2002 Higher School Certificate Solutions

MATHEMATICS EXTENSION 2

QUESTION 1

(a) Let $u = \sec x$

$du = \sec x \tan x \, dx$

$$\int \sec^3 x \tan x \, dx = \int \sec^2 x (\sec x \tan x) \, dx$$
$$= \int u^2 \, du$$
$$= \frac{1}{3}u^3 + c$$
$$= \frac{1}{3}\sec^3 x + c.$$

(b) $$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{x^2+2x+1+1}$$
$$= \int \frac{dx}{(x+1)^2+1}$$
$$= \tan^{-1}(x+1) + c.$$

(c) Let $\dfrac{x}{(x+3)(x-1)} = \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$x = A(x-1) + B(x+3).$

Let $x = 1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}.$

Let $x = -3 \Rightarrow -3 = -4A \Rightarrow A = \frac{3}{4}.$

$$\therefore \int \frac{x \, dx}{(x+3)(x-1)}$$
$$= \frac{1}{4}\int \frac{3}{x+3} + \frac{1}{x-1} \, dx$$
$$= \frac{3}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + c.$$

(d) $$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$
$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \frac{d}{dx}(\cos x) \, dx$$
$$= \left(e^{\frac{\pi}{2}} \cos\frac{\pi}{2} - e^0 \cos 0\right) + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$
$$= -1 + \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \frac{d}{dx}(\sin x) \, dx$$
$$= -1 + e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = e^{\frac{\pi}{2}} - 1$$
$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right).$$

(e) $t = \tan\frac{\theta}{2}$

$\therefore \theta = 2\tan^{-1} t$

$$d\theta = \frac{2 \, dt}{1+t^2}.$$

When $\theta = \frac{\pi}{2}$, $t = 1$.

When $\theta = 0$, $t = 0$.

Now $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt$$
$$= \int_0^1 \frac{2}{2+2t^2+1-t^2} \, dt$$
$$= \int_0^1 \frac{2}{3+t^2} \, dt$$
$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$
$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0\right)$$
$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6}$$
$$= \frac{\pi}{3\sqrt{3}}.$$

QUESTION 2

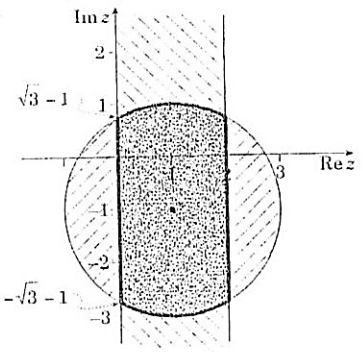
(a) $z = 1+2i, \quad w = 1+i$

(i) $$z\bar{w} = (1+2i)(1-i)$$
$$= 1 - 2i^2 + 2i - i$$
$$= 3 + i.$$

(ii) $$\frac{1}{w} = \frac{\bar{w}}{w\bar{w}}$$
$$= \frac{1-i}{1^2+1^2}$$
$$= \frac{1}{2} - \frac{1}{2}i.$$

(b) $0 \le \operatorname{Re} z \le 2$ – vertical strip

$|z-1+i| \le 2$ – circle centre $1-i$, radius 2



(c) $P(z) = z^3 + rz^2 + sz + 20.$ r, s are real.

(i) Since all of the coefficients of $P(z)$ are real, any complex (that is, not real) roots occur in conjugate pairs.

So if $2+i$ is a root, so is its conjugate $2-i$.

(ii) *METHOD 1*

Let the other root be α.

$$(2+i)(2-i)\alpha = -\frac{20}{1} \quad \text{(Product of roots)}$$
$$(4-i^2)\alpha = -20$$
$$5\alpha = -20$$
$$\alpha = -4.$$

$$\therefore P(z) = [z-(-4)][z-(2+i)][z-(2-i)]$$
$$= (z+4)(z^2-4z+5).$$

METHOD 2

Combining the known factors:

$$(z-2-i)(z-2+i) = (z-2)^2 + 1$$
$$= z^2 - 4z + 5$$

$$\therefore P(z) = (z^2-4z+5)(z+4),$$

where the factor $z+4$ has been found by equating the coefficient of z^3 and the constant.

(d) Prove $(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$.

Let $n = 1$: LHS $= (\cos\theta - i\sin\theta)^1$
$= \cos\theta - i\sin\theta.$

RHS $= \cos(1\times\theta) - i\sin(1\times\theta)$
$= \cos\theta - i\sin\theta$
$=$ LHS.

So the result is true when $n = 1$.

Assume the result is true for $n = k$, that is, assume $(\cos\theta - i\sin\theta)^k = \cos(k\theta) - i\sin(k\theta)$.

We need to prove that the result is true for $n = k+1$, that is, we need to prove

$(\cos\theta - i\sin\theta)^{k+1} = \cos(k+1)\theta - i\sin(k+1)\theta.$

Now $(\cos\theta - i\sin\theta)^{k+1}$
$= (\cos\theta - i\sin\theta)^k(\cos\theta - i\sin\theta)$
$= (\cos k\theta - i\sin k\theta)(\cos\theta - i\sin\theta)$, by assumption,
$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) - i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$
$= \cos(k\theta+\theta) - i\sin(k\theta+\theta)$
$= \cos(k+1)\theta - i\sin(k+1)\theta$, as required.

We have shown that if the result is true for $n = k$, then it is true for $n = k+1$. Since it is true for $n = 1$, it is true for $n = 1+1 = 2$, and hence for $n = 2+1 = 3$, and so on. By the principle of mathematical induction, the result is true for all positive integers n.

(e) (i) $1 - z = 1 - 2\cos\theta - 2i\sin\theta$

$\therefore \overline{1-z} = (1-2\cos\theta) + 2i\sin\theta$

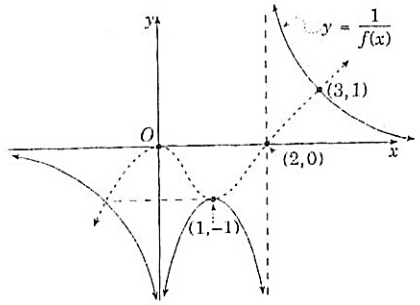
(ii) $$\frac{1}{1-z} = \frac{1}{1-z} \times \frac{\overline{1-z}}{\overline{1-z}}$$
$$= \frac{\overline{1-z}}{|1-z|^2}$$
$$= \frac{(1-2\cos\theta)+2i\sin\theta}{(1-2\cos\theta)^2 + 4\sin^2\theta}$$
$$= \frac{(1-2\cos\theta)+2i\sin\theta}{1-4\cos\theta+4\cos^2\theta+4\sin^2\theta}$$
$$= \frac{(1-2\cos\theta)+2i\sin\theta}{5-4\cos\theta}.$$

$$\therefore \operatorname{Re}\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{5-4\cos\theta}.$$

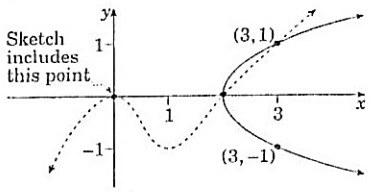
(iii) $$\operatorname{Im}\left(\frac{1}{1-z}\right) = \frac{2\sin\theta}{5-4\cos\theta}.$$

QUESTION 3

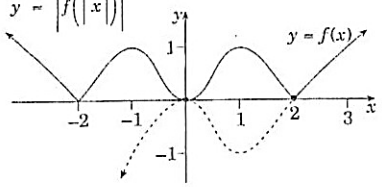
(a) (i) $y = \dfrac{1}{f(x)}$



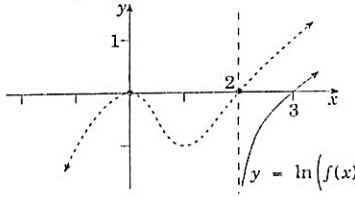
(ii) $y^2 = f(x)$



(iii) $y = \left|f\left(|x|\right)\right|$



(iv) $y = \ln\left(f(x)\right)$



(b) (i) To find the gradient at P:

$xy = c^2$

$x\dfrac{dy}{dx} + y = 0$, differentiating implicitly,

$\dfrac{dy}{dx} = -\dfrac{y}{x}$

$= -\dfrac{\frac{c}{p}}{cp}$

$= -\dfrac{1}{p^2}$.

OR $y = \dfrac{c^2}{x}$

$\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

$= -\dfrac{c^2}{(cp)^2}$

$= -\dfrac{1}{p^2}$.

OR $x = cp$, $y = \dfrac{c}{p}$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dp}}{\frac{dx}{dp}}$, differentiating parametrically,

$= -\dfrac{\frac{c}{p^2}}{c}$

$= -\dfrac{1}{p^2}$.

The tangent at P is

$y - \dfrac{c}{p} = -\dfrac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$. —①

(ii) *METHOD 1*

The tangent at Q is $x + q^2y = 2cq$. —②

Solving ① and ② simultaneously,

① − ②: $\left(p^2 - q^2\right)y = 2c(p - q)$

$y = \dfrac{2c}{p + q}$. —③

∴ Substitute ③ into ①:

$x + \dfrac{2cp^2}{p + q} = 2cp$

$x = \dfrac{2cp}{p + q}(p + q - p)$

$= \dfrac{2cpq}{p + q}$

∴ T is $\left(\dfrac{2cpq}{p + q}, \dfrac{2c}{p + q}\right)$.

METHOD 2

Since T is given, it is enough to show that T lies on PT. Since the form of T is symmetric with respect to p and q, it follows that if T lies on PT it also must lie on QT, and hence on the intersection of PT and QT. So, substituting

$T = \left(\dfrac{2cpq}{p + q}, \dfrac{2c}{p + q}\right)$ into ①,

LHS $= \dfrac{2cpq}{p + q} + p^2\dfrac{2c}{p + q}$

$= \dfrac{2cp(p + q)}{p + q}$

$= 2cp$

$=$ RHS, as required.

(iii) Substitute $(cq, 0)$ into ①:

$cq = 2cp$

$q = 2p$. —④

At T, $x = \dfrac{2cpq}{p + q}$

$= \dfrac{4cp^2}{3p}$, using ④

$= \dfrac{4cp}{3}$.

$y = \dfrac{2c}{p + q} = \dfrac{2c}{3p}$

∴ $xy = \dfrac{4cp}{3} \times \dfrac{2c}{3p}$,

that is, $xy = \dfrac{8c^2}{9}$.

Since $\dfrac{8c^2}{9}$ is a constant, this is the equation of a rectangular hyperbola with eccentricity $e = \sqrt{2}$.

QUESTION 4

(a) (i) At P, $3 - x^2 = x + x^2$

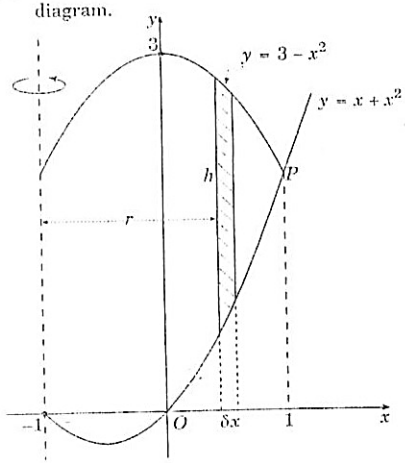
$2x^2 + x - 3 = 0$

$(2x + 3)(x - 1) = 0$

∴ $x = 1, -\dfrac{3}{2}$.

Since P is in the first quadrant, $x = 1$.

(ii) Define r, h and δx as shown in the diagram.



$h = \left(3 - x^2\right) - \left(x + x^2\right)$

$= 3 - x - 2x^2$.

$\delta V = 2\pi rh\ \delta x$

$= 2\pi(x + 1)\left(3 - x - 2x^2\right)\delta x$.

∴ $V = \displaystyle\int_{-1}^{1} 2\pi(x + 1)\left(3 - x - 2x^2\right)dx$

$= 2\pi\displaystyle\int_{-1}^{1}\left(3 + 2x - 3x^2 - 2x^3\right)dx$.

(iii) $V = 2\pi\displaystyle\int_{-1}^{1}\left(3 - 3x^2\right)dx + 2\pi\int_{-1}^{1}\left(2x - 2x^3\right)dx$

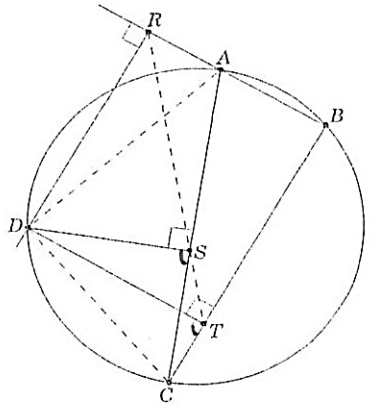
$= 4\pi\displaystyle\int_{0}^{1}\left(3 - 3x^2\right)dx + 0$, since $2x - 2x^3$ is an odd function.

$= 4\pi\left[3x - x^3\right]_0^1$

$= 4\pi(3 - 1)$

$= 8\pi$.

(b)



(i) In $DSAR$,

$\angle DSA + \angle ARD = 180°$ (since both right $\angle$s).

∴ Points D, S, A, R are concyclic (opp. $\angle$s are supplementary).

∴ $\angle DSA = \angle DAR$ ($\angle$s at circumference of circle $DSAR$ subtended by same arc DR).

(ii) In $DSTC$,

$\angle DSC = \angle DTC$ (since both right $\angle$s).

∴ Points D, S, T, C are concyclic (converse of $\angle$s at circumference subtended by the same arc are equal).

∴ $\angle DST = \pi - \angle DCT$ (opp. $\angle$s of cyclic quadrilateral $DSTC$ are supplementary).

(iii) $\angle DSR = \angle DAR$, (from (i)),

but $\angle DAR = \angle DCT$ (ext. $\angle$ of cyclic quad. $DABC$ equals int. opp. $\angle$).

∴ $\angle DSR = \angle DCT$

$\angle DST = \pi - \angle DCT$, (from (ii)),

∴ $\angle DSR + \angle DST = \pi$

∴ R, S, T are collinear (adjacent $\angle$s are supplementary).

(c) (i) To get a number exceeding 400, the first digit must be 4 or greater.

$\mathrm{P}(x > 400) = \dfrac{6}{9} = \dfrac{2}{3}$.

(ii) For any set of 3 digits, there is only one arrangement in which the digits are in descending order.

No. of possible arrangements $= 3!$

∴ P(digits are in descending order) $= \dfrac{1}{3!}$

$= \dfrac{1}{6}$.

QUESTION 5

(a) Let $P(x) = 4x^3 - 27x + k$ and let α be the double root. This means that $P(\alpha) = 0$ and $P'(\alpha) = 0$.

From $P'(\alpha) = 0$, we have $12\alpha^2 - 27 = 0$

$$\alpha^2 = \frac{9}{4}$$

$$\alpha = \pm\frac{3}{2}.$$

From $P(\alpha) = 0$, we have

$$\begin{aligned} k &= 27\alpha - 4\alpha^3 \\ &= \left(27 - 4\alpha^2\right) \times \alpha \\ &= \left(27 - 9\right) \times \left(\pm\frac{3}{2}\right) \\ &= \pm 27. \end{aligned}$$

(b) (i) *METHOD 1* (Substitution)

Let $P(x) = x^3 - 5x^2 + 5$ and let $y = x - 1$.

Then if $P(x) = 0$ has roots α, β and γ, then $P(y+1) = 0$ will have roots $\alpha - 1$, $\beta - 1$ and $\gamma - 1$.

$$(y+1)^3 - 5(y+1)^2 + 5 = 0$$

$$y^3 + 3y^2 + 3y + 1 - 5y^2 - 10y - 5 + 5 = 0$$

$$y^3 - 2y^2 - 7y + 1 = 0,$$

or equivalently, $x^3 - 2x^2 - 7x + 1 = 0.$

METHOD 2 (Sums and products of roots)

From the original equation,

$$\alpha + \beta + \gamma = 5$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 0$$

$$\alpha\beta\gamma = -5.$$

For the new polynomial equation,

$$\begin{aligned} (\alpha - 1) + (\beta - 1) + (\alpha - 1) &= \alpha + \beta + \gamma - 3 \\ &= 5 - 3 \\ &= 2. \end{aligned}$$

$$\begin{aligned} &(\alpha - 1)(\beta - 1) + (\beta - 1)(\gamma - 1) + (\gamma - 1)(\alpha - 1) \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha + \beta + \gamma) + 3 \\ &= 0 - 2 \times 5 + 3 \\ &= -7. \end{aligned}$$

$$\begin{aligned} &(\alpha - 1)(\beta - 1)(\gamma - 1) \\ &= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma) - 1 \\ &= -5 - 0 + 5 - 1 \\ &= -1. \end{aligned}$$

So the new polynomial could be

$$x^3 - 2x^2 + (-7)x - (-1) = 0$$

or $\quad x^3 - 2x^2 - 7x + 1 = 0.$

(ii) *METHOD 1* (Substitution)

Let $y = x^2$, so that $P\left(\sqrt{y}\right) = 0$ will have the desired roots.

$$\left(\sqrt{y}\right)^3 - 5\left(\sqrt{y}\right)^2 + 5 = 0$$

$$y\sqrt{y} - 5y + 5 = 0$$

$$y\sqrt{y} = 5y - 5$$

$$y^3 = (5y - 5)^2$$

$$y^3 = 25y^2 - 50y + 25$$

$$y^3 - 25y^2 + 50y - 25 = 0,$$

or equivalently, $x^3 - 25x^2 + 50x - 25 = 0.$

METHOD 2 (Sums and products of roots)

For the new polynomial equation,

$$\begin{aligned} &\alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= 5^2 - 2 \times 0 \\ &= 25. \end{aligned}$$

$$\begin{aligned} &\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= 0^2 - 2 \times (-5) \times 5 \\ &= 50. \end{aligned}$$

$$\begin{aligned} \alpha^2\beta^2\gamma^2 &= (\alpha\beta\gamma)^2 \\ &= (-5)^2 \\ &= 25. \end{aligned}$$

So the new polynomial equation could be

$$x^3 - 25x^2 + 50x - 25 = 0.$$

(iii) Since α, β and γ are roots, we have

$$\alpha^3 - 5\alpha^2 + 5 = 0$$

$$\beta^3 - 5\beta^2 + 5 = 0$$

$$\gamma^3 - 5\gamma^2 + 5 = 0$$

$$\therefore \left(\alpha^3 + \beta^3 + \gamma^3\right) - 5\left(\alpha^2 + \beta^2 + \gamma^2\right) + 15 = 0$$

$$\begin{aligned} \therefore \quad \alpha^3 + \beta^3 + \gamma^3 &= 5\left(\alpha^2 + \beta^2 + \gamma^2\right) - 15 \\ &= 5 \times 25 - 15, \text{ from (ii)}, \\ &= 110. \end{aligned}$$

(c) (i) $\mathcal{E}$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Differentiating w.r.t. x:

$$\therefore \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

At $P\left(x_1, y_1\right)$, $m = -\dfrac{b^2x_1}{a^2y_1}$.

$$\therefore \text{ Tangent is } \quad y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\frac{y_1y - y_1^2}{b^2} + \frac{x_1x - x_1^2}{a^2} = 0$$

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1, \qquad \text{—①}$$

since $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$, as P lies on $\mathcal{E}$.

(ii) Let Q be the point $\left(x_2, y_2\right)$.

Following (i), the tangent at Q is

$$\frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1. \qquad \text{—②}$$

Since $\left(x_0, y_0\right)$ lies on both ① and ②,

$$\frac{x_1x_0}{a^2} + \frac{y_1y_0}{b^2} = 1 \text{ and } \frac{x_2x_0}{a^2} + \frac{y_2y_0}{b^2} = 1.$$

Hence $\left(x_1, y_1\right)$ and $\left(x_2, y_2\right)$ both satisfy

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

Since $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$ is the equation of a line, and since it passes through both P and Q, it is the chord of contact from T.

(iii) S is $(ae, 0)$

$$\therefore \quad m_{TS} = \frac{y_0}{x_0 - ae}.$$

At R, $\quad x = \dfrac{a}{e}$

$$\therefore \quad \frac{x_0}{a^2}\cdot\frac{a}{e} + \frac{y_0y}{b^2} = 1, \quad \text{from (ii), since } R \text{ lies on } PQ.$$

$$\frac{y_0y}{b^2} = 1 - \frac{x_0}{ae}$$

$$y = \frac{ae - x_0}{ae}\cdot\frac{b^2}{y_0}$$

$$\therefore \quad R \text{ is } \left(\frac{a}{e}, \frac{-b^2(x_0 - ae)}{aey_0}\right)$$

$$\therefore \quad m_{SR} = \frac{\dfrac{b^2(x_0 - ae)}{aey_0}}{a\left(e - \dfrac{1}{e}\right)}$$

$$= \frac{b^2(x_0 - ae)}{aey_0}\cdot\frac{1}{a\left(e - \frac{1}{e}\right)}$$

$$= \frac{b^2(x_0 - ae)}{a^2y_0\left(e^2 - 1\right)}$$

$$= \frac{b^2(x_0 - ae)}{y_0\left(-b^2\right)}, \text{ since } b^2 = a^2\left(1 - e^2\right),$$

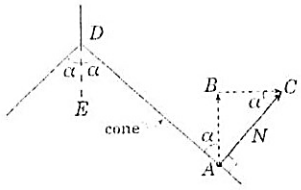
$$= -\frac{x_0 - ae}{y_0}$$

$$\therefore \quad m_{TS}\cdot m_{SR} = \frac{y_0}{x_0 - ae}\cdot\left(-\frac{x_0 - ae}{y_0}\right) = -1$$

$$\therefore \quad TS \perp SR.$$

QUESTION 6

(a) (i)



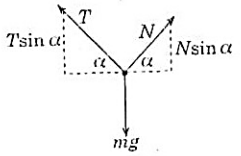
Let N have vertical component $\overrightarrow{AB}$ and horizontal component $\overrightarrow{BC}$, as shown.

$$\begin{aligned} \angle BCA &= 90° - \angle BAC \\ &= \angle BAD \\ &= \angle ADE \text{ (alternate } \angle\text{s)} \\ &= \alpha. \end{aligned}$$

$$\therefore \quad AB = N\sin\alpha.$$

That is, the vertical component of N is $N\sin\alpha$.

(ii)



Both T and N are at an angle of α with the horizontal. So their vertical components are $T\sin\alpha$ and $N\sin\alpha$.

For the vertical components:

$\sum F = 0$ (Newton's law, no acceleration in the vertical direction).

$$T\sin\alpha + N\sin\alpha - mg = 0$$

$$(T + N)\sin\alpha = mg$$

$$T + N = \frac{mg}{\sin\alpha},$$

as required.

For the horizontal components:

$\sum F = m\omega^2r$ (Newton's law, with centripetal acceleration).

$$T\cos\alpha - N\cos\alpha = m\omega^2\ell\cos\alpha$$

(since $r = \ell\cos\alpha$)

$$\therefore \quad T - N = m\omega^2\ell.$$

(iii)
$$\left.\begin{aligned} T + N &= \frac{mg}{\sin\alpha} \\ T - N &= m\omega^2\ell \end{aligned}\right\}$$

When $N = 0$, $\quad \dfrac{mg}{\sin\alpha} = m\omega^2\ell$

$$\omega^2 = \frac{g}{\ell\sin\alpha}$$

$$\omega = \sqrt{\frac{g}{\ell\sin\alpha}}.$$

(b) $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n\theta \; d\theta.$

(i) $I_1 = \displaystyle\int_0^{\frac{\pi}{4}} \tan\theta \; d\theta$

$$= \left[-\ln\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= \ln(\cos 0) - \ln\left(\cos\frac{\pi}{4}\right)$$

$$= \ln 1 - \ln\frac{1}{\sqrt{2}}$$

$$= \ln\sqrt{2}$$

$$= \frac{1}{2}\ln 2.$$

(ii) $I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\left(\sec^2\theta - 1\right)\,d\theta$, provided $n \geq 2$,

$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\sec^2\theta\;\;d\theta - I_{n-2}$

$\therefore\;\; I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\sec^2\theta\;\;d\theta$

$= \left[\dfrac{\tan^{n-1}\theta}{n-1}\right]_0^{\frac{\pi}{4}}$

$= \dfrac{1}{n-1} - 0$

$= \dfrac{1}{n-1}.$

(iii) For $0 < \theta < \dfrac{\pi}{4}$

$0 < \tan\theta < 1$, since $\tan\theta$ is monotonic increasing on the domain,

$0 < \tan^2\theta < 1$

$0 < \tan^n\theta < \tan^{n-2}\theta$, since $\tan^{n-2}\theta > 0$,

$\therefore \int_0^{\frac{\pi}{4}} \tan^n\theta\;\;d\theta < \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\;\;d\theta$,

that is, $I_n < I_{n-2}$ —①

$\therefore\;\; 2I_n < I_{n-2} + I_n$

$2I_n < \dfrac{1}{n-1}$, from (ii),

$I_n < \dfrac{1}{2(n-1)}.$

Replacing n by $n+2$ in ①, we have

$I_{n+2} < I_n$

$I_{n+2} + I_n < 2I_n$

$\dfrac{1}{n+2-1} < 2I_n$, from (ii),

$\dfrac{1}{2(n+1)} < I_n.$

Hence $\dfrac{1}{2(n+1)} < I_n < \dfrac{1}{2(n-1)}.$

(iv) $I_5 = \dfrac{1}{4} - I_3$, from (ii),

$= \dfrac{1}{4} - \left(\dfrac{1}{2} - I_1\right)$, from (ii),

$= -\dfrac{1}{4} + \dfrac{1}{2}\ln 2$, from (i),

$= \dfrac{2\ln 2 - 1}{4}.$

From (iii), we have

$\dfrac{1}{2\times 6} < I_5 < \dfrac{1}{2\times 4}$

$\dfrac{1}{12} < \dfrac{2\ln 2 - 1}{4} < \dfrac{1}{8}$

$\dfrac{1}{3} < 2\ln 2 - 1 < \dfrac{1}{2}$

$\dfrac{4}{3} < 2\ln 2 < \dfrac{3}{2}$

$\dfrac{2}{3} < \ln 2 < \dfrac{3}{4}$, as required.

QUESTION 7

(a) (i) Since $V = Ay$

$\dfrac{dV}{dt} = A\dfrac{dy}{dt}.$

As $\dfrac{dV}{dt} = -k\sqrt{y}$ (given)

$A\dfrac{dy}{dt} = -k\sqrt{y}$

ie. $\dfrac{dy}{dt} = -\dfrac{k}{A}\sqrt{y}.$

(ii) $\therefore\;\; \int_{y_0}^{y} \dfrac{dy}{\sqrt{y}} = -\dfrac{k}{A}\int_0^t dt$

$\left[2\sqrt{y}\right]_{y_0}^{y} = -\dfrac{k}{A}\left[t\right]_0^t$

$2\sqrt{y} - 2\sqrt{y_0} = -\dfrac{kt}{A}$

$\sqrt{y} = \sqrt{y_0} - \dfrac{kt}{2A}.$ —①

Equation ① can also be obtained as follows:

$\dfrac{dt}{dy} = -\dfrac{A}{k\sqrt{y}}$

$t = -\dfrac{2A\sqrt{y}}{k} + C.$ —②

Using $y = y_0$ when $t = 0$,

$0 = -\dfrac{2A\sqrt{y_0}}{k} + C$

$\therefore\;\; C = \dfrac{2A\sqrt{y_0}}{k}.$ —③

Substitute ③ into ②:

$t = -\dfrac{2A}{k}\left(\sqrt{y_0} - \sqrt{y}\right)$

$\sqrt{y} = \sqrt{y_0} - \dfrac{kt}{2A}.$ —①

Using $y = 0$ when $t = T$,

$0 = \sqrt{y_0} - \dfrac{kt}{2A}$

$\therefore\;\; \dfrac{k}{2A} = \dfrac{\sqrt{y_0}}{T}.$ —④

Substitute ④ into ①:

$\sqrt{y} = \sqrt{y_0} - \sqrt{y_0}\,\dfrac{t}{T}$

$= \sqrt{y_0}\left(1 - \dfrac{t}{T}\right).$

Hence $y = y_0\left(1 - \dfrac{t}{T}\right)^2$ as required.

[N.B. The condition $0 \leq t \leq T$ is necessary since $0 < y \leq y_0$ and the differential equation only applies on this domain.]

(iii) Using $y = \dfrac{y_0}{2}$ when $t = 10$ in (ii),

$\dfrac{y_0}{2} = y_0\left(1 - \dfrac{10}{T}\right)^2$

$\dfrac{1}{\sqrt{2}} = 1 - \dfrac{10}{T}.$

[N.B. The positive square root, since $t \leq T$ implies $\dfrac{10}{T} \leq 1$.]

$\dfrac{10}{T} = 1 - \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{2}-1}{\sqrt{2}}$

$\therefore\;\; T = \dfrac{10\sqrt{2}}{\sqrt{2}-1}$

$= 10\sqrt{2}\left(\sqrt{2}+1\right)$

$= 10\left(2+\sqrt{2}\right)$ seconds.

(b) (i) $\overrightarrow{OP_1} = \overrightarrow{OP_0} + \overrightarrow{P_0P_1}$

$\therefore\;\; z_0 + z_1 = z_0 + \overrightarrow{P_0P_1}$

$\overrightarrow{P_0P_1} = z_1.$

Similarly, $\overrightarrow{P_1P_2} = z_2$ and $\overrightarrow{P_2P_3} = z_3$

$\theta_0 = \arg\overrightarrow{P_0P_1} - \arg\overrightarrow{OP_0}$

$= \arg z_1 - \arg z_0$

$= \arg\left[\text{cis}(\alpha+\beta)\right] - \arg\left[\text{cis}\,\alpha\right]$

$= \alpha + \beta - \alpha$

$= \beta.$

Similarly,

$\theta_1 = \arg\overrightarrow{P_1P_2} - \arg\overrightarrow{P_0P_1}$

$= \arg z_2 - \arg z_1$

$= \arg\left[\text{cis}(\alpha+2\beta)\right] - \arg\left[\text{cis}(\alpha+\beta)\right]$

$= \alpha + 2\beta - (\alpha+\beta)$

$= \beta,$

and $\theta_2 = \arg z_3 - \arg z_2$

$= \beta.$

(ii) In ΔOP_0P_1 and $\Delta P_0P_1P_2$

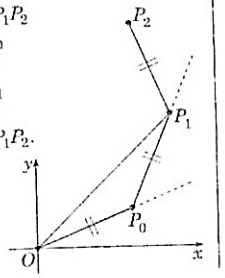
$\angle OP_0P_1 = \pi - \theta_0$

$= \pi - \beta$

$\angle P_0P_1P_2 = \pi - \theta_1$

$= \pi - \beta$

$\therefore\;\; \angle OP_0P_1 = \angle P_0P_1P_2.$



$OP_0 = P_0P_1 = P_1P_2$ (given)

$\therefore\;\; \Delta OP_0P_1 \equiv \Delta P_0P_1P_2$ (SAS)

$\therefore\;\; \angle P_0OP_1 = \angle P_0P_2P_1$ (corresp. $\angle$s of congruent Δs)

$\therefore\;\; OP_0P_1P_2$ is a cyclic quadrilateral, since angles subtended by P_0P_1 at O and P_2 are equal.

Note: $\angle P_0OP_1 = \angle P_0P_2P_1 = \dfrac{\beta}{2}$ since both ΔOP_0P_1 and $\Delta P_0P_1P_2$ are isosceles.

(iii) Following a similar argument to that used in (ii). In $\Delta P_0P_1P_2$ and $\Delta P_1P_2P_3$,

$\angle P_1P_2P_3 = \angle P_0P_1P_2$ (both $\pi - \beta$)

$P_0P_1 = P_1P_2 = P_2P_3$ (given)

$\therefore\;\; \Delta P_0P_1P_2 \equiv \Delta P_1P_2P_3$ (SAS)

$\therefore\;\; \angle P_1P_0P_2 = \angle P_1P_3P_2$ (corresp. $\angle$s of congruent Δs)

$\therefore\;\; P_0P_1P_2P_3$ are concyclic (since $\angle$s subtended by P_1P_2 at P_0 and P_3 are equal).

Using this and (ii), we conclude that the circle $P_0P_1P_2$ also goes through O and P_3, and hence O, P_0, P_1, P_2 and P_3 are concyclic.

(iv) From (iii) we know that $OP_0P_1P_2P_3$ is a concyclic pentagon. We also know that

$OP_0 = P_0P_1 = P_1P_2 = P_2P_3$

$= \left|z_i\right|$ for $i = 0, 1, 2$ or 3

$= 1.$

Now from $z_0 + z_1 + z_2 + z_3 + z_4 = 0$,

we obtain $z_4 = -\left(z_0 + z_1 + z_2 + z_3\right)$

$= -\overrightarrow{OP_3}.$

So $OP_3 = \left|z_4\right| = 1.$

Therefore $OP_0P_1P_2P_3$ is a concyclic pentagon with equal sides, and hence is a regular pentagon. The exterior angle is β. The sum of exterior angles $= 2\pi$

$5\beta = 2\pi$

$\beta = \dfrac{2\pi}{5}.$

Since $\beta < \dfrac{\pi}{2}$, $5\beta < \dfrac{5\pi}{2} < 4\pi$.

Therefore we can eliminate the possibility that $5\beta = 4\pi, 6\pi, \ldots$.

So $\beta = \dfrac{2\pi}{5}$ is the only possible value of β.

QUESTION 8

(a) (i) $\sin(2m+1)\theta$

$= \text{Im}\left[\cos(2m+1)\theta + i\sin(2m+1)\theta\right]$

$= \text{Im}\left[(\cos\theta + i\sin\theta)^{2m+1}\right]$

$$= \text{Im}\left[(\cos\theta)^{2m+1} + \binom{2m+1}{1}\cos^{2m}\theta\, i\sin\theta + \binom{2m+1}{2}\cos^{2m-1}\theta\, i^2\sin^2\theta + \binom{2m+1}{3}\cos^{2m-2}\theta\, i^3\sin^3\theta + \cdots + \binom{2m+1}{2m+1} i^{2m+1}\sin^{2m+1}\theta\right]$$

$$= \binom{2m+1}{1}\cos^{2m}\theta\sin\theta - \binom{2m+1}{3}\cos^{2m-2}\theta\sin^3\theta + \cdots + (-1)^m\sin^{2m+1}\theta.$$

The last term is found from

$$\binom{2m+1}{2m+1} i^{2m}\sin^{2m+1}\theta \cdot i = \left(1^2\right)^m\sin^{2m+1}\theta\cdot i = (-1)^m\sin^{2m+1}\theta\cdot i.$$

(ii) Divide both sides of (i) by $\sin^{2m+1}\theta$.

$$\therefore \frac{\sin(2m+1)\theta}{\sin^{2m+1}\theta} = \binom{2m+1}{1}\frac{\cos^{2m}\theta\sin\theta}{\sin^{2m}\theta\sin\theta} - \binom{2m+1}{3}\frac{\cos^{2m-2}\theta\sin^3\theta}{\sin^{2m-2}\theta\sin^3\theta} + \cdots + (-1)^m\frac{\sin^{2m+1}\theta}{\sin^{2m+1}\theta}$$

$$= \binom{2m+1}{1}\cot^{2m}\theta - \binom{2m+1}{3}\cot^{2(m-1)}\theta + \cdots + (-1)^m.$$

Let $x = \cot^2\theta$:

$$\therefore \frac{\sin(2m+1)\theta}{\sin^{2m+1}\theta} = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \cdots + (-1)^m = p(x).$$

$p(x) = 0$ gives roots to the polynomial in x.

$$\therefore \frac{\sin(2m+1)\theta}{\sin^{2m+1}\theta} = 0$$

$$\therefore \sin(2m+1)\theta = 0, \text{ where } \sin^{2m+1}\theta \neq 0 \Rightarrow \theta \neq 0$$

$$\therefore (2m+1)\theta = k\pi,\ k = 1, 2, \ldots, m \quad [\text{N.B. } k \neq 0 \text{ since } \theta \neq 0.]$$

$$\therefore \theta = \frac{k\pi}{(2m+1)},\ k = 1, 2, \ldots, m.$$

$\therefore p(x) = 0$ has m distinct roots:

$$\alpha_k = \cot^2\left(\frac{k\pi}{2m+1}\right),\ k = 1, 2, \ldots, m.$$

The roots are distinct because $\cot^2\theta$ is monotonic decreasing for $0 < \theta < \frac{\pi}{2}$.

That is, $0 < \frac{k\pi}{2m+1} < \frac{\pi}{2}$

$$0 < k < m + \frac{1}{2}.$$

Note also that other solutions for $k > m$ are either not possible (when k is a multiple of $2m+1$) or are repetitions due to periodicity

$$\left[\cot^2\left(\frac{k\pi}{2m+1}\right) = \cot^2\left(\frac{(k+2m+1)\pi}{2m+1}\right)\right]$$

or symmetry

$$\left[\cot^2\left(\frac{k\pi}{2m+1}\right) = \cot^2\left(\frac{(2m+1-k)\pi}{2m+1}\right)\right].$$

(iii) $\therefore \cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \cdots + \cot^2\left(\frac{m\pi}{2m+1}\right)$

$$= \sum_{k=1}^{m}\alpha_k \quad [\text{N.B. Sum of roots of } p(x).]$$

$$= -\frac{-\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{(2m+1)(2m)(2m-1)}{3!}\times\frac{1}{2m+1} = \frac{m(2m-1)}{3}.$$

(iv) $\cot\theta < \frac{1}{\theta}, \quad 0 < \theta < \frac{\pi}{2}.$

Substitute $\theta = \frac{k\pi}{2m+1}$:

$$\cot\left(\frac{k\pi}{2m+1}\right) < \frac{2m+1}{k\pi}$$

$$\left[\text{N.B. } k = 1, \ldots, m \Rightarrow 0 < \theta < \frac{\pi}{2}\right].$$

$$\cot^2\left(\frac{k\pi}{2m+1}\right) < \frac{(2m+1)^2}{k^2\pi^2}, \text{ since } \cot\left(\frac{k\pi}{2m+1}\right) > 0 \text{ for } k = 1, \ldots, m.$$

Now apply this to each of the terms in the expression in (iii):

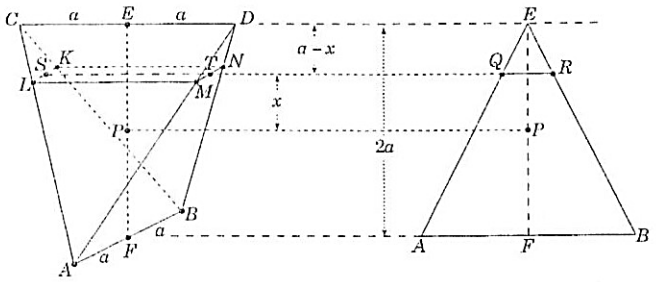
$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2} + \frac{(2m+1)^2}{2^2\pi^2} + \cdots + \frac{(2m+1)^2}{m^2\pi^2}$$

$$= \frac{(2m+1)^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2}\right)$$

$$\therefore \frac{\pi^2}{6} < \frac{(2m+1)^2}{2m(2m-1)}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2}\right),$$

as required.

(b) Let Q, R, S, T be the midpoints of the sides of rectangle $KLMN$ as shown.

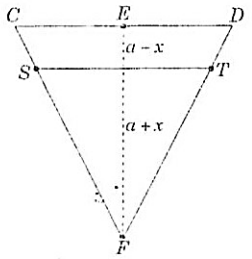


(i) $\Delta EQR \parallel\!\parallel \Delta EAB$

$$\therefore \frac{QR}{AB} = \frac{a-x}{2a}$$

$QR = a - x$, since $AB = 2a$.

$KL = a - x$, since $KL = QR$.



Similarly, $\Delta FST \parallel\!\parallel \Delta FCD$

$$\therefore \frac{ST}{CD} = \frac{a+x}{2a}.$$

$ST = a + x$, since $CD = 2a$,

$LM = a + x$, since $LM = ST$.

$\therefore$ Area of $KLMN = (a - x)(a + x) = a^2 - x^2$.

(ii) Volume of tetrahedron $ABCD$

$$= \int_{-a}^{a}\left(a^2 - x^2\right)dx = 2\int_0^a\left(a^2 - x^2\right)dx = 2\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\left(a^3 - \frac{a^3}{3}\right) = \frac{4a^3}{3}.$$

END OF MATHEMATICS EXTENSION 2 SOLUTIONS