Multiple choice questions

Q1 (B)

Q2 (A), Equations of directrices $x=\pm\frac{a}{e}=\pm\frac{a^2}{\sqrt{a^2+b^2}}$

$=\pm\frac{25}{\sqrt{25+4}}=\pm\frac{25}{\sqrt{29}}$

Q3 (D), since $|z|<1,|z^2|<|z|$, and $\arg(z^2)=2\arg(z)$

Q4 (A), Let $x=\alpha+1,\therefore\alpha=x-1,\therefore 4(x-1)^3+(x-1)^2$

$-3(x-1)+5=4x^3-11x^2+\ldots$

Q5 (B), z lies between 2 circles of centre 1, radii $\frac{\pi}{4}$ and $\frac{\pi}{3}$

Q6 (D), by completing the square $x^2-6x+5=(x-3)^2-4$

Q7 (C), $v=r\omega$, where $\omega=2\pi f=2\pi\times10=20\pi$,

$\therefore v=100\pi$

Q8 (C), Area $=(2y)^2=4y^2=4(16-x^2)$

Q9 (B)

Q10 (A), = total – (6 pp in a room) – (5 pp in a room)

$=4^6-{}^4C_1-{}^4C_1{}^6C_5{}^3C_1=4020$

Question 11

(a) (i) $z+\bar{w}=2-i\sqrt{3}+(1-i\sqrt{3})=3-2i\sqrt{3}$

(ii) $w=2\operatorname{cis}\frac{\pi}{3}$.

(iii) $w^{24}=2^{24}\operatorname{cis}8\pi=2^{24}$

(b) $\frac{x^2+8x+11}{(x-3)(x^2+2)}=\frac{4}{x-3}+\frac{-3x-1}{x^2+2}$,

$\therefore A=4,B=-3,C=-1$

(c) $z^2+4iz+5=(z-i)(z+5i)$

(d) $\int_0^1 x^3\sqrt{1-x^2}\,dx$

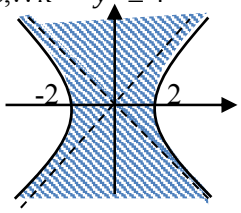
Let $u^2=1-x^2,udu=-xdx.$

When $x=0,u=1$; when $x=1,u=0$

$$I=-\int_1^0(1-u^2)u^2du=\int_0^1\left(u^2-u^4\right)du=\left[\frac{u^3}{3}-\frac{u^5}{5}\right]_0^1=\frac{2}{15}$$

(e) Let $z=x+iy,z^2+\bar{z}^2=(x+iy)^2+(x-iy)^2$

$=2\left(x^2-y^2\right)\le8,\therefore x^2-y^2\le4$



Question 12

(a) Let $t=\tan\frac{x}{2},dt=\frac{1}{2}\sec^2\frac{x}{2}dx,\therefore dx=\frac{2dt}{1+t^2}$

When $x=0,t=0$. When $x=\frac{\pi}{2},t=1$

$$\int_0^{\frac{\pi}{2}}\frac{1}{4+5\cos x}dx=\int_0^1\frac{1}{4+\frac{5(1-t^2)}{1+t^2}}\frac{2dt}{1+t^2}$$

$$=\int_0^1\frac{2}{9-t^2}dt=2\int_0^1\frac{1}{(3-t)(3+t)}dt=\frac{1}{3}\int_0^1\left(\frac{1}{3-t}+\frac{1}{3+t}\right)dt$$

$$=\frac{1}{3}\left[\ln\frac{3+t}{3-t}\right]_0^1=\frac{1}{3}\ln2$$

(b) Differentiate $\ln y-\ln(1000-y)=\frac{x}{50}-\ln3$

$$\frac{y'}{y}+\frac{y'}{1000-y}=\frac{1}{50}$$

$$y'\left(\frac{1000-y+y}{y(1000-y)}\right)=\frac{1}{50}$$

$$y'=\frac{y}{50}\frac{1000-y}{1000}=\frac{y}{50}\left(1-\frac{y}{1000}\right)$$

(c) Using the shells method

$\partial V\approx2\pi rh\partial x=2\pi(4-x)e^x\partial x$

$V=2\pi\int_1^3(4-x)e^xdx$

Let $u=4-x,du=-dx$

Let $dv=e^x,v=e^x$

$$\therefore\int_1^3(4-x)e^xdx=\left[(4-x)e^x\right]_1^3+\int_1^3e^xdx$$

$$=\left[(4-x)e^x+e^x\right]_1^3=\left[(5-x)e^x\right]_1^3=2e^3-4e.$$

$$\therefore V=2\pi\left(2e^3-4e\right)=4\pi\left(e^3-2e\right)$$

(d) (i) $m=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{-c}{p^2}}{c}=-\frac{1}{p^2}$

Eqn of tangent at $P,y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$

$p^2y-cp=-x+cp$

$\therefore x+p^2y=2cp$

(ii) Let $x=0,y=\frac{2c}{p},\therefore B\left(0,\frac{2c}{p}\right)$

Let $y=0,x=2cp,\therefore A(2cp,0)$

Midpoint of AB has coordinates $x=\dfrac{0+2cp}{p}=cp$,

$y=\dfrac{\frac{2c}{p}+0}{2}=\dfrac{c}{p}$, which are the coordinates of P.

$\therefore PA=PB$

Since $\angle AOB=90°, A, B$ and O are on a circle with centre P.

(iii) Similarly, $QD=QC$

$\therefore \dfrac{AP}{AB}=\dfrac{DQ}{DC}=\dfrac{1}{2}$

$\therefore AD \,//\, PQ \,//\, BC$ (When two or more parallel lines are cut off by 2 transversals, the cut-off segments in one transversal are proportional to the corresponding segments in the other transversal)

Question 13

(a) (i) $I_n=\int_0^1 (1-x^2)^{\frac{n}{2}}dx$

Let $u=(1-x^2)^{\frac{n}{2}}, du=-nx(1-x^2)^{\frac{n}{2}-1}dx$

Let $dv=dx, v=x$

$I_n=\left[x(1-x^2)^{\frac{n}{2}}\right]_0^1+n\int_0^1 x^2(1-x^2)^{\frac{n}{2}-1}dx$

$=n\int_0^1 (x^2-1+1)(1-x^2)^{\frac{n}{2}-1}dx$

$=-n\int_0^1 (1-x^2)^{\frac{n}{2}}dx+n\int_0^1 (1-x^2)^{\frac{n}{2}-1}dx$

$=-nI_n+nI_{n-2}.$

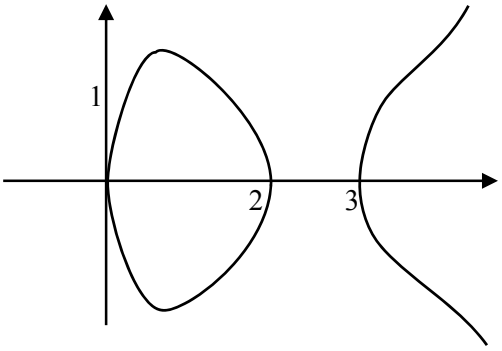
$\therefore (n+1)I_n=nI_{n-2}.$

$\therefore I_n=\dfrac{n}{n+1}I_{n-2}.$

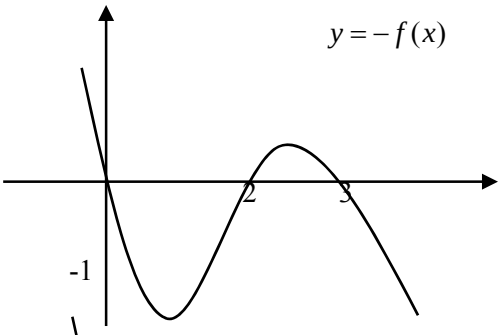
(ii) $I_5=\dfrac{5}{6}I_3, I_3=\dfrac{3}{4}I_1, I_1=\int_0^1\sqrt{1-x^2}dx=\dfrac{1}{4}$ area of a circle of radius $1=\dfrac{\pi}{4}$.

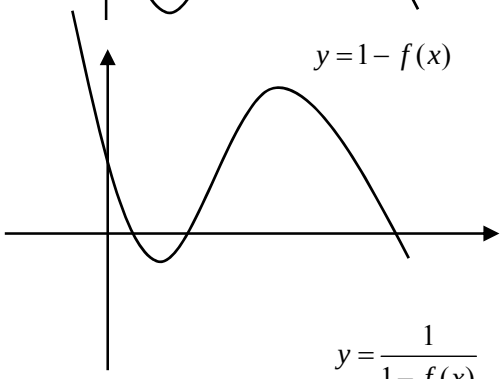
$\therefore\ I_5=\dfrac{5}{6}\dfrac{3}{4}\dfrac{\pi}{4}=\dfrac{5\pi}{32}$

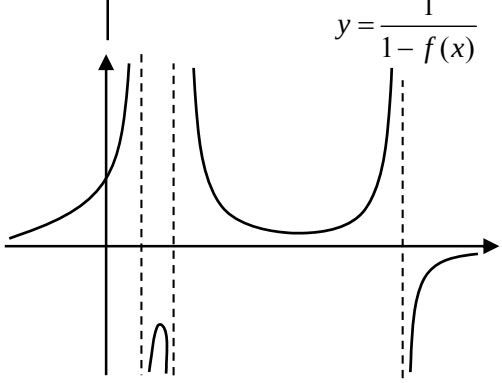
(b) (i) $y^2=f(x)$



(ii) $y=\dfrac{1}{1-f(x)}$







(c) (i) $\angle ABC = \angle ABD + \angle DBC = \alpha + \beta$

since $\angle ABD = \angle ACD = \beta$ and $\angle DBC = \angle DAC = \alpha$ (angles subtending the same arc are equal)

$\angle ACB = 90°$ (semi-circle angle)

$\therefore \sin(\alpha+\beta) = \dfrac{AC}{AB} = \dfrac{AC}{2r}$

$\therefore AC = 2r\sin(\alpha+\beta).$

(ii) In ΔABD (noting $\angle ADB = 90°$, semicircle angle),

$\sin\beta = \dfrac{AD}{AB} = \dfrac{AD}{2r}, \therefore AD = 2r\sin\beta$

In $\Delta ADE, \cos\alpha = \dfrac{AE}{AD},$

$\therefore AE = AD\cos\alpha = 2r\cos\alpha\sin\beta$

(iii) Similarly, $EC = 2r\cos\beta\sin\alpha$

In $\Delta ABC, \sin(\alpha+\beta) = \dfrac{AC}{AB} = \dfrac{AE+EC}{2r}$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Question 14

(a) Exact area $= \int_1^t \ln x dx = \left[x\ln x - x\right]_1^t = t\ln t - t + 1.$

Area of triangle $= \dfrac{1}{2}(t-1)\ln t$

Exact area > Area of triangle, $\therefore t\ln t - t + 1 > \dfrac{1}{2}(t-1)\ln t$

$2t\ln t - 2t + 2 > (t-1)\ln t$

$(t+1)\ln t > 2t - 2$

$\therefore \ln t > \dfrac{2(t-1)}{t+1}$ for $t > 1$

(b) Let $n = 2, |z_2| = |1+i| = \sqrt{2}, \therefore$ true for $n = 2.$

Assume $|z_n| = \sqrt{n}$ for some integer n.

Required to prove that $|z_{n+1}| = \sqrt{n+1}.$

$z_{n+1} = z_n\left(1 + \dfrac{i}{|z_n|}\right) = z_n\left(1 + \dfrac{i}{\sqrt{n}}\right)$

$\therefore |z_{n+1}| = \sqrt{n}\sqrt{1+\dfrac{1}{n}} = \sqrt{n}\sqrt{\dfrac{n+1}{n}} = \sqrt{n+1}.$

$\therefore$ By the principle of Mathematical Induction, it's true for all $n \geq 2$.

(c) (i) $\sec^{2n}\theta = (1+\tan^2\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta$

(ii) $\sec^8\theta = \sec^6\theta\sec^2\theta = \sum_{k=0}^{3}\binom{3}{k}\tan^{2k}\theta\sec^2\theta$

$= \left\{\binom{3}{0} + \binom{3}{1}\tan^2\theta + \binom{3}{2}\tan^4\theta + \binom{3}{3}\tan^6\theta\right\}\sec^2\theta$

$\therefore \int\sec^8\theta d\theta = \binom{3}{0}\tan\theta + \binom{3}{1}\dfrac{\tan^3\theta}{3} + \binom{3}{2}\dfrac{\tan^5\theta}{5}$

$+ \binom{3}{3}\dfrac{\tan^7\theta}{7} + C$

$= \tan\theta + \tan^3\theta + \dfrac{3}{5}\tan^5\theta + \dfrac{1}{7}\tan^7\theta + C$

(d) (i) $\angle A$ is common

$\dfrac{AD}{AC} = \dfrac{3}{6} = \dfrac{1}{2}, \dfrac{AE}{AB} = \dfrac{4}{8} = \dfrac{1}{2}, \therefore \dfrac{AD}{AC} = \dfrac{AE}{AB}$

$\therefore \Delta ABC \,|||\, \Delta AED$ (SAS)

(ii) $\angle B = \angle AED$ (corresponding angles in similar Δs)

$\therefore BCED$ is a cyclic quad (interior angle = opposite exterior angle)

(iii) $\cos\angle DEA = \dfrac{3^2+4^2-3^2}{2\times 3\times 4} = \dfrac{2}{3}$

$\therefore \cos\angle DEC = \cos(\pi - \angle DEA) = -\cos\angle DEA = -\dfrac{2}{3}$

$\therefore CD^2 = 3^2 + 2^2 - 2\times 3\times 2\times -\dfrac{2}{3} = 21$

$\therefore CD = \sqrt{21}$

(iv) Let O be the centre,

$\angle DOC = 2(\pi - \angle DEC)$ (angle at the centre is twice angle on the circumference)

$\cos\angle DOC = \cos 2(\pi - \angle DEC)$

$= \cos 2\angle DEC$

$= 2\cos^2\angle DEC - 1$

$= 2\times\dfrac{4}{9} - 1 = -\dfrac{1}{9}.$

In $\Delta DOC, CD^2 = OC^2 + OD^2 - 2OC\times OD\times\cos\angle DOC$

$\therefore 21 = r^2 + r^2 + \dfrac{2}{9}r^2 = \dfrac{20}{9}r^2$

$\therefore r^2 = \dfrac{189}{20}.$

$\therefore r = \sqrt{\dfrac{189}{20}} = \dfrac{3\sqrt{105}}{10}.$

Question 15

(a) Area $A = \dfrac{1}{2}|z||w|\sin(\theta-\phi)$, assuming $\theta > \phi$.

$z\bar{w} - w\bar{z} = |z|\text{cis}\,\theta|w|\text{cis}(-\phi) - |w|\text{cis}\,\phi|z|\text{cis}(-\theta)$

$= |z||w|\left(\text{cis}\,\theta\,\text{cis}(-\phi) - \text{cis}\,\phi\,\text{cis}(-\theta)\right)$

$= |z||w|\left(\text{cis}(\theta-\phi) - \text{cis}(\phi-\theta)\right)$

$= |z||w|2i\sin(\theta-\phi)$, since $\cos(\theta-\phi) = \cos(\phi-\theta)$

and $\sin(\theta-\phi) = -\sin(\phi-\theta)$

$\therefore z\bar{w} - w\bar{z} = 4iA.$

(b) (i) $P(x) = ax^4 + bx^3 + cx^2 + e$

$P(1) = -3, \therefore a + b + c + e = -3$ (1)

$P(-1) = 0, \therefore a - b + c + e = 0$ (2)

$P'(x) = 4ax^3 + 3bx^2 + 2cx$

$P'(-1) = 0, \therefore -4a + 3b - 2c = 0$ (3)

(1) − (2) gives $2b = -3, \therefore b = -\dfrac{3}{2}$

Sub to (3), $-4a - \dfrac{9}{2} - 2c = 0$

$\therefore 4a + 2c = -\dfrac{9}{2}$

(ii) When $x = 1, m = 4a + 3b + 2c = 3b + (4a + 2c)$

$= 3\times -\dfrac{3}{2} - \dfrac{9}{2} = -9$

(c) (i) $0.7^4 = 0.24$

(ii) Pr(at least 3 cars complete 4 days)

$= 1 - \Pr(0,1,2 \text{ cars complete 4 days})$

$= 1 - 0.76^8 - {}^8C_1 0.76^7 0.24 - {}^8C_2 0.76^6 0.24^2$

(d) (i) Terminal velocity occurs when $\dot{v} = 0, \therefore mg = kv^2$

$\therefore v_T = \sqrt{\dfrac{mg}{k}}$

(ii) When the ball goes up, $mv\dfrac{dv}{dx} = -(mg + kv^2)$,

taking the direction upward positive

$-\int_u^0 \dfrac{mv\,dv}{mg + kv^2} = \int_0^H dx$

$H = -\dfrac{m}{2k}\left[\ln(mg + kv^2)\right]_u^0 = \dfrac{m}{2k}\ln\dfrac{mg + ku^2}{mg}$

$= \dfrac{v_T^2}{2g}\ln\left(1 + \dfrac{u^2}{\frac{mg}{k}}\right) = \dfrac{v_T^2}{2g}\ln\left(1 + \dfrac{u^2}{v_T^2}\right)$

(iii) When the ball goes down, $mv\dfrac{dv}{dx} = mg - kv^2$

$$\int_0^w \frac{mvdv}{mg-kv^2} = \int_0^H dx$$

$$H = -\frac{m}{2k}\left[\ln(mg-kv^2)\right]_0^w = -\frac{m}{2k}\ln\frac{mg-kw^2}{mg}$$

$$= -\frac{v_T^{\,2}}{2g}\ln\left(1-\frac{w^2}{v_T^{\,2}}\right) = \ln\left(\frac{v_T^{\,2}}{v_T^{\,2}-w^2}\right)$$

But $H = \dfrac{v_T^{\,2}}{2g}\ln\left(1+\dfrac{u^2}{v_T^{\,2}}\right) = \dfrac{v_T^{\,2}}{2g}\ln\left(\dfrac{v_T^{\,2}+u^2}{v_T^{\,2}}\right)$

$$\therefore \ln\left(\frac{v_T^{\,2}+u^2}{v_T^{\,2}}\right) = \ln\left(\frac{v_T^{\,2}}{v_T^{\,2}-w^2}\right)$$

$$\therefore \frac{v_T^{\,2}+u^2}{v_T^{\,2}} = \frac{v_T^{\,2}}{v_T^{\,2}-w^2}$$

$$\therefore -v_T^{\,2}w^2 + v_T^{\,2}u^2 - w^2u^2 = 0$$

$$\therefore v_T^{\,2}w^2 + w^2v^2 = v_T^{\,2}u^2$$

$$\therefore \frac{1}{u^2} + \frac{1}{v_T^{\,2}} = \frac{1}{w^2}$$

Question 16

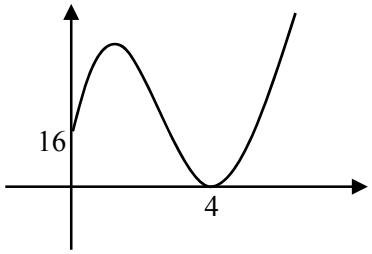
(a) $P(x) = 2x^3 - 15x^2 + 24x + 16$

$P'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4)$

$= 6(x-1)(x-4)$

$= 0$ when $x = 1, 4$

$P(0) = 16, P(1) = 27,$ and $P(4) = 0, \therefore$ For $x \geq 0,$ the minimum of $P(x)$ is 0, which occurs at $x = 4$.



(ii) For $x \geq 0, 2x^3 - 15x^2 + 24x + 16 \geq 0$

$2x^3 + 10x^2 + 24x + 16 \geq 25x^2$

By trial and error, $2x^3 + 10x^2 + 24x + 16 = 0$ when $x = -1$.

$\therefore 2x^3 + 10x^2 + 24x + 16 = (x+1)(2x^2 + 8x + 16)$

$= (x+1)\left(x^2 + (x+4)^2\right).$

$\therefore (x+1)\left(x^2 + (x+4)^2\right) \geq 25x^2$

(iii) $\therefore x^2 + (x+4)^2 \geq \dfrac{25x^2}{x+1}$

Let $x = m+n, (m+n)^2 + (m+n+4)^2 \geq \dfrac{25(m+n)^2}{m+n+1}$

$\geq \dfrac{25(m^2+n^2+2mn)}{m+n+1} \geq \dfrac{25(2mn+2mn)}{m+n+1},$

since $m^2 + n^2 \geq 2mn.$

$\therefore (m+n)^2 + (m+n+4)^2 \geq \dfrac{100mn}{m+n+1}$

(b) (i) $PS + PS' = 2a, \therefore P$ lies on the ellipse with foci S and S'.

(ii) $SP = ePM,$ where M lies on the directrix

$= e\left(\dfrac{a}{e} - a\cos\theta\right) = a(1 - e\cos\theta)$

(iii) $\sin\beta = \dfrac{QS'}{PS'} = \dfrac{a\cos\theta + ae}{a(1+e\cos\theta)} = \dfrac{e+\cos\theta}{1+e\cos\theta},$

as $SP = a(1-e\cos\theta)$ and $PS + PS' = 2a.$

(iv) Similarly, $\sin\angle QPS = \dfrac{QS}{PS} = \dfrac{e-\cos\theta}{1-e\cos\theta}$

Resolving the forces vertically,

$T\sin\beta - T\sin\angle QPS = mg$

$T\left(\dfrac{e+\cos\theta}{1+e\cos\theta} - \dfrac{e-\cos\theta}{1-e\cos\theta}\right) = mg$

$\therefore mg = \dfrac{T}{1-e^2\cos^2\theta}\Big(e+\cos\theta - e^2\cos\theta - e\cos^2\theta$

$-(e-\cos\theta+e^2\cos\theta-e\cos^2\theta)\Big)$

$= \dfrac{T}{1-e^2\cos^2\theta}\left(2\cos\theta - 2e^2\cos\theta\right)$

$= \dfrac{2T(1-e^2)\cos\theta}{1-e^2\cos^2\theta}.$ (1)

(v) Resolving the forces horizontally,

$T\cos\beta + T\cos\angle QPS = mr\omega^2$

$T\left(\dfrac{PQ}{PS'} + \dfrac{PQ}{PS}\right) = mr\omega^2$

$\therefore mr\omega^2 = PQ\times T\dfrac{PS+PS'}{PS\times PS'}$

$= PQ\times T\dfrac{2a}{a(1-e\cos\theta)a(1+e\cos\theta)}$

$= PQ\times T\dfrac{2a}{a^2(1-e^2\cos\theta)}$

But $PQ = y_P = b\sin\theta = a\sqrt{1-e^2}\sin\theta$

$\therefore mr\omega^2 = \dfrac{2T\sqrt{1-e^2}\sin\theta}{1-e^2\cos\theta}.$ (2)

(vi) $\dfrac{(2)}{(1)}$ gives $\dfrac{r\omega^2}{g} = \dfrac{\sqrt{1-e^2}\sin\theta}{(1-e^2)\cos\theta} = \dfrac{\tan\theta}{\sqrt{1-e^2}}$

$\therefore \tan\theta = \dfrac{r\omega^2\sqrt{1-e^2}}{g}$