2013 HSC Examination

Mathematics Extension 2

SOLUTIONS

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Multiple Choice

1. B
2. A
3. D
4. A
5. B
6. D
7. C
8. C
9. B
10. A

Brief Explanations

Question 1 Basic integral.

Question 2 Manipulate and calculate as usual.

Question 3 The argument is doubled and the modulus is less since $|z|<1$.

Question 4 Standard $u=\alpha+1\Rightarrow\alpha=u-1$, then $P(\alpha)=0$.

Question 5 The region is clearly an annulus, so (B).

Question 6 Complete the square then use table of standard integrals.

Question 7 Use the fact that circumference is $2\pi r$.

Question 8 Standard slicing problem, where the base length is $2y$, so the area is $4y^2$.

Question 9 We know the limit as $x\to 0$ is 1, and the function is even, so (B).

Question 10 First find the total, ignoring restrictions, then subtract the 'excess' cases where the rooms have more than 4 people.

For the total, each person has 4 choices, so 4^6.

Case #1: 5 in one room, 1 in another. Choose 1 person to be 'left out', so we have $\binom{6}{1}$.

5 people have 4 rooms to choose from, then last person has 3 rooms, so 4×3.

Case #2: All 6 in one room. Four rooms to choose from, so 4 ways.

Hence the total is $4^6-6\times 4\times 3-4=4020$.

Written Response

Question 11 (a) (i)

$$\begin{aligned} z+\overline{w} &= 2-i\sqrt{3}+1-i\sqrt{3} \\ &= 3-2i\sqrt{3} \end{aligned}$$

Question 11 (a) (ii)

$$w = 2\,\text{cis}\left(\frac{\pi}{3}\right)$$

Question 11 (a) (iii)

$$\begin{aligned} w^{24} &= 2^{24}\,\text{cis}\left(\frac{24\pi}{3}\right) \quad ...(\text{by De Moivre's Theorem}) \\ &= 2^{24}\,\text{cis}\left(8\pi\right) \\ &= 2^{24} \quad ...(\text{since } \text{cis}(2k\pi)=1, \text{ for all } k\in\mathbb{Z}\ ) \end{aligned}$$

Question 11 (b)

$$\frac{x^2+8x+11}{(x-3)(x^2+2)} \equiv \frac{4}{x-3}+\frac{-3x-1}{x^2+2}$$

Question 11 (c)

By inspection, $z=i$ is a solution. We can see that the sum of roots is $-4i$, so let the other root be α.

$$\begin{aligned} \alpha+i &= -4i \\ \alpha &= -5i \end{aligned}$$

Hence, $P(z)=(z+5i)(z-i)$.

Question 11 (d)

We use the substitution $u = x^2 \Rightarrow \frac{1}{2}du = x\,dx$.

$x = 1 \quad \Rightarrow u = 1$
$x = 0 \quad \Rightarrow u = 0$

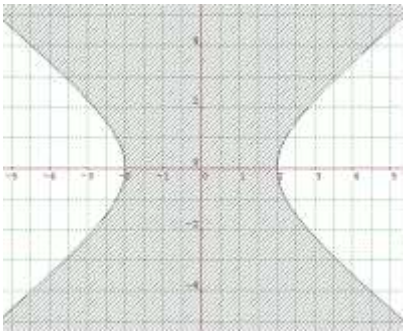
Hence we have

$$
\begin{aligned}
I &= \frac{1}{2}\int_0^1 u\sqrt{1-u}\,du \\
&= -\frac{1}{2}\int_0^1 -u\sqrt{1-u}\,du \\
&= -\frac{1}{2}\int_0^1 (1-u-1)\sqrt{1-u}\,du \\
&= -\frac{1}{2}\int_0^1 (1-u)\sqrt{1-u} - \sqrt{1-u}\,du \\
&= -\frac{1}{2}\int_0^1 (1-u)^{\frac{3}{2}} - (1-u)^{\frac{1}{2}}\,du \\
&= -\frac{1}{2}\left[-\frac{2}{5}(1-u)^{\frac{3}{2}} + \frac{2}{3}(1-u)^{\frac{3}{2}}\right]_0^1 \\
&= -\frac{1}{2}\left(\frac{2}{5} - \frac{2}{3}\right) \\
&= \frac{2}{15}
\end{aligned}
$$

Question 11 (e)

Letting $z = x + iy$, we have $x^2 + 2ixy - y^2 + x^2 - 2ixy - y^2 \le 8 \Rightarrow x^2 - y^2 \le 4$

This is a rectangular hyperbola with *x* intercepts at $x = \pm 2$.



Question 12 (a)

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$
$$= \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)dx$$
$$= \frac{1}{2}\left(1+t^2\right)dx$$
$$dx = \frac{2\,dt}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx = \int_0^1 \frac{1}{4+5\times\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2}$$
$$= \int_0^1 \frac{2}{4\left(1+t^2\right)+5\left(1-t^2\right)}\,dt$$
$$= \int_0^1 \frac{2}{4+4t^2+5-5t^2}\,dt$$
$$= \int_0^1 \frac{2}{\left(3-t\right)\left(3+t\right)}\,dt$$
$$= 2\int_0^1 \frac{1/6}{3-t} + \frac{1/6}{3+t}\,dt$$
$$= \frac{1}{3}\int_0^1 \frac{1}{3-t} + \frac{1}{3+t}\,dt$$
$$= \frac{1}{3}\left[-\ln\left(3-t\right)+\ln\left(3+t\right)\right]_0^1$$
$$= \frac{1}{3}\ln\left(\frac{3+t}{3-t}\right)\Bigg|_0^1$$
$$= \frac{1}{3}\ln 2$$

Question 12 (b)

Differentiate both sides with respect to x.

$$\frac{y'}{y}+\frac{y'}{1000-y}=\frac{1}{50}$$

$$y'\left(\frac{1}{y}+\frac{1}{1000-y}\right)=\frac{1}{50}$$

$$y'\left(\frac{1000-y+y}{y(1000-y)}\right)=\frac{1}{50}$$

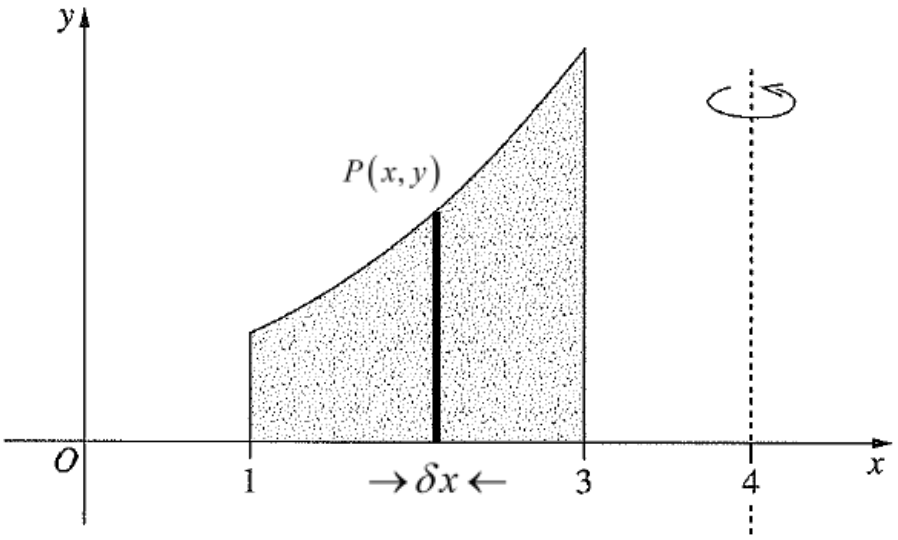
$$y'\left(\frac{1000}{y(1000-y)}\right)=\frac{1}{50}$$

$$y'=\frac{y}{50}\left(\frac{1000-y}{1000}\right)$$

$$=\frac{y}{50}\left(1-\frac{y}{1000}\right)$$

Question 12 (c)

We use the method of cylindrical shells.



We can see that the radius of the shell will be $4-x$ and that its height will be y.

Unfolding the shell, we roughly have a rectangular prism of height y, length $2\pi(4-x)$ and thickness δx. Hence, the volume of the shell is

$$\delta V \approx 2\pi(4-x)\,y\,\delta x$$
$$= 2\pi(4-x)e^x\,\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{1 \le x \le 3} 2\pi(4-x)e^x\,\delta x$$
$$= 2\pi \int_1^3 (4-x)e^x\,dx$$
$$= 2\pi \left[e^x(4-x)\Big|_1^3 + \int_1^3 e^x dx \right]$$
$$= 2\pi \left[e^3 - 3e + e^3 - e \right]$$
$$= 2\pi \left[2e^3 - 4e \right]$$
$$= 4\pi e\left(e^2 - 2\right)$$

Question 12 (d) (i)

We first use parametric differentiation.

$$\frac{dy}{dx} = \frac{dy/dp}{dx/dp}$$
$$= \frac{-c/p^2}{c}$$
$$= -\frac{1}{p^2}$$

Now we proceed with the point/gradient formula as usual, using P.

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$
$$p^2 y - cp = -x + cp$$
$$x + p^2 y = 2cp$$

Question 12 (d) (ii)

We know that *OB* makes a right angle with *OA*, since they are the coordinate axes. We must now prove that *P* is the midpoint of *AB*, so we first find *A* and *B*.

For *A*, let $y = 0$, so $x = 2cp$ and hence $A(2cp, 0)$.

For *B*, let $x = 0$, so $p^2 y = 2cp \Rightarrow y = \frac{2c}{p}$ and hence $B\left(0, \frac{2c}{p}\right)$.

Clearly, *P* is the midpoint of *AB*.

Question 12 (d) (iii)

Since *C* and *D* are *x* and *y* intercepts of the tangent at *Q*, we can apply the result of (ii) to deduce that *C*, *Q* and *D* lie on a circle centred at *Q*.

Since *P* and *Q* are the centres of their respective circles, we have the ratios

$$\frac{AP}{AB} = \frac{DQ}{DC} = \frac{1}{2}$$

However, notice that the intervals *PQ* and *BC* form intercepts on transversals, so since the ratios are the same, we can deduce that *BC* is parallel to *PQ*.

Question 13 (a) (i)

We use Integration by Parts, with the following choices.

$u = \left(1-x^2\right)^{\frac{n}{2}}$ $\qquad\qquad$ $dv = dx$

$du = -nx\left(1-x^2\right)^{\frac{n-2}{2}}\,dx$ $\qquad\qquad$ $v = x$

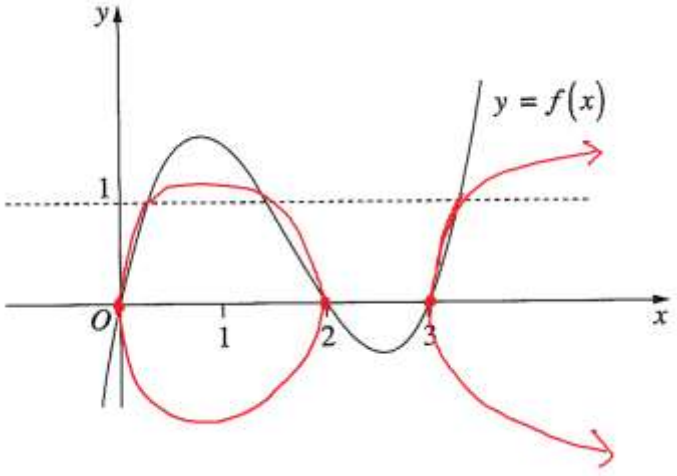
$$\begin{aligned} I_n &= x\left(1-x^2\right)^{\frac{n}{2}}\Big|_0^1 + n\int_0^1 x^2\left(1-x^2\right)^{\frac{n}{2}-1}dx \\ &= n\int_0^1 x^2\left(1-x^2\right)^{\frac{n}{2}-1}dx \\ &= -n\int_0^1 \left(1-x^2-1\right)\left(1-x^2\right)^{\frac{n}{2}-1}dx \\ &= -n\int_0^1 \left(1-x^2\right)^{\frac{n}{2}} - \left(1-x^2\right)^{\frac{n}{2}-1}dx \\ &= -nI_n + nI_{n-2} \\ \left(n+1\right)I_n &= nI_{n-2} \\ I_n &= \frac{n}{n+1}I_{n-2} \end{aligned}$$

Question 13 (a) (ii)

$I_5 = \frac{5}{6}I_3 = \frac{5}{6}\times\frac{3}{4}I_1 = \frac{5}{6}\times\frac{3}{4}\times\frac{\pi}{4} = \frac{5\pi}{32}$, using the area of a quarter circle with radius 1.

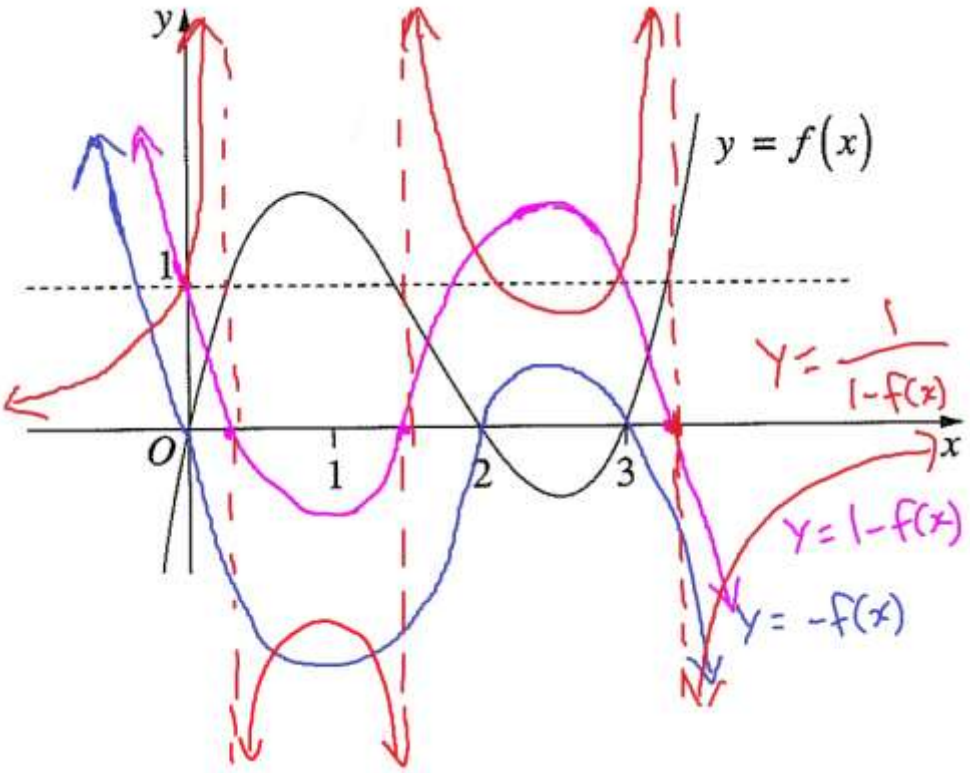
Question 13 (b) (i)

This is essentially the square root curve, except reflected about the x axis.



Question 13 (b) (i)

We first draw $y = 1 - f(x)$, then we draw its reciprocal curve.



Question 13 (c) (i)

We need to somehow get $\alpha + \beta$, and the only way is by somehow considering the exterior angle of ΔADC, constructed by extending *CD* to say *F*. But notice if we do this, that is also the opposite exterior angle of the cyclic quadrilateral *ABCD*, so $\angle ADF = \angle ABC = \alpha + \beta$.

$\angle ACB = 90^\circ$ since *AB* is a diameter and *C* is a point on the circle. Now, we use trig ratios.

$$\begin{aligned}\sin(\angle ABC) &= \sin(\alpha + \beta) \\ &= \frac{AC}{2r} \\ AC &= 2r\sin(\alpha + \beta)\end{aligned}$$

Question 13 (c) (ii)

$AE = AD\cos\alpha$, so we must prove that $AD = 2r\sin\beta$.

However, if we construct *BD*, we see that $\angle ADB = 90^\circ$ since *AB* is a diameter. Furthermore, $\angle ABD = \beta$ since it is subtended by chord *AD*, which subtends β, so $\sin\beta = \dfrac{AD}{2r}$ and hence we have the result.

Question 13 (c) (iii)

We know that $AE = 2r\cos\alpha\sin\beta$. Similarly, $EC = 2r\sin\alpha\cos\beta$.

But $AC = AE + EC$, so we have

$$2r\sin(\alpha+\beta) = 2r\cos\alpha\sin\beta + 2r\sin\alpha\cos\beta$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

Question 14 (a)

We can see that the area under the curve A_C is greater than the area of the right angled triangle A_T.

$$\begin{aligned} A_C &= \int_1^t \ln x\,dx \\ &= x\ln x - x\Big|_1^t \quad \ldots(\text{using Integration by Parts}) \\ &= t\ln t - t + 1 \end{aligned}$$

$$A_T = \frac{1}{2}(t-1)\ln t$$

But we know that $A_C > A_T$ for all $t > 1$, so

$$t\ln t - t + 1 > \frac{1}{2}(t-1)\ln t$$
$$2t\ln t - 2t + 2 > t\ln t - \ln t$$
$$t\ln t + \ln t > 2t - 2$$
$$\ln t > 2\left(\frac{t-1}{t+1}\right)$$

Question 14 (b)

Base Case: $n = 2$

$|z_2| = \sqrt{2}$, which is indeed true since $|1+i| = \sqrt{2}$.

Induction Hypothesis: $k \Rightarrow k+1$

$|z_k| = \sqrt{k}$.

Inductive Step: $k \Rightarrow k+1$

$$z_{k+1} = z_k \left(1 + \frac{i}{|z_k|}\right)$$

$$\begin{aligned} |z_{k+1}| &= \left| z_k \left(1 + \frac{i}{|z_k|}\right) \right| \\ &= |z_k| \left| 1 + \frac{i}{|z_k|} \right| \\ &= \sqrt{k} \left| 1 + \frac{i}{\sqrt{k}} \right| \\ &= \left| \sqrt{k} + i \right| \\ &= \sqrt{k+1} \end{aligned}$$

And thus proven by Induction.

Question 14 (c) (i)

$$\begin{aligned} \sec^{2n}\theta &= \left(\sec^2\theta\right)^n \\ &= \left(1 + \tan^2\theta\right)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \tan^{2k}\theta \end{aligned}$$

Question 14 (c) (ii)

$$\begin{aligned}\sec^8\theta &= \sec^6\theta\sec^2\theta \\ &= \sum_{k=0}^{3}\binom{n}{k}\tan^{2k}\theta\sec^2\theta \\ &= \left[\binom{n}{0}+\binom{n}{1}\tan^2\theta+\binom{n}{2}\tan^4\theta+\binom{n}{3}\tan^6\theta\right]\sec^2\theta\end{aligned}$$

$$\begin{aligned}\int\sec^8\theta\,d\theta &= \int\left[\binom{3}{0}+\binom{3}{1}\tan^2\theta+\binom{3}{2}\tan^4\theta+\binom{3}{3}\tan^6\theta\right]\sec^2\theta \\ &= \tan\theta+\tan^3\theta+\frac{3}{5}\tan^5\theta+\frac{1}{7}\tan^7\theta+C\end{aligned}$$

Question 14 (d) (i)

$$\frac{AD}{AC}=\frac{AE}{AB}=\frac{1}{2}$$

$\angle BAC$ is common.

Hence, $\Delta ABC \,|||\, \Delta AED$, since two sides are in ratio and the included angle is equal.

Question 14 (d) (ii)

Since $\Delta ABC \,|||\, \Delta AED$, we can deduce that $\angle DEA = \angle DBC$. But one is the opposite exterior angle of the other, so *BCED* must be a cyclic quadrilateral, by the converse of the opposite interior angle theorem.

Question 14 (d) (iii)

We use the Cosine Rule on $\theta = \angle DCE$ in both ΔDCE and ΔDCA.

In ΔDCE:

$$\cos\theta = \frac{2^2 + CD^2 - 3^2}{2 \times 2 \times CD} = \frac{CD^2 - 5}{4 \times CD}$$

In ΔDCA:

$$\cos\theta = \frac{6^2 + CD^2 - 3^2}{2 \times 6 \times CD} = \frac{27 + CD^2}{12 \times CD}$$

Equate the two

$$\frac{CD^2 - 5}{4 \times CD} = \frac{27 + CD^2}{12 \times CD}$$

$$CD^2 - 5 = \frac{27 + CD^2}{3}$$

$$3 \times CD^2 - 15 = 27 + CD^2$$

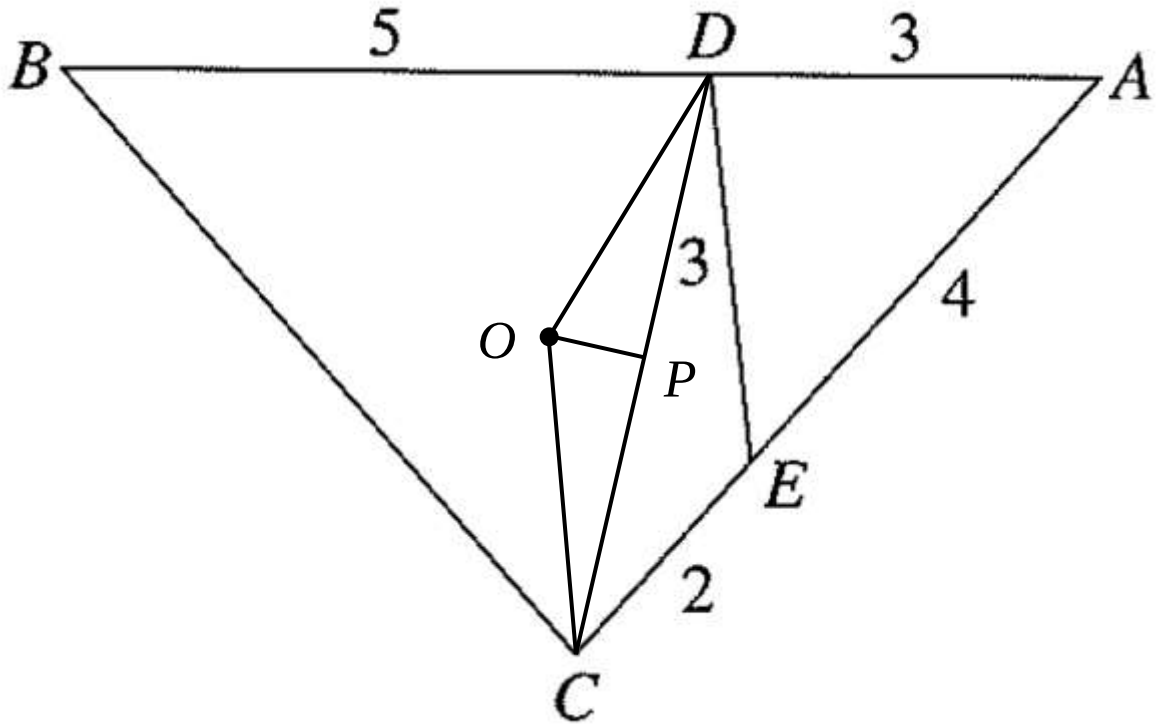
$$2 \times CD^2 = 42$$

$$CD^2 = 21$$

$$CD = \sqrt{21}$$

Question 14 (d) (iv)

First let *O* be the centre of the circle and construct chord *CD* and perpendicular bisector *OP*.



By similar triangle ratios, we can deduce that $BC = 6$ and so we can work out that

$$\cos\left(\angle ABC\right) = \frac{8^2 + 6^2 - 6^2}{2 \times 8 \times 6} = \frac{2}{3}$$

But $\angle ABC = \frac{1}{2}\angle COD$ since angles at the centre are twice angles at circumference.

$$\cos\left(\angle COD\right) = \cos\left(2 \times \angle ABC\right) = 2\left(\frac{2}{3}\right)^2 - 1 = -\frac{1}{9}$$

Now, we use the Cosine Rule on ΔCOD.

$$\cos\left(\angle COD\right) = \frac{2r^2 - 21}{2r^2}$$

$$\frac{2r^2 - 21}{2r^2} = -\frac{1}{9}$$

And solving for *r* yields $r = \frac{3\sqrt{105}}{10}$

Question 15 (a)

Let $z = A\text{cis}\,\theta$ and $w = B\text{cis}\,\phi$.

Since the area of the triangle is *A*, we can deduce that $A = \frac{1}{2}AB\sin(\theta - \phi)$.

$$\begin{aligned} z\bar{w} - w\bar{z} &= A\text{cis}\,\theta \times B\text{cis}(-\phi) - A\text{cis}(-\theta) \times B\text{cis}\,\phi \\ &= AB\text{cis}(\theta - \phi) - AB\text{cis}(\phi - \theta) \\ &= AB\text{cis}(\theta - \phi) - AB\overline{\text{cis}(\theta - \phi)} \qquad \ldots\left(\text{since cis}(-\alpha) = \overline{\text{cis}(\alpha)}\right) \\ &= 2ABi \times \text{Im}(\text{cis}(\theta - \phi)) \\ &= 2ABi\sin(\theta - \phi) \\ &= 4iA \end{aligned}$$

Question 15 (b) (i)

Let the roots of the polynomial be $-1, -1, \alpha, \beta$.

From the sum of roots in triples, we have

$$-\alpha\beta - \alpha\beta + \alpha + \beta = 0$$
$$\alpha + \beta = 2\alpha\beta$$

We now use the sum of roots.

$$-2 + \alpha + \beta = -b$$
$$\alpha + \beta = 2 - b$$

From the product of roots, we have $\alpha\beta = e$, so $2 - b = e$ and thus $b + e = 2$.

We know that $P(1) = 3$, $P(-1) = 0$ and $P'(-1) = 0$ since it has a double root at $x = -1$.

$P(1) = a + b + c + e = -3$ and $P(-1) = a - b + c + e = 0$

Subtracting the two equations, we have $2b = -3 \Rightarrow b = -\frac{3}{2}$.

Differentiate the polynomial so we can let $P'(-1) = 0$, by the Multiple Root Theorem.

$P'(x) = 4ax^3 + 3bx^2 + 2cx$ and thus $P'(-1) = -4a + 3b - 2c = 0$

Hence we have

$$4a + 2c = 3b = -\frac{9}{2}$$

Question 15 (b) (ii)

The slope of the tangent when $x=1$ is $P'(1)=4a+3b+2c$.

But since $4a+2c=-\frac{9}{2}$, we have

$$\begin{aligned} P'(1) &= -\frac{9}{2}+3\left(-\frac{3}{2}\right) \\ &= -\frac{9}{2}-\frac{9}{2} \\ &= -9 \end{aligned}$$

Question 15 (c) (i)

Since it completes all four days, we have $0.7\times 0.7\times 0.7\times 0.7 = 0.7^4$.

Question 15 (c) (i)

The required scenario is the converse of the following cases. Let $p=0.7^4$.

Case 1: No cars complete.

$$P(X=0)=\binom{8}{0}p^0(1-p)^8$$

Case 2: 1 car completes.

$$P(X=1)=\binom{8}{1}p^1(1-p)^7$$

Case 3: 2 cars complete.

$$P(X=2)=\binom{8}{2}p^2(1-p)^6$$

Hence, the probability is $1-\sum_{k=0}^{2}P(X=k)$

Question 15 (d) (i)

For terminal velocity, let $m\dot{v} = 0$.

$$kv_T^{\,2} = mg$$
$$v_T^{\,2} = \frac{mg}{k}$$
$$v_T = \sqrt{\frac{mg}{k}}$$

Question 15 (d) (ii)

We use the formula $\dot{v} = v \times \frac{dv}{dx}$, but the required equation of motion is $m\dot{v} = -mg - kv^2$

$$m \times v \times \frac{dv}{dx} = -mg - kv^2$$
$$\int_0^H dx = -m \int_u^0 \frac{v}{mg + kv^2}\, dv$$
$$H = -\frac{m}{2k} \ln\left(mg + kv^2\right)\Big|_u^0$$
$$H = -\frac{m}{2k} \ln\left(\frac{mg}{mg + ku^2}\right)$$
$$= \frac{m}{2k} \ln\left(\frac{mg + ku^2}{mg}\right)$$
$$= \frac{v_T^{\,2}}{2g} \ln\left(1 + \frac{u^2}{v_T^{\,2}}\right)$$

Question 15 (d) (iii)

We derive similar equations for w.

$$m \times v \times \frac{dv}{dx} = mg - kv^2$$

$$\int_0^H dx = m \int_0^w \frac{v}{mg - kv^2} dv$$

$$H = -\frac{m}{2k} \ln\left(mg - kv^2\right)\Big|_0^w$$

$$H = -\frac{m}{2k} \ln\left(\frac{mg - kw^2}{mg}\right)$$

$$= -\frac{v_T^{\,2}}{2g} \ln\left(1 - \frac{w^2}{v_T^{\,2}}\right)$$

$$= -\frac{v_T^{\,2}}{2g} \ln\left(\frac{v_T^{\,2} - w^2}{v_T^{\,2}}\right)$$

$$= \frac{v_T^{\,2}}{2g} \ln\left(\frac{v_T^{\,2}}{v_T^{\,2} - w^2}\right)$$

Equate the two heights.

$$\frac{v_T^{\,2}}{2g} \ln\left(\frac{v_T^{\,2}}{v_T^{\,2} - w^2}\right) = \frac{v_T^{\,2}}{2g} \ln\left(1 + \frac{u^2}{v_T^{\,2}}\right)$$

$$\frac{v_T^{\,2}}{v_T^{\,2} - w^2} = 1 + \frac{u^2}{v_T^{\,2}}$$

$$v_T^{\,4} = \left(v_T^{\,2} + u^2\right)\left(v_T^{\,2} - w^2\right)$$

$$= v_T^{\,4} - v_T^{\,2} w^2 + u^2 v_T^{\,2} - u^2 w^2$$

$$u^2 v_T^{\,2} = v_T^{\,2} w^2 + u^2 w^2$$

$$\frac{1}{w^2} = \frac{1}{u^2} + \frac{1}{v_T^{\,2}}$$

Question 16 (a) (i)

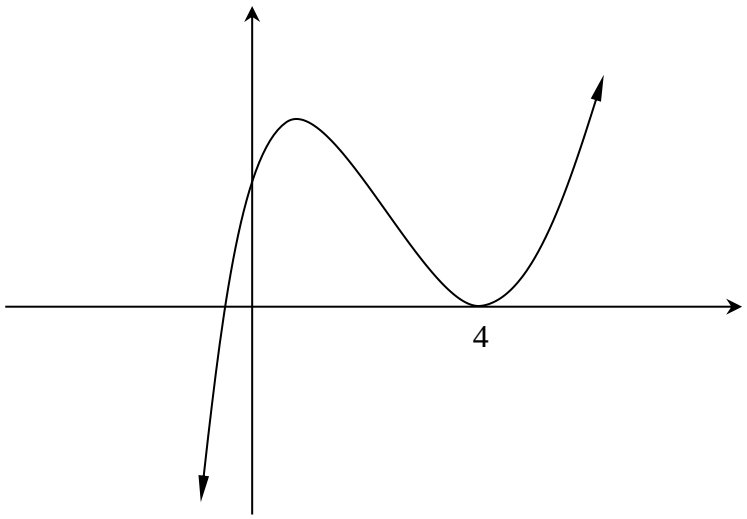
To find the minimum value, let the first derivative be equal to 0.

$$\begin{aligned} 6x^2 - 30x + 24 &= 0 \\ x^2 - 5x + 4 &= 0 \\ (x-1)(x-4) &= 0 \\ x &= 1,\ 4 \end{aligned}$$

Since it is a positive cubic, we know that the minimum value must come after the maximum value, so $x = 4$ must be where the minimum occurs. Hence, it is $(4, 0)$, by finding $P(4)$.

Question 16 (a) (ii)

From (i), we found that $x = 4$ is a double root, so we know that our curve looks like this now.



Hence, we see that for all $x \geq 0$, the curve is positive, so $P(x) \geq 0$.

We now manipulate our polynomial.

$$\begin{aligned} 2x^3 - 15x^2 + 24x + 16 &\geq 0 \\ 2x^3 + 10x^2 + 24x + 16 &\geq 25x^2 \\ 2(x^3 + 5x^2 + 12x + 8) &\geq 25x^2 \\ 2(x+1)(x^2 + 4x + 8) &\geq 25x^2 \\ (x+1)(2x^2 + 8x + 16) &\geq 25x^2 \\ (x+1)(x^2 + (x+4)^2) &\geq 25x^2 \end{aligned}$$

Question 16 (a) (iii)

From our expression in (ii), we have $x^2+\left(x+4\right)^2 \geq \dfrac{25x^2}{x+1}$, since $x \geq 0$.

Setting $x = m+n$, we have

$$\begin{aligned}\left(m+n\right)^2+\left(m+n+4\right)^2 &\geq \frac{25\left(m+n\right)^2}{m+n+1}\\ &= \frac{25\left(m^2+n^2+2mn\right)}{m+n+1}\\ &\geq \frac{25\left(2mn+2mn\right)}{m+n+1} \qquad \ldots\left(\text{since } m^2+n^2 \geq 2mn\right)\\ &= \frac{100mn}{m+n+1}\end{aligned}$$

Question 16 (b) (i)

Since the length of the string is 2*a*, so $PS+PS' = 2a$, and $SS' = 2ae$, where $0<e<1$, we can deduce that *P* lies on an ellipse with foci *S* and S'.

Question 16 (b) (ii)

$$\begin{aligned}PS &= ePM\\ &= e\left(\frac{a}{e}-a\cos\theta\right)\\ &= a\left(1-e\cos\theta\right)\end{aligned}$$

Question 16 (b) (iii)

$$\begin{aligned}\sin\beta &= \frac{QS'}{PS'}\\ &= \frac{a\cos\theta+ae}{ePM'}\\ &= \frac{a\left(\cos\theta+e\right)}{a\left(1+e\cos\theta\right)}\\ &= \frac{e+\cos\theta}{1+e\cos\theta}\end{aligned}$$

Question 16 (b) (iv)

Resolving forces, we have $T\sin\beta - T\sin\left(\angle QPS\right) = mg$.

Now, we must find $\sin\left(\angle QPS\right)$, which is very similar to (iii).

$$\begin{aligned}\sin\left(\angle QPS\right) &= \frac{QS}{PS} \\ &= \frac{ae - a\cos\theta}{ePM} \\ &= \frac{ae - a\cos\theta}{a\left(1 - e\cos\theta\right)} \\ &= \frac{e - \cos\theta}{1 - e\cos\theta}\end{aligned}$$

Substituting this back in, we have

$$T \times \frac{e + \cos\theta}{1 + e\cos\theta} - T \times \frac{e - \cos\theta}{1 - e\cos\theta} = mg$$

$$T\left(\frac{\left(e + \cos\theta\right)\left(1 - e\cos\theta\right) - \left(e - \cos\theta\right)\left(1 + e\cos\theta\right)}{1 - e^2\cos^2\theta}\right) = mg$$

$$T\left(\frac{e - e^2\cos\theta + \cos\theta - e\cos^2\theta - e - e^2\cos\theta + \cos\theta - e\cos^2\theta}{1 - e^2\cos^2\theta}\right) = mg$$

$$2T\cos\theta\left(\frac{1 - e^2}{1 - e^2\cos^2\theta}\right) = mg$$

Question 16 (b) (v)

Resolving forces horizontally, we have $T\cos\beta + T\cos\left(\angle QPS\right) = mr\omega^2$.

$$T\cos\beta + T\cos\left(\angle QPS\right) = mr\omega^2$$

$$T\left(\frac{PQ}{PS'} + \frac{PQ}{PS}\right) = mr\omega^2$$

$$PQ \times T\left(\frac{PS + PS'}{PS \times PS'}\right) = mr\omega^2$$

$$PQ \times T\left(\frac{2a}{a\left(1 - e\cos\theta\right) \times a\left(1 + e\cos\theta\right)}\right) = mr\omega^2$$

$$PQ \times T\left(\frac{2}{a\left(1 - e^2\cos^2\theta\right)}\right) = mr\omega^2$$

$$\left(\frac{b\sin\theta \times 2T}{a\left(1 - e^2\cos^2\theta\right)}\right) = mr\omega^2 \qquad \ldots\left(\text{where } b \text{ is the length of the minor axis}\right)$$

$$\left(\frac{a\sqrt{1 - e^2}\sin\theta \times 2T}{a\left(1 - e^2\cos^2\theta\right)}\right) = mr\omega^2 \qquad \ldots\left(\text{since } b^2 = a^2\left(1 - e^2\right)\right)$$

Question 16 (b) (vi)

Divide the expressions from (iv) and (v).

$$\frac{r\omega^2}{g} = \frac{\sqrt{1 - e^2}\sin\theta}{\left(1 - e^2\right)\cos\theta}$$

$$= \frac{\sqrt{1 - e^2}}{1 - e^2}\tan\theta$$

$$= \frac{\tan\theta}{\sqrt{1 - e^2}}$$

$$\tan\theta = \frac{r\omega^2}{g}\sqrt{1 - e^2}$$