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NSW BOS Mathematics Extension 2 Solutions 2008
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Q1a Let $u = 5+x^3$, $\frac{du}{dx} = 3x^2$, $\int \frac{x^2}{(5+x^3)^2}dx = \int \frac{1}{3u^2}\frac{du}{dx}du$

$= \int \frac{1}{3u^2}du = -\frac{1}{3u} + c = -\frac{1}{3(5+x^3)} + c$.

Q1b $\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{dx}{\sqrt{(2x)^2+1}} = \frac{1}{2}\ln\left(2x+\sqrt{4x^2+1}\right) + c$.

Q1c

$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx = \left[x\tan^{-1}x\right]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

Q1d Let $u = \sqrt{2x-1}$ for $2x-1>0$, $\frac{du}{dx} = \frac{1}{\sqrt{2x-1}}$,

$x = \frac{1}{2}(u^2+1)$.

When $x=1$, $u=1$; when $x=2$, $u=\sqrt{3}$.

$\int_1^2 \frac{dx}{x\sqrt{2x-1}} = \int_1^2 \frac{1}{\frac{1}{2}(u^2+1)}\frac{du}{dx}dx = \int_1^{\sqrt{3}} \frac{2}{1+u^2}du$

$= 2\left[\tan^{-1}u\right]_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$.

Q1e $\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}dx = \int_0^1 \left(\frac{4-2x}{2-2x+x^2} + \frac{-2x}{2-x^2}\right)dx$

$= \int_0^1 \left(\frac{-(-2+2x)+2}{2-2x+x^2} + \frac{-2x}{2-x^2}\right)dx$

$= \int_0^1 \left(\frac{-(-2+2x)}{2-2x+x^2} + \frac{2}{2-2x+x^2} + \frac{-2x}{2-x^2}\right)dx$

$= \int_0^1 \left(\frac{-(-2+2x)}{2-2x+x^2} + \frac{2}{1+(1-x)^2} + \frac{-2x}{2-x^2}\right)dx$

$= \left[-\ln(2-2x+x^2) - 2\tan^{-1}(1-x) + \ln(2-x^2)\right]_0^1$

$= \ln 2 + 2\tan^{-1}1 - \ln 2 = \frac{\pi}{2}$.

Q2a $(1+2i)(1-3i) = a+ib$, $1-3i+2i-6i^2 = a+ib$,
$7-i = a+ib$, $\therefore a=7$, $b=-1$.

Q2bi

$\frac{1+i\sqrt{3}}{1+i} = \frac{1+i\sqrt{3}}{1+i}\times\frac{1-i}{1-i} = \frac{1-i+i\sqrt{3}+\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} + i\frac{\sqrt{3}-1}{2}$.

Q2bii $1+i\sqrt{3} = 2cis\frac{\pi}{3}$, $1+i = \sqrt{2}cis\frac{\pi}{4}$.

$\frac{1+i\sqrt{3}}{1+i} = \frac{2cis\frac{\pi}{3}}{\sqrt{2}cis\frac{\pi}{4}} = \sqrt{2}cis\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sqrt{2}cis\frac{\pi}{12}$.

Q2biii $\frac{1+i\sqrt{3}}{1+i} = \sqrt{2}cis\frac{\pi}{12}$,

$\therefore \frac{1+\sqrt{3}}{2} + i\frac{\sqrt{3}-1}{2} = \sqrt{2}\cos\frac{\pi}{12} + i\sqrt{2}\sin\frac{\pi}{12}$,

$\therefore \sqrt{2}\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2}$. Hence $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

Q2biv $\left(\frac{1+i\sqrt{3}}{1+i}\right)^{12} = \left(\sqrt{2}cis\frac{\pi}{12}\right)^{12} = 2^6 cis\left(12\times\frac{\pi}{12}\right) = 2^6 cis\pi$

$= 2^6\cos\pi = -64$.

Q2c $z^2 + \bar{z}^2 = 8$, $x^2 + i2xy - y^2 + x^2 - i2xy - y^2 = 8$,

$\therefore \frac{x^2}{4} - \frac{y^2}{4} = 1$. Rectangular hyperbola centred at the origin with asymptotes $y = \pm x$ and intercepts $(-2,0)$ and $(2,0)$.

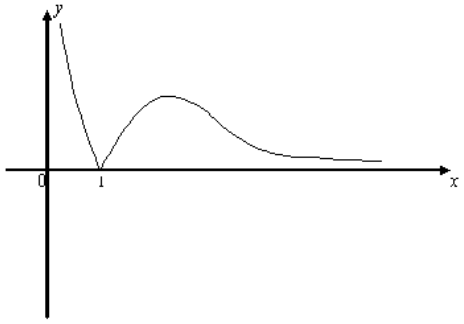
Q2di M is $\frac{1}{2}(\omega z + \bar{\omega}z)$

$= \frac{z}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right) = z\cos\frac{2\pi}{3} = -\frac{z}{2}$.

Q2dii Diagonals of a parallelogram bisect each other. M is the midpoint of diagonal PS.
Let α be the complex number represented by S.

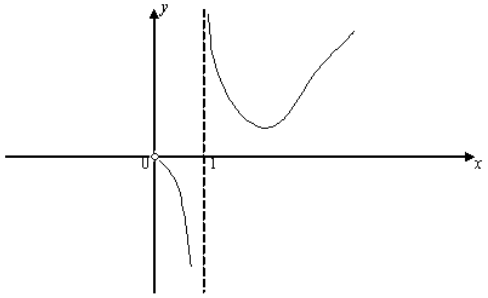
$\therefore \frac{1}{2}(z+\alpha) = -\frac{z}{2}$. Hence $\alpha = -2z$.

Q3ai $y = |g(x)|$

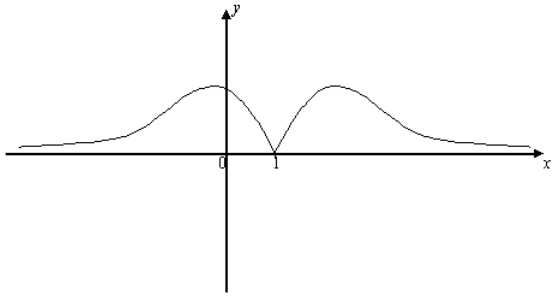


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Q3aii $y = \dfrac{1}{g(x)}$



Q3aiii $y = f(x) = \begin{cases} g(x), & x \geq 1 \\ g(2-x), & x < 1 \end{cases}$



Q3bi Since $z^2 \geq 0$ and $z^4 \geq 0$ for real z,
$\therefore p(z) = 1 + z^2 + z^4 > 0$ for real z. Hence $p(z)$ has no real zeros.

Q3bii α is a zero of $p(z)$, $\therefore 1 + \alpha^2 + \alpha^4 = 0$,
$\therefore (1-\alpha^2)(1+\alpha^2+\alpha^4) = 0$, $1 - \alpha^6 = 0$. Hence $\alpha^6 = 1$.

Q3biii $p(\alpha^2) = 1 + (\alpha^2)^2 + (\alpha^2)^4 = 1 + \alpha^4 + \alpha^8 = 1 + \alpha^4 + \alpha^6\alpha^2$
$= 1 + \alpha^2 + \alpha^4 = 0$, $\therefore \alpha^2$ is a zero of $p(z)$.

Q3ci For $n \geq 0$, $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n}\theta d\theta$.

For $n \geq 1$, $I_n + I_{n-1} = \displaystyle\int_0^{\frac{\pi}{4}} \left(\tan^{2n}\theta + \tan^{2n-2}\theta\right) d\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta\left(\tan^2\theta + 1\right) d\theta = \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta \sec^2\theta d\theta$.

Let $u = \tan\theta$, $\dfrac{du}{d\theta} = \sec^2\theta$, $u = 0$ when $\theta = 0$ and $u = 1$ when
$\theta = \dfrac{\pi}{4}$. $\therefore I_n + I_{n-1} = \displaystyle\int_0^1 u^{2n-2} du = \left[\frac{1}{2n-1}u^{2n-1}\right]_0^1 = \frac{1}{2n-1}$.

$\therefore I_n = \dfrac{1}{2n-1} - I_{n-1}$.

Q3cii $I_0 = \displaystyle\int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}$

$I_3 = \dfrac{1}{5} - I_2 = \dfrac{1}{5} - \left(\dfrac{1}{3} - I_1\right) = \dfrac{1}{5} - \left(\dfrac{1}{3} - (1 - I_0)\right) = \dfrac{1}{5} - \dfrac{1}{3} + 1 - \dfrac{\pi}{4}$

$= \dfrac{52 - 15\pi}{60}$.

Q3d Radius $r = \ell\sin\alpha$.
Speed of particle $P = v = \omega r = \omega\ell\sin\alpha$.
Vertically: $T\cos\alpha - mg = 0$ .....(1)

Horizontally: $T\sin\alpha = \dfrac{mv^2}{r} = \dfrac{m\omega^2\ell^2\sin^2\alpha}{\ell\sin\alpha}$ .....(2)

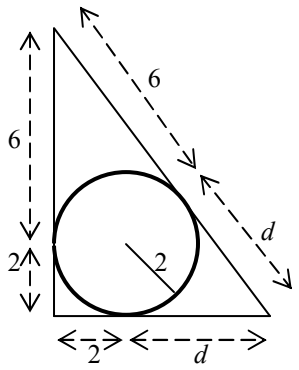
Solve (1) and (2) simultaneously, $\omega^2 = \dfrac{g}{\ell\cos\alpha}$.

Q4ai Area of $\Delta LOM = \dfrac{1}{2}kr$.

Q4aii Area of $\Delta KLM = A = \dfrac{1}{2}kr + \dfrac{1}{2}\ell r + \dfrac{1}{2}mr$

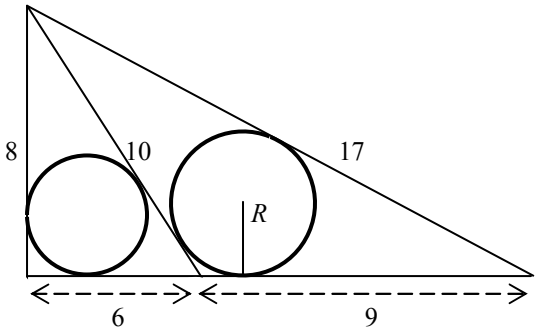
$= \dfrac{1}{2}(k + \ell + m)r = \dfrac{1}{2}P\,r.$

Q4aiii



$A = \dfrac{1}{2}P\,r$, $\dfrac{1}{2}(2+d)8 = \dfrac{1}{2}(8 + (2+d) + (6+d))2$, $\therefore d = 4$.
$\therefore 2 + d = 6$ units from the foot of the fence.

Q4aiv Use Pythagoras theorem to find the side lengths.



$A = \dfrac{1}{2}P\,r$, $\dfrac{1}{2}\times 9\times 8 = \dfrac{1}{2}(9 + 17 + 10)R$, $\therefore R = 2$ units.

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Q4bi $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

By implicit differentiation, $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$.

Gradient of tangent at $P(x_1,y_1)$ is $m=-\frac{b^2x_1}{a^2y_1}$.

Equation of tangent is $y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$.

Expand and rearrange to $\frac{x_1}{a^2}x+\frac{y_1}{b^2}y=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$.

Since $P(x_1,y_1)$ is a point on the ellipse, $\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ and the equation of tangent at $P(x_1,y_1)$ can be expressed as $\frac{x_1}{a^2}x+\frac{y_1}{b^2}y=1$.

Q4bii Similarly, equation of tangent at $Q(x_2,y_2)$ can be expressed as $\frac{x_2}{a^2}x+\frac{y_2}{b^2}y=1$.

Let $T(u,v)$ be the intersection of the two tangents.

$\therefore \frac{x_1}{a^2}u+\frac{y_1}{b^2}v=1$ and $\frac{x_2}{a^2}u+\frac{y_2}{b^2}v=1$.

$\therefore \frac{x_1}{a^2}u+\frac{y_1}{b^2}v-\frac{x_2}{a^2}u-\frac{y_2}{b^2}v=0$.

Rearrange to $\frac{(x_1-x_2)}{a^2}u+\frac{(y_1-y_2)}{b^2}v=0$.

$\therefore T(u,v)$ lies on the line $\frac{(x_1-x_2)}{a^2}x+\frac{(y_1-y_2)}{b^2}y=0$.

Q4biii M is the midpoint of PQ, $\therefore M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

$\frac{(x_1-x_2)}{a^2}x+\frac{(y_1-y_2)}{b^2}y=0$ is a line through the origin O.

M satisfies the line because

$$\frac{(x_1-x_2)}{a^2}\times\frac{(x_1+x_2)}{2}+\frac{(y_1-y_2)}{b^2}\times\frac{(y_1+y_2)}{2}$$

$$=\frac{1}{2}\left(\frac{x_1^2-x_2^2}{a^2}+\frac{y_1^2-y_2^2}{b^2}\right)=\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}\right)\right)$$

$$=\frac{1}{2}(1-1)=0.$$

$\therefore M$ lies on the line OT, i.e. O, M and T are collinear.

Q5ai $P=\frac{21000}{7+3e^{-\frac{t}{3}}}$,

$$\frac{dP}{dt}=-\frac{21000}{\left(7+3e^{-\frac{t}{3}}\right)^2}\times\left(-e^{-\frac{t}{3}}\right)=\frac{21000e^{-\frac{t}{3}}}{\left(7+3e^{-\frac{t}{3}}\right)^2}.$$

$$\frac{1}{3}\left(1-\frac{P}{3000}\right)P=\frac{1}{3}\left(1-\frac{7}{7+3e^{-\frac{t}{3}}}\right)\frac{21000}{7+3e^{-\frac{t}{3}}}=\frac{1}{3}\left(\frac{3e^{-\frac{t}{3}}}{7+3e^{-\frac{t}{3}}}\right)\frac{21000}{7+3e^{-\frac{t}{3}}}$$

$$=\frac{21000}{\left(7+3e^{-\frac{t}{3}}\right)^2}.$$

$\therefore P$ satisfies $\frac{dP}{dt}=\frac{1}{3}\left(1-\frac{P}{3000}\right)P$.

Q5aii Today: $t=0$, $P=\frac{21000}{7+3e^{-\frac{t}{3}}}=\frac{21000}{7+3e^0}=2100$.

Q5aiii Eventually: $t\to\infty$, $e^{-\frac{t}{3}}\to 0$, $P\to\frac{21000}{7}=3000$.

Q5aiv Today: $t=0$, $P=2100$ and $\frac{dP}{dt}=\frac{1}{3}\left(1-\frac{P}{3000}\right)P$

$$=\frac{1}{3}\left(1-\frac{2100}{3000}\right)2100=210.$$

Annual % rate of growth today $=\frac{210}{2100}\times 100\%=10\%$.

Q5bi $p(x)=x^{n+1}-(n+1)x+n$, $p(1)=0$, $\therefore p(x)$ has a zero at $x=1$.

$p'(x)=(n+1)x^n-(n+1)$, $p'(1)=0$, $\therefore$ the zero at $x=1$ is a double zero.

Q5bii $p'(x)=(n+1)x^n-(n+1)$, $p''(x)=n(n+1)x^{n-1}\geq 0$ for $x\geq 0$. $\therefore p(x)$ is concave upward for $x\geq 0$. Also $p(x)$ touches the x-axis at $x=1$ (double zero), $\therefore p(x)\geq 0$ for $x\geq 0$.

Q5biii $p(x)=x^4-4x+3$ when $n=3$.

$p(x)$ has a double zero at $x=1$,

$\therefore p(x)=x^4-4x+3=(x-1)^2q(x)=(x^2-2x+1)(x^2+2x+3)$.

$\therefore p(x)=(x-1)^2(x^2+2x+3)$.

Q5ci $(x-a)^2=b^2-h^2$, $x=a\pm\sqrt{b^2-h^2}$.

$x_1=a-\sqrt{b^2-h^2}$ and $x_2=a+\sqrt{b^2-h^2}$.

Q5cii Area of cross-section $=\pi x_2^2-\pi x_1^2$

$$=\pi\left[\left(a+\sqrt{b^2-h^2}\right)^2-\left(a-\sqrt{b^2-h^2}\right)^2\right]=4\pi a\sqrt{b^2-h^2}.$$

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Q5ciii Volume of torus:

$$V = 2\times\int_0^b 4\pi a\sqrt{b^2-h^2}\,dh = 8\pi a\int_0^b \sqrt{b^2-h^2}\,dh$$

$$= 8\pi a\times\frac{1}{4}\pi b^2 = 2\pi^2 ab^2 .$$

Note: $\int_0^r \sqrt{r^2-x^2}\,dx$ gives $\frac{1}{4}$ of the area of a circle of radius r.

Q6a Given $p(z) = z^3 + az^2 + bz + c$ has zeros 1, $-\omega$ and $-\overline{\omega}$, where $\omega^3 = 1$ and $\text{Im}(\omega) > 0$.

The roots of $z^3 = 1$ lie on the unit circle, and are separated by $\frac{2\pi}{3}$. $\therefore \omega = cis\frac{2\pi}{3}$ and $\overline{\omega} = cis\left(-\frac{2\pi}{3}\right)$.

Hence $\omega + \overline{\omega} = 2\cos\frac{2\pi}{3} = -1$ and $\omega\overline{\omega} = cis0 = 1$.

$$\therefore z^3 + az^2 + bz + c = (z-1)(z+\omega)(z+\overline{\omega})$$

$$= (z-1)\left(z^2 + (\omega+\overline{\omega})z + \omega\overline{\omega}\right) = (z-1)(z^2 - z + 1)$$

$$= z^3 - 2z^2 + 2z - 1 .$$

Q6bi $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

By implicit differentiation, $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{b^2x}{a^2y}$.

Gradient of tangent at $P(a\sec\theta, b\tan\theta)$:

$$m = \frac{dy}{dx} = \frac{b^2x}{a^2y} = \frac{b\sec\theta}{a\tan\theta}.$$

Equation of tangent: $y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$.

Expand and simplify: $bx\sec\theta - ay\tan\theta = ab\sec^2\theta - ab\tan^2\theta$

$$= ab(\sec^2\theta - \tan^2\theta) = ab .$$

$$\therefore bx\sec\theta - ay\tan\theta - ab = 0 .$$

Q6bii Shortest distance between line $ax + by + c = 0$ and point (u,v) is $\frac{|au+bv+c|}{\sqrt{a^2+b^2}}$.

$$\therefore SR = \frac{|abe\sec\theta - ab|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} = \frac{ab|e\sec\theta - 1|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} .$$

$P(a\sec\theta, b\tan\theta)$ is on the right branch of the hyperbola ($e > 1$),

$\therefore a\sec\theta \ge a$, $\sec\theta \ge 1$ and $SR = \frac{ab(e\sec\theta - 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$.

Q6biii $S'R' = \frac{|-abe\sec\theta - ab|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}} = \frac{ab|-e\sec\theta - 1|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$

$$= \frac{ab(e\sec\theta + 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}} .$$

$$SR \times S'R' = \frac{ab(e\sec\theta - 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}} \times \frac{ab(e\sec\theta + 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$$

$$= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2\tan^2\theta + b^2\sec^2\theta} = \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2\tan^2\theta + a^2(e^2-1)\sec^2\theta}$$

$$= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2(e^2\sec^2\theta + \tan^2\theta - \sec^2\theta)}$$

$$= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2(e^2\sec^2\theta - 1)} = b^2 .$$

Q6ci

$$\frac{r}{r-1}\left[\frac{1}{{}^{n-1}C_{r-1}} - \frac{1}{{}^{n}C_{r-1}}\right] = \frac{r}{r-1}\left[\frac{(n-r)!(r-1)!}{(n-1)!} - \frac{(n-r+1)!(r-1)!}{n!}\right]$$

$$= \frac{r(r-1)!}{r-1}\left[\frac{n(n-r)!}{n(n-1)!} - \frac{(n-r+1)!}{n!}\right] = \frac{r!}{r-1}\left[\frac{n(n-r)! - (n-r+1)!}{n!}\right]$$

$$= \frac{r!}{r-1}\left[\frac{n(n-r)! - (n-r+1)(n-r)!}{n!}\right] = \frac{(n-r)!r!}{r-1}\left[\frac{n-(n-r+1)}{n!}\right]$$

$$= \frac{(n-r)!r!}{r-1}\left[\frac{r-1}{n!}\right] = \frac{(n-r)!r!}{n!} = \frac{1}{{}^{n}C_{r}}$$

Q6cii From Q6ci, $\frac{1}{{}^{n}C_{r}} = \frac{r}{r-1}\left[\frac{1}{{}^{n-1}C_{r-1}} - \frac{1}{{}^{n}C_{r-1}}\right]$.

$$\therefore \frac{1}{{}^{r}C_{r}} + \frac{1}{{}^{r+1}C_{r}} + \ldots + \frac{1}{{}^{m}C_{r}}$$

$$= \frac{r}{r-1}\left[\frac{1}{{}^{r-1}C_{r-1}} - \frac{1}{{}^{r}C_{r-1}} + \frac{1}{{}^{r}C_{r-1}} - \frac{1}{{}^{r+1}C_{r-1}} + \ldots + \frac{1}{{}^{m-1}C_{r-1}} - \frac{1}{{}^{m}C_{r-1}}\right]$$

$$= \frac{r}{r-1}\left[1 - \frac{1}{{}^{m}C_{r-1}}\right].$$

Q6ciii As $m \to \infty$, $\frac{1}{{}^{m}C_{r-1}} \to 0$, $\sum_{n=r}^{m}\frac{1}{{}^{n}C_{r}} \to \frac{r}{r-1}$.

Q7ai $p_s = 3\times\frac{{}^{n}C_3\times{}^{n}C_0\times{}^{n}C_0}{{}^{3n}C_3} = 3\times\frac{{}^{n}C_3}{{}^{3n}C_3}$.

Q7aii $p_d = \frac{{}^{n}C_1\times{}^{n}C_1\times{}^{n}C_1}{{}^{3n}C_3} = \frac{({}^{n}C_1)^3}{{}^{3n}C_3}$.

Q7aiii $p_m = 3!\times\frac{{}^{n}C_2\times{}^{n}C_1\times{}^{n}C_0}{{}^{3n}C_3} = 6\times\frac{{}^{n}C_2\times{}^{n}C_1}{{}^{3n}C_3}$.

Q7aiv $p_s : p_d : p_m = 3\times{}^{n}C_3 : ({}^{n}C_1)^3 : 6\times{}^{n}C_2\times{}^{n}C_1$

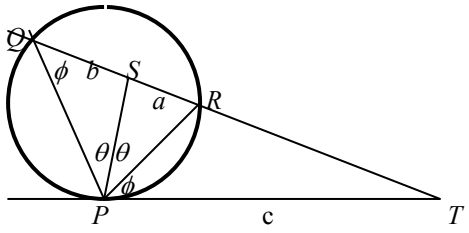
$$= 3\times\frac{n(n-1)(n-2)}{3!} : n^3 : 6\times\frac{n(n-1)}{2!}\times n$$

$$\approx \frac{n^3}{2} : n^3 : 3n^3 = 1:2:6 \text{ for large } n.$$

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Q7bi



Let $\angle PQR = \phi$.

$\angle TSP = \theta + \phi$ (Exterior angle equals sum of opposite interior angles).

$\angle RPT = \phi$ (Equal angles in alternate segments).

$\angle TPS = \theta + \phi$ (See diagram).

$\therefore \angle TSP = \angle TPS$.

Q7bii $TS = TP = c$ (ΔTSP is isosceles), $\therefore RT = c - a$ and $QT = b + c$.

$\angle TRP = 2\theta + \phi$ (Exterior angle equals sum of opposite interior angles).

$\angle TPQ = 2\theta + \phi$ (See diagram).

$\therefore \angle TRP = \angle TPQ$. $\therefore \Delta TRP$ and ΔTPQ are similar.

Hence $\dfrac{c}{c-a} = \dfrac{b+c}{c}$, $c^2 = (b+c)(c-a)$.

Expand and simplify, $bc = ca + ab$.

Both sides divided by abc, $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$.

Q7ci $v = b - (b - v_0)e^{-\alpha t}$, v_0, b and $\alpha \in R^+$, and $v_0 < b$.

$\dfrac{dv}{dt} = \alpha(b - v_0)e^{-\alpha t} = \alpha(b - v)$.

Q7cii b represents the velocity of the current.

Q7ciii $v = \dfrac{dx}{dt} = b - (b - v_0)e^{-\alpha t}$,

$x = \int \left[b - (b - v_0)e^{-\alpha t}\right] dt = bt + \dfrac{(b - v_0)e^{-\alpha t}}{\alpha} + c$.

At the start of the drift, $t = 0$, $x = 0$, $\therefore c = -\dfrac{b - v_0}{\alpha}$, and

$x = bt + \dfrac{b - v_0}{\alpha}\left(e^{-\alpha t} - 1\right)$.

Given $v = b - (b - v_0)e^{-\alpha t}$, $\therefore e^{-\alpha t} = \dfrac{b - v}{b - v_0}$, $t = \dfrac{1}{\alpha}\log_e\left(\dfrac{b - v_0}{b - v}\right)$

and $e^{-\alpha t} - 1 = \dfrac{v_0 - v}{b - v_0}$.

$\therefore x = \dfrac{b}{\alpha}\log_e\left(\dfrac{b - v_0}{b - v}\right) + \dfrac{v_0 - v}{\alpha}$.

Q7civ Given $v_0 = \dfrac{b}{10} = 0.1b$, when $v = \dfrac{b}{2} = 0.5b$,

$x = \dfrac{b}{\alpha}\log_e\left(\dfrac{b - 0.1b}{b - 0.5b}\right) + \dfrac{0.1b - 0.5b}{\alpha} = \dfrac{b}{\alpha}(\log_e 1.8 - 0.4) \approx 0.19\dfrac{b}{\alpha}$.

Q8a The statement: $\cos\theta + \cos 3\theta + \ldots + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$.

When $n = 1$, $RHS = \dfrac{\sin 2\theta}{2\sin\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta = LHS$. $\therefore$ the statement is true when $n = 1$.

Assume it is true when $n = k$,

i.e. $\cos\theta + \cos 3\theta + \ldots + \cos(2k-1)\theta = \dfrac{\sin 2k\theta}{2\sin\theta}$.

When $n = k + 1$,

$LHS = \cos\theta + \cos 3\theta + \ldots + \cos(2k-1)\theta + \cos(2(k+1)-1)$

$= \cos\theta + \cos 3\theta + \ldots + \cos(2k-1)\theta + \cos(2k+1)\theta$,

$RHS = \dfrac{\sin 2(k+1)\theta}{2\sin\theta} = \dfrac{\sin(2k\theta + 2\theta)}{2\sin\theta}$

$= \dfrac{\sin 2k\theta\cos 2\theta + \cos 2k\theta\sin 2\theta}{2\sin\theta}$

$= \dfrac{\sin 2k\theta(1 - 2\sin^2\theta) + 2\cos 2k\theta\sin\theta\cos\theta}{2\sin\theta}$

$= \dfrac{\sin 2k\theta}{2\sin\theta} + \dfrac{2\cos 2k\theta\sin\theta\cos\theta - 2\sin 2k\theta\sin^2\theta}{2\sin\theta}$

$= \dfrac{\sin 2k\theta}{2\sin\theta} + \cos 2k\theta\cos\theta - \sin 2k\theta\sin\theta$

$= \dfrac{\sin 2k\theta}{2\sin\theta} + \cos(2k\theta + \theta)$

$= \cos\theta + \cos 3\theta + \ldots + \cos(2k-1)\theta + \cos(2k+1)\theta = LHS$.

$\therefore$ the statement is true for $n = k + 1$.

Hence it is true for $n \geq 1$.

Q8bi

$A = \sum_{k=1}^{n} A_k = 2\pi R^2 \sin\delta\left[\cos\dfrac{\delta}{2} + \cos\dfrac{3\delta}{2} + \ldots + \cos\dfrac{(2n-1)\delta}{2}\right]$

$= 2\pi R^2 \sin\delta\left[\dfrac{\sin n\delta}{2\sin\frac{\delta}{2}}\right] = 2\pi R^2\left(2\sin\dfrac{\delta}{2}\cos\dfrac{\delta}{2}\right)\left[\dfrac{\sin n\delta}{2\sin\frac{\delta}{2}}\right]$

$= 2\pi R^2\cos\dfrac{\delta}{2}\sin n\delta$.

Since $n\delta = \dfrac{\pi}{2}$, $\therefore \dfrac{\delta}{2} = \dfrac{\pi}{4n}$, and hence $A = 2\pi R^2\cos\dfrac{\pi}{4n}$.

Q8bii As $n \to \infty$, $\cos\dfrac{\pi}{4n} \to 1$, $A \to 2\pi R^2$ ($\frac{1}{2}$ of the surface area of a sphere).

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Q8ci $f(t)=\sin(a+nt)\sin b-\sin a\sin(b-nt)$, where $a>0$, $b>0$, $a+b<\pi$ and $n\neq 0$.

$f(0)=\sin(a+0)\sin b-\sin a\sin(b-0)=0$.

$f'(t)=n\cos(a+nt)\sin b+n\sin a\cos(b-nt)$

$f''(t)=-n^2\sin(a+nt)\sin b-n^2\sin a\sin(b-nt)$

$=-n^2(\sin(a+nt)\sin b+\sin a\sin(b-nt))$

$\therefore f''(t)=-n^2 f(t)$.

Q8cii

$f(t)=(\sin a\cos nt+\cos a\sin nt)\sin b-\sin a(\sin b\cos nt-\cos b\sin nt)$

$=\sin a\cos nt\sin b+\cos a\sin nt\sin b-\sin a\sin b\cos nt+\sin a\cos b\sin nt$

$=\cos a\sin nt\sin b+\sin a\cos b\sin nt$

$=(\sin a\cos b+\cos a\sin b)\sin nt$

$=\sin(a+b)\sin nt$.

Q8ciii $\dfrac{\sin(a+nt)}{\sin(b-nt)}=\dfrac{\sin a}{\sin b}$, $\sin(b-nt)\neq 0$ and $\sin b\neq 0$.

$\therefore \sin(a+nt)\sin b-\sin a\sin(b-nt)=0$,

i.e. $\sin(a+b)\sin nt=0$.

Since $0<a+b<\pi$, $\sin(a+b)\neq 0$,

$\therefore \sin nt=0$ and $\sin(b-nt)\neq 0$.

Hence $nt=k\pi$, $\therefore t=\dfrac{k\pi}{n}$ where $k\in J$.

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