d\phi}\left(\frac{\sin\phi}{\cos^2\phi}\right)=\frac{\cos^3\phi+2\sin^2\phi\cos\phi}{\cos^4\phi}=\frac{\cos^2\phi+2\sin^2\phi}{\cos^3\phi}$$

However, since ϕ is in Quadrants 1 & 4, the denominator is always positive. The numerator is always positive regardless, so $\frac{d}{d\phi}\left(\frac{\sin\phi}{\cos^2\phi}\right)>0$ and hence, the function is increasing.

Question 15 (c) (iv)

As *v* increases, we have $\frac{\sin\phi}{\cos^2\phi}\to\frac{lk}{m}$ from below since $\frac{\sin\phi}{\cos^2\phi}<\frac{lk}{m}$ (given).

But since $\frac{\sin\phi}{\cos^2\phi}$ is an increasing function (from (iii)), it must be the case that ϕ is increasing.

Question 16 (a) (i)

$\angle APX = \angle ADP$ (alternate segment theorem)

$\angle ADP = \angle DPQ$ (alternate angles, $AD \parallel PQ$)

$\therefore \angle APX = \angle ADP = \angle DPQ = \alpha$.

Question 16 (a) (ii)

Similarly to (i), we have $\angle BPY = \angle BCP = \angle CPR = \beta$.

By vertically opposite angles, we have $\angle XPY = \angle QPR = \gamma$.

If we sum all the angles around *P*, they add to 2π (angle of revolution).

$$2\left(\frac{\pi}{2} + \alpha + \beta + \gamma\right) = 2\pi$$

$$\alpha + \beta + \gamma = \frac{\pi}{2}$$

So since we have four right angles intersecting at a common point *P*, it must be the case that they are all coming from straight lines. Hence points *A*, *P* and *C* are collinear.

Question 16 (a) (iii)

By the alternate segment theorem, we have $\angle DAP = \alpha + \gamma$ so by summing the angles in ΔAPD, we have $2\alpha + \gamma = \frac{\pi}{2}$. Similarly for ΔBPC, we have $2\beta + \gamma = \frac{\pi}{2}$.

Subtracting the two equations, we get $\alpha = \beta$ and therefore $\angle DAP = \angle CBP$. By the converse of the "angles on the same arc are equal" property, we have *ABCD* being cyclic.

Question 16 (b) (i)

Consider the geometric series

$$\begin{aligned} 1-x^2+x^4-x^6+\ldots+\left(-x^2\right)^{n-1} &= \frac{1-\left(-x^2\right)^n}{1+x^2} \\ &= \frac{1}{1+x^2}-\frac{\left(-1\right)^n x^{2n}}{1+x^2} \end{aligned}$$

So upon re-arranging, we have

$$\frac{1}{1+x^2}-\left(1-x^2+x^4-x^6+\ldots+\left(-1\right)^{n-1}x^{2n-2}\right)=\frac{\left(-1\right)^n x^{2n}}{1+x^2}$$

So we now consider the right hand side, more specifically its absolute value.

$$\begin{aligned} \left|\frac{\left(-1\right)^n x^{2n}}{1+x^2}\right| &= \left|\frac{x^{2n}}{1+x^2}\right| && \ldots\left(\text{since we are considering absolute values}\right) \\ &\le \left|x^{2n}\right| && \ldots\left(\text{since } 1+x^2\ge 1\right) \\ &= x^{2n} && \ldots\left(\text{since } x^{2n}\ge 0\right) \end{aligned}$$

So hence

$$-x^{2n}\le\frac{\left(-1\right)^n x^{2n}}{1+x^2}\le x^{2n}$$

in a similar manner to how if $\left|f\left(x\right)\right|\le k$, then $-k\le f\left(x\right)\le k$.

Hence we have the result

$$-x^{2n}\le\frac{1}{1+x^2}-\left(1-x^2+x^4-x^6+\ldots+\left(-1\right)^{n-1}x^{2n-2}\right)\le x^{2n}$$

Question 16 (b) (ii)

Integrate both sides over the domain $0 \le x \le 1$.

$$-\frac{1}{2n+1} \le \int_0^1 \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1} x^{2n-2}\right) \le \frac{1}{2n+1}$$

$$-\frac{1}{2n+1} \le \tan^{-1} x - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{n-1}}{2n-1} x^{2n-1} \right]_0^1 \le \frac{1}{2n+1}$$

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^{n-1}}{2n-1} \right]_0^1 \le \frac{1}{2n+1}$$

Question 16 (b) (iii)

Taking the limit as $n \to \infty$ to each term, we notice that both the left and right-most expression converge to 0.

As the term in the middle is 'sandwiched' between them, it too must also converge to 0 (Squeeze Law).

Hence, $\frac{\pi}{4} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^{n-1}}{2n-1} \right] \to 0$ as $n \to \infty$, and thus the result (Leibniz Series).

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$$

Question 16 (c)

We use the old "plus one minus one" trick first.

$$I = \int \frac{(1+\ln x) - 1}{(1+\ln x)^2} dx$$

$$= \int \frac{(1+\ln x) - x/x}{(1+\ln x)^2} dx$$

However, notice that this is the perfect form of the quotient rule, if we are differentiating $\frac{x}{1+\ln x}$.

So the integral is simply $\frac{x}{1+\ln x} + C$.