2011 NSW BOS Mathematics Extension 2 Solutions

Free download from www.itute.com **© Copyright 2011 itute.com**

Q1a Let $u=\ln x$ and $\frac{dv}{dx}=x$, $\therefore\ \frac{du}{dx}=\frac{1}{x}$ and $v=\frac{x^2}{2}$

$$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\int\frac{x^2}{2}\times\frac{1}{x}dx=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c$$

Q1b Let $u=x+1$, $\therefore\ x=u-1$, $\frac{du}{dx}=1$

$$\int_0^3 x\sqrt{x+1}dx=\int_0^3(u-1)\sqrt{u}\,\frac{du}{dx}dx=\int_1^4\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$$

$$=\left[\frac{2u^{\frac{5}{2}}}{5}-\frac{2u^{\frac{3}{2}}}{3}\right]_1^4=\frac{64}{5}-\frac{16}{3}-\left(\frac{2}{5}-\frac{2}{3}\right)=\frac{116}{15}$$

Q1ci $\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x-1}=\frac{1}{x^2(x-1)}$,

$$\frac{ax(x-1)+b(x-1)+cx^2}{x^2(x-1)}=\frac{1}{x^2(x-1)},$$

$\therefore\ ax(x-1)+b(x-1)+cx^2=1$

$\therefore\ b=-1$, $a+c=0$, $b-a=0$, $\therefore\ a=-1$ and $c=1$

Q1cii $\int\frac{1}{x^2(x-1)}dx=\int\left(\frac{-1}{x}+\frac{-1}{x^2}+\frac{1}{x-1}\right)dx$

$$=-\ln|x|+\frac{1}{x}+\ln|x-1|+c=\ln\left|\frac{x-1}{x}\right|+\frac{1}{x}+c$$

Q1d $\int\cos^3\theta d\theta=\int\cos^2\theta\cos\theta d\theta=\int(1-\sin^2\theta)\cos\theta d\theta$

$$=\int(1-u^2)\frac{du}{d\theta}d\theta\quad\text{(where } u=\sin\theta\text{ and }\frac{du}{d\theta}=\cos\theta\text{)}$$

$$=u-\frac{1}{3}u^3+c=\sin\theta-\frac{1}{3}\sin^3\theta+c$$

Q1e $\int_{-1}^1\frac{1}{5-2t+t^2}dt=\int_{-1}^1\frac{1}{4+t^2-2t+1}dt=\int_{-1}^1\frac{1}{4+(t-1)^2}dt$

$$=\frac{1}{2}\int_{-1}^1\frac{2}{4+(t-1)^2}dt=\left[\frac{1}{2}\tan^{-1}\left(\frac{t-1}{2}\right)\right]_{-1}^1=\left(0-\frac{1}{2}\tan^{-1}(-1)\right)=\frac{\pi}{8}$$

Q2ai $\overline{w}+z=(2+3i)+(3+4i)=5+7i$

Q2aii $|w|=\sqrt{2^2+(-3)^2}=\sqrt{13}$

Q2aiii $\frac{w}{z}=\frac{w\bar{z}}{z\bar{z}}=\frac{(2-3i)(3-4i)}{(3+4i)(3-4i)}=\frac{-6-17i}{25}=-\frac{6}{25}-\frac{17}{25}i$

Q2bi $z=\left(1+\sqrt{3}i\right)+\left(\sqrt{3}+i\right)=\left(1+\sqrt{3}\right)+\left(1+\sqrt{3}\right)i$

Q2bii $Arg\left(\sqrt{3}+i\right)=\frac{\pi}{6}$, $Arg\left(1+\sqrt{3}i\right)=\frac{\pi}{3}$

$$\theta=\pi-\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{5\pi}{6}$$

Q2c $z^3=8$, one obvious solution is $z=2$, the other two solutions lie on the circle of radius 2 centred at the origin. The three solutions are separated from each other by $\frac{2\pi}{3}$.

The solutions: $z=2=2cis0$, $2cis\left(-\frac{2\pi}{3}\right)$, $2cis\left(\frac{2\pi}{3}\right)$.

Q2di $(\cos\theta+i\sin\theta)^3$

$$=(\cos\theta)^3+3(\cos\theta)^2(i\sin\theta)+3(\cos\theta)(i\sin\theta)^2+(i\sin\theta)^3$$

$$=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$$

$$=\cos^3\theta-3\cos\theta\sin^2\theta+i\left(3\cos^2\theta\sin\theta-\sin^3\theta\right)$$

Q2dii $(\cos\theta+i\sin\theta)^3=\cos3\theta+i\sin3\theta$ (de Moivre's theorem)

$\therefore\ \cos3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$ (real parts)

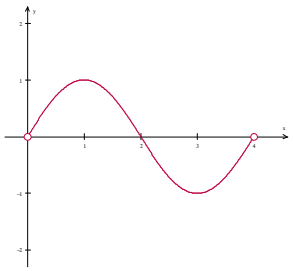
$$\cos3\theta=\cos^3\theta-3\cos\theta(1-\cos^2\theta)=4\cos^3\theta-3\cos\theta$$

$$\therefore\ \cos^3\theta=\frac{1}{4}\cos3\theta+\frac{3}{4}\cos\theta$$

Q2diii $4\cos^3\theta-3\cos\theta=1$, $\cos3\theta=1$, $3\theta=2k\pi$,

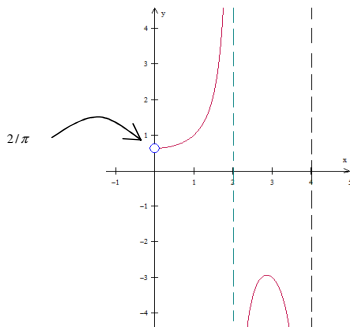
$\theta=\frac{2\pi}{3}$ is the smallest positive value when $k=1$.

Q3ai $y=\sin\frac{\pi}{2}x$, $0<x<4$



Q3aii $\lim_{x\to0}\frac{x}{\sin\frac{\pi}{2}x}=\frac{2}{\pi}\lim_{x\to0}\frac{\frac{\pi}{2}x}{\sin\frac{\pi}{2}x}=\frac{2}{\pi}$

Q3aiii



 © Copyright 2011 itute.com

Q3b

Base of isosceles triangle $= 2\cos x$

Height $= \sqrt{1-\cos^2 x} = \sin x$

Area $= \frac{1}{2} \times 2\cos x \times \sin x = \frac{1}{2}\sin 2x$

Volume of solid $= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2x\,dx = \left[-\frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$ cubic units

Q3c To prove $(2n)! \geq 2^n (n!)^2$ for all positive integers n.

For $n = 1$, $2! \geq 2^1 (1!)^2$ is true.

Assume $(2k)! \geq 2^k (k!)^2$, i.e. $\frac{(2k)!}{2^k (k!)^2} \geq 1$

For $n = k+1$,

$$\frac{(2n)!}{2^n (n!)^2} = \frac{(2k+2)!}{2^{k+1}(k+1)!(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{2 \times 2^k (k+1)^2 (k!)^2}$$

$$= \frac{2(k+1)(2k+1)(2k)!}{2 \times 2^k (k+1)^2 (k!)^2} = \frac{(2k+1)(2k)!}{(k+1)2^k (k!)^2} \geq 1 \text{ is true.}$$

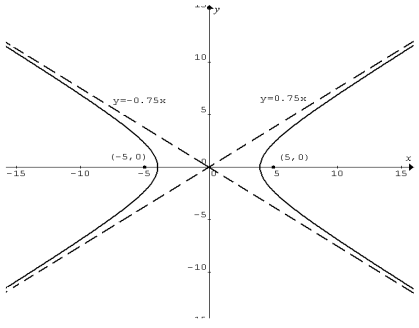
$\therefore (2n)! \geq 2^n (n!)^2$ is true for all positive integers n.

Q3di $b^2 = a^2(e^2-1)$, $9 = 16(e^2-1)$, $e = \frac{5}{4}$

Q3dii Foci are $(\pm ae, 0) = (\pm 5, 0)$

Q3diii $y = \pm\frac{b}{a}x$, $y = \pm\frac{3}{4}x$

Q3div



Q3dv $b^2 = a^2(e^2-1)$, $\frac{b}{a} = \sqrt{e^2-1}$, as $e \to \infty$, $\frac{b}{a} \to \infty$, the two asymptotes become vertical and the same line as the y-axis, also the two branches of the general hyperbola become a vertical line along the y-axis.

Q4ai $|z-a|^2 - |z-b|^2 = 1$, $|x-a+iy|^2 - |x-b+iy|^2 = 1$

$(x-a)^2 + y^2 - (x-b)^2 - y^2 = 1$, $-2ax + a^2 + 2bx - b^2 = 1$

$2(b-a)x = b^2 - a^2 + 1$, $x = \frac{(b-a)(b+a)+1}{2(b-a)} = \frac{a+b}{2} + \frac{1}{2(b-a)}$

Q4aii The locus of z is the vertical line through the x-axis at $x = \frac{a+b}{2} + \frac{1}{2(b-a)}$.

Q4bi $\angle ABC + \angle ADC = 180°$ (opposite angles of a cyclic quadrilateral)

$\angle ADG + \angle ADC = 180°$ (supplementary angles)

$\angle ABC + \angle AFG = 180°$ (co-interior angles)

$\therefore \angle ADG + \angle AFG = 180°$, $\therefore$ *FADG* is cyclic.

Q4bii Alternate angles are equal.

Q4biii *FADG* is cyclic, $\therefore \angle GFD = \angle GAD$

From part ii, $\angle GFD = \angle AED$, $\therefore \angle GAD = \angle AED$

$\therefore$ *GA* is a tangent to the circle *ABCD* because the angle between *GA* (tangent) and *AD* (chord) equals the angle in the alternate segment.

Q4ci $y = Af(t) + Bg(t)$, $\frac{dy}{dt} = Af'(t) + Bg'(t)$

$$\frac{d^2y}{dt^2} = Af''(t) + Bg''(t)$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y$$

$$= A(f''(t) + 3f'(t) + 2f(t)) + B(g''(t) + 3g'(t) + 2g(t))$$

$$= A \times 0 + B \times 0 = 0$$

$\therefore y = Af(t) + Bg(t)$ is also a solution.

Q4aii $y = e^{kt}$, $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$, $k^2e^{kt} + 3ke^{kt} + 2e^{kt} = 0$

$(k^2 + 3k + 2)e^{kt} = 0$, $(k+2)(k+1)e^{kt} = 0$.

Since $e^{kt} > 0$, $\therefore k = -2$ or -1.

Q4ciii $y = Ae^{-2t} + Be^{-t}$, $\frac{dy}{dt} = -2Ae^{-2t} - Be^{-t}$

When $t = 0$, $y = 0$, $\therefore A + B = 0$ and

$\frac{dy}{dt} = 1$, $\therefore -2A - B = 1$

$\therefore A = -1$ and $B = 1$

Q5ai $a = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$

Horizontal component of resultant force = ma

$F\sin\theta - N\sin\theta = m\omega^2 r$

Vertical component of resultant force $= 0$

$F\cos\theta + N\cos\theta - mg = 0$, $\therefore F\cos\theta + N\cos\theta = mg$

Q5aii $F - N = \frac{m\omega^2 r}{\sin\theta}$, $F + N = \frac{mg}{\cos\theta}$

$\therefore 2N = \frac{mg}{\cos\theta} - \frac{m\omega^2 r}{\sin\theta}$, $\therefore N = \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r\,\text{cosec}\,\theta$

© Copyright 2011 itute.com

itute.com

Q5aiii $\cos\theta = \frac{h}{R}$, $\sin\theta = \frac{r}{R}$

To remain in contact, $N \geq 0$.

$\therefore \frac{1}{2}mg\sec\theta - \frac{1}{2}m\omega^2 r\,\text{cosec}\,\theta \geq 0$

$\therefore \frac{1}{2}mg\frac{R}{h} - \frac{1}{2}m\omega^2 r\frac{R}{r} \geq 0$, $\frac{g}{h} - \omega^2 \geq 0$, $\omega^2 \leq \frac{g}{h}$, $\therefore \omega \leq \sqrt{\frac{g}{h}}$.

Q5b $\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r}$

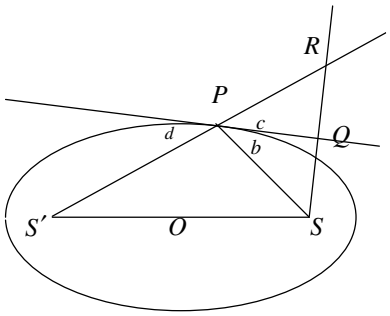
$= \frac{p(1+q)(1+r) + q(1+r)(1+p) - r(1+p)(1+q)}{(1+p)(1+q)(1+r)}$

$= \frac{(p+q-r) + 2pq + pqr}{(1+p)(1+q)(1+r)}$

Since p, q, r are positive real numbers and $p+q \geq r$, $\therefore$ the numerator and denominator are both positive,

$\therefore \frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0$

Q5ci



$\angle d = \angle b$ (reflective property), $\angle d = \angle c$ (vertically opposite angles), $\therefore \angle b = \angle c$, $\angle PQR = \angle PQS = 90°$, PQ is a common side, $\therefore \Delta PRQ$ and ΔPSQ are congruent and $SQ = RQ$

Q5cii ΔPRQ and ΔPSQ are congruent, $\therefore SP = PR$

Since $S'P + SP = 2a$, $\therefore S'P + PR = S'R = 2a$

Q5ciii O is the midpoint of SS' and Q is the midpoint of SR, $\therefore OQ = \frac{1}{2}S'R = a$, $\therefore Q$ is at a constant distance a from the origin O, i.e. it lies on the circle $x^2 + y^2 = a^2$.

Q6ai $m\frac{dv}{dt} = mg - kv^2$, when $v = v_T$, $\frac{dv}{dt} = 0$

$mg - kv_T^2 = 0$, $v_T = \sqrt{\frac{mg}{k}}$

Q6aii $\frac{dv}{dt} = \frac{mg - kv^2}{m}$, $\frac{dt}{dv} = \frac{1}{g}\frac{\frac{mg}{k}}{\frac{mg}{k} - v^2} = \frac{v_T^2}{g}\frac{1}{v_T^2 - v^2}$

$t = \frac{v_T^2}{g}\int \frac{1}{v_T^2 - v^2}\,dv = \frac{v_T}{2g}\int\left(\frac{1}{v_T - v} + \frac{1}{v_T + v}\right)dv$

$= \frac{v_T}{2g}\left(-\ln(v_T - v) + \ln(v_T + v) + c\right)$

$= \frac{v_T}{2g}\left(\ln\frac{v_T + v}{v_T - v} + c\right)$

At $t = 0$, $v = v_0$, $\frac{v_T}{2g}\left(\ln\frac{v_T + v_0}{v_T - v_0} + c\right) = 0$, $\therefore c = -\ln\frac{v_T + v_0}{v_T - v_0}$

$\therefore t = \frac{v_T}{2g}\left(\ln\frac{v_T + v}{v_T - v} - \ln\frac{v_T + v_0}{v_T - v_0}\right) = \frac{v_T}{2g}\left(\ln\frac{(v_T + v)(v_T - v_0)}{(v_T - v)(v_T + v_0)}\right)$

Q6aiii Jac: $\Delta t = \left[\frac{v_T}{2g}\left(\ln\frac{v_T + v}{v_T - v}\right)\right]_{\frac{1}{3}v_T}^{\frac{2}{3}v_T} = \frac{v_T}{2g}\ln\frac{5}{2}$

Gil: $\Delta t = \left[\frac{v_T}{2g}\left(\ln\left|\frac{v_T + v}{v_T - v}\right|\right)\right]_{3v_T}^{\frac{3}{2}v_T} = \frac{v_T}{2g}\ln\frac{5}{2}$

$\therefore$ the same time for Jac's speed to double and Gil's speed to halve.

Q6bi $y = (f(x))^3$, $\frac{dy}{dx} = 3(f(x))^2 f'(x)$

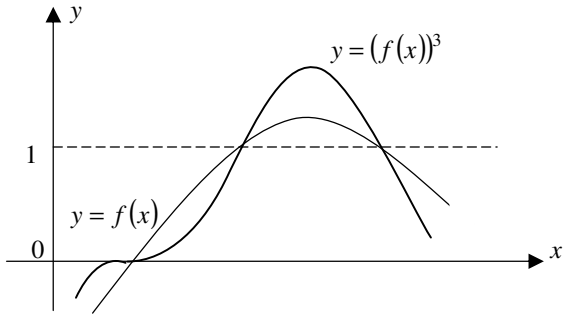
At $x = a$, $\frac{dy}{dx} = 3(f(a))^2 f'(a)$

If $f(a) = 0$ or $f'(a) = 0$, $\frac{dy}{dx} = 0$, $\therefore y = (f(x))^3$ has a stationary point at $x = a$.

Q6bii If $f(a) = 0$, $(f(a))^3 = 0$, $\therefore$ both $y = f(x)$ and $y = (f(x))^3$ have an x-intercept at $x = a$.

For $y = f(x)$, $(a,0)$ is not a stationary point. For $y = (f(x))^3$, it is a stationary point and a triple root, $\therefore (a,0)$ is a horizontal point of inflexion of $y = (f(x))^3$.

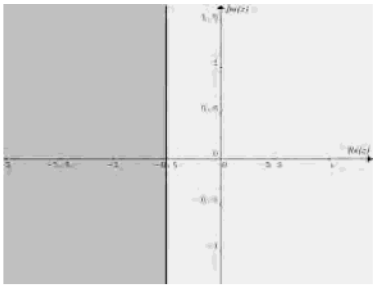
Q6biii



 © Copyright 2011 itute.com

Q6c $\left|1+\frac{1}{z}\right| \le 1,\ \left|\frac{z+1}{z}\right| \le 1,\ \frac{|z+1|}{|z|} \le 1,\ |z+1| \le |z|$

$|z+1| \le |z|$ defines a region where the distance of z from $z=-1$ is less than or equal to its distance from $z=0$. The region is to the left of the line (including the line) $\text{Re}(z) = -\frac{1}{2}$.



Q7a *Volume of a cylindrical shell* $= 2\pi rhdx = 2\pi(1-x)ydx$

$$= 2\pi(1-x)\frac{x}{1+x^2}dx = 2\pi\left(\frac{x}{1+x^2} - 1 + \frac{1}{1+x^2}\right)dx$$

$$\text{Volume of solid} = \int_0^1 2\pi\left(\frac{x}{1+x^2} - 1 + \frac{1}{1+x^2}\right)dx$$

$$= 2\pi\left[\frac{1}{2}\ln\left(1+x^2\right) - x + \tan^{-1}x\right]_0^1$$

$$= 2\pi\left(\frac{1}{2}\ln 2 - 1 + \frac{\pi}{4}\right)$$

Q7bi $I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}dx$. Let $u = 4-x$.

$$I = -\int_3^1 \frac{\cos^2\left(\frac{\pi}{8}(4-u)\right)}{u(4-u)}du = \int_1^3 \frac{\cos^2\left(\frac{\pi}{2} - \frac{\pi}{8}u\right)}{u(4-u)}du = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}du$$

Q7bii $2I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}dx + \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}du$

$$2I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right) + \sin^2\left(\frac{\pi}{8}x\right)}{x(4-x)}dx = \int_1^3 \frac{1}{x(4-x)}dx$$

$$= \frac{1}{4}\int_1^3 \left(\frac{1}{x} + \frac{1}{4-x}\right)dx = \frac{1}{4}\left[\ln\frac{x}{4-x}\right]_1^3 = \frac{1}{4}\left(\ln 3 - \ln\frac{1}{3}\right) = \frac{1}{2}\ln 3$$

$\therefore\ I = \frac{1}{4}\ln 3$

Q7ci $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ y = mx + c,\ \frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$$b^2x^2 + a^2(mx+c)^2 - a^2b^2 = 0$$

$$\left(b^2 + a^2m^2\right)x^2 + \left(2a^2mc\right)x + a^2\left(c^2 - b^2\right) = 0$$

Only one point of intersection, $\Delta = 0$,

$$\left(2a^2mc\right)^2 - 4\left(b^2 + a^2m^2\right)a^2\left(c^2 - b^2\right) = 0$$

$$4a^4m^2c^2 - 4a^2b^2c^2 - 4a^4m^2c^2 + 4a^2b^4 + 4a^4b^2m^2 = 0$$

$\therefore\ a^2m^2 + b^2 = c^2$

Q7cii y-intercept of ℓ is $C(0,c)$, x-intercept of ℓ is $R\left(-\frac{c}{m}, 0\right)$

$$RC = \sqrt{\frac{c^2}{m^2} + c^2},\ RS = \left|-\frac{c}{m} - ae\right| = \left|\frac{c}{m} + ae\right|$$

ΔCOR and ΔSQR are similar, $\therefore\ \frac{QS}{OC} = \frac{RS}{RC}$

$$\therefore\ \frac{QS}{|c|} = \frac{\left|\frac{c}{m} + ae\right|}{\sqrt{\frac{c^2\left(1+m^2\right)}{m^2}}} = \frac{\left|\frac{c}{m} + ae\right|}{\frac{|c|}{|m|}\sqrt{1+m^2}},\ \therefore\ QS = \frac{|mae + c|}{\sqrt{1+m^2}}$$

Q7ciii $QS = \frac{|mae+c|}{\sqrt{1+m^2}}$ and given $Q'S' = \frac{|mae-c|}{\sqrt{1+m^2}}$,

$$QS \times Q'S' = \frac{|mae+c|}{\sqrt{1+m^2}} \times \frac{|mae-c|}{\sqrt{1+m^2}} = \frac{\left|m^2a^2e^2 - c^2\right|}{1+m^2}$$

$$= \frac{\left|m^2a^2e^2 - a^2m^2 - b^2\right|}{1+m^2} = \frac{\left|-m^2a^2\left(1-e^2\right) - b^2\right|}{1+m^2} = \frac{\left|-m^2b^2 - b^2\right|}{1+m^2}$$

$$= \frac{b^2\left(1+m^2\right)}{1+m^2} = b^2$$

Q8a $I_m = \int_0^1 x^m\left(x^2-1\right)^5 dx = \int_0^1 x^{m-1}x\left(x^2-1\right)^5 dx$

Integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Let $u = x^{m-1}$, $\frac{du}{dx} = (m-1)x^{m-2}$

Let $\frac{dv}{dx} = x\left(x^2-1\right)^5$, $v = \frac{1}{12}\left(x^2-1\right)^6$

$$\therefore\ I_m = \left[\frac{1}{12}x^{m-1}\left(x^2-1\right)^6\right]_0^1 - \int_0^1 \frac{1}{12}\left(x^2-1\right)^6(m-1)x^{m-2}dx$$

$$I_m = -\frac{m-1}{12}\int_0^1 \left(x^2-1\right)^6 x^{m-2}dx = -\frac{m-1}{12}\int_0^1 \left(x^2-1\right)^5\left(x^m - x^{m-2}\right)dx$$

$$\therefore\ I_m = -\frac{m-1}{12}\left(\int_0^1 x^m\left(x^2-1\right)^5 dx - \int_0^1 x^{m-2}\left(x^2-1\right)^5 dx\right)$$

$$I_m = -\frac{m-1}{12}\left(I_m - I_{m-2}\right),\ -12I_m = mI_m - I_m - (m-1)I_{m-2}$$

$$-(m+11)I_m = -(m-1)I_{m-2},\ \therefore\ I_m = \frac{m-1}{m+11}I_{m-2}$$

© Copyright 2011 itute.com

Q8bi Pr*(each ball is selected exactly once)* $= \dfrac{7!}{7^7} = \dfrac{6!}{7^6}$

Q8bii Pr*(at least one ball is not selected)*
$= 1 -$ Pr*(none is not selected)*
$= 1 -$ Pr*(each ball is selected exactly once)*
$= 1 - \dfrac{6!}{7^6}$

Q8biii There are ${}^7C_6$ ways to select 6 different balls out of 7, and ${}^6C_1$ ways to select the remaining ball out of the 6 already selected balls. The 7 balls can be arranged in $\dfrac{7!}{2!}$ ways.

$\therefore$ Pr*(exactly one of the balls is not selected)*

$$= \frac{{}^7C_6 \times {}^6C_1 \times \frac{7!}{2!}}{7^7} = \frac{3 \times 6!}{7^5}$$

Q8ci β is a root of $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} ... + a_1 z + a_0$

$\therefore P(\beta) = \beta^n + a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} ... + a_1\beta + a_0 = 0$

$\therefore \beta^n = -a_{n-1}\beta^{n-1} - a_{n-2}\beta^{n-2} - ... - a_1\beta - a_0$

$$\left|\beta^n\right| = \left|a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + ... + a_1\beta + a_0\right|$$

$$\le \left|a_{n-1}\beta^{n-1}\right| + \left|a_{n-2}\beta^{n-2}\right| + ... + \left|a_1\beta\right| + \left|a_0\right|$$

$$= \left|a_{n-1}\right|\left|\beta^{n-1}\right| + \left|a_{n-2}\right|\left|\beta^{n-2}\right| + ... + \left|a_1\right|\left|\beta\right| + \left|a_0\right|$$

$$\le M\left|\beta^{n-1}\right| + M\left|\beta^{n-2}\right| + ... + M\left|\beta\right| + M$$

$$\therefore \left|\beta^n\right| \le M\left(\left|\beta^{n-1}\right| + \left|\beta^{n-2}\right| + ... + \left|\beta\right| + 1\right)$$

$$\therefore \left|\beta\right|^n \le M\left(\left|\beta\right|^{n-1} + \left|\beta\right|^{n-2} + ... + \left|\beta\right| + 1\right)$$

Q8cii M is the maximum value of $\left|a_{n-1}\right|$, $\left|a_{n-2}\right|$, …, $\left|a_0\right|$.

If $0 < \left|\beta\right| < 1$, $\left|\beta\right| < 1 + M$ since $M > 0$.

If $\left|\beta\right| > 1$, $\left|\beta\right|^n - 1 = \left(\left|\beta\right| - 1\right)\left(\left|\beta\right|^{n-1} + \left|\beta\right|^{n-2} + ... + \left|\beta\right| + 1\right)$

$$\therefore \left(\left|\beta\right|^{n-1} + \left|\beta\right|^{n-2} + ... + \left|\beta\right| + 1\right) = \frac{\left(\left|\beta\right|^n - 1\right)}{\left(\left|\beta\right| - 1\right)}$$

$$\therefore \left|\beta\right|^n \le M\left(\left|\beta\right|^{n-1} + \left|\beta\right|^{n-2} + ... + \left|\beta\right| + 1\right) = M\frac{\left(\left|\beta\right|^n - 1\right)}{\left(\left|\beta\right| - 1\right)} < M\frac{\left|\beta\right|^n}{\left|\beta\right| - 1}$$

$$\therefore 1 < \frac{M}{\left|\beta\right| - 1},\ \left|\beta\right| - 1 < M,\ \therefore \left|\beta\right| < 1 + M$$

$\therefore$ for any root β of $P(z)$, $\left|\beta\right| < 1 + M$

Q8d $S(x) = \sum_{k=0}^{n} c_k\left(x + \dfrac{1}{x}\right)^k$, $\dfrac{S(x)}{c_n} = \sum_{k=0}^{n} \dfrac{c_k}{c_n}\left(x + \dfrac{1}{x}\right)^k$ which is in the form of $P(z)$.

Let M be the maximum value of $\left|\dfrac{c_k}{c_n}\right|$.

Given $\left|c_k\right| \le \left|c_n\right|$, $\therefore \left|\dfrac{c_k}{c_n}\right| \le 1$, $\therefore M \le 1$.

For any root $\beta = \alpha + \dfrac{1}{\alpha}$ of $\dfrac{S(x)}{c_n} = 0$, $\left|\beta\right| < 1 + M$

$$\therefore \left|\alpha + \frac{1}{\alpha}\right| < 1 + M \le 2$$

$$\therefore \left(\frac{\alpha^2 + 1}{\alpha}\right)^2 < 4,\ \left(\alpha^2 + 1\right)^2 < 4\alpha^2,\ \left(\alpha^2 + 1\right)^2 - 4\alpha^2 < 0$$

$$\therefore \left(\alpha^2 - 1\right)^2 < 0$$

No real α can satisfy the last inequality, $\therefore \dfrac{S(x)}{c_n} = 0$ and hence $S(x) = 0$ has no real solutions.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.

© Copyright 2011 itute.com