2001 HIGHER SCHOOL CERTIFICATE SOLUTIONS

MATHEMATICS EXTENSION 2

QUESTION 1

(a) Let $u = \tan x$. Then $du = \sec^2 x\ dx$.

When $x = 0$, $u = 0$. When $x = \frac{\pi}{4}$, $u = 1$.

Therefore $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x\ dx = \int_0^1 u^3\ du$

$$= \left[\frac{u^4}{4}\right]_0^1$$

$$= \frac{1}{4}.$$

Note that the substitution $u = \tan^2 x$ also works.

(b) $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+1}} = \int \frac{dx}{\sqrt{(x-2)^2-3}}$

$$= \int \frac{du}{\sqrt{u^2-3}} \quad (\text{where } u = x-2)$$

$$= \ln\left(u + \sqrt{u^2-3}\right) + c \quad (\text{from table})$$

$$= \ln\left[x-2+\sqrt{(x-2)^2-3}\right] + c.$$

(c) $\displaystyle\int_e^4 \frac{\ln x}{x^2}\ dx = \int_e^4 x^{-2} \ln x\ dx.$

Let $u = \ln x$, $dv = x^{-2}\ dx$.

Then $du = \frac{1}{x}\ dx$, $v = \frac{x^{-1}}{-1} = -\frac{1}{x}$.

Therefore

$$\int_e^4 \frac{\ln x}{x^2}\ dx = \left[-\frac{1}{x}\ln x\right]_e^4 - \int_e^4 -\frac{1}{x}\cdot\frac{1}{x}\ dx$$

$$= \left[\frac{\ln x}{x}\right]_e^4 + \int_e^4 x^{-2}\ dx$$

$$= \left(\frac{\ln e}{e} - \frac{\ln 4}{4}\right) + \left[-\frac{1}{x}\right]_e^4$$

$$= \frac{1}{e} - \frac{\ln 4}{4} - \frac{1}{4} + \frac{1}{e}$$

$$= \frac{2}{e} - \frac{1}{4}(1 + \ln 4).$$

(d) Let $u = \sqrt{x-1} = (x-1)^{\frac{1}{2}}$.

Then $du = \frac{1}{2}(x-1)^{-\frac{1}{2}}\ dx = \frac{dx}{2\sqrt{x-1}}$.

$\therefore \quad \frac{dx}{\sqrt{x-1}} = 2\ du$ and $u^2 = x-1$

$\therefore \quad 1 + x = u^2 + 2.$

When $x = 2$, $u = 1$.

When $x = 3$, $u = \sqrt{2}$.

Therefore

$$\int_2^3 \frac{1+x}{\sqrt{x-1}}\ dx = \int_1^{\sqrt{2}} 2\left(u^2+2\right)\ du$$

$$= 2\left[\frac{u^3}{3} + 2u\right]_1^{\sqrt{2}}$$

$$= 2\left[\left(\frac{2\sqrt{2}}{3} + 2\sqrt{2}\right) - \left(\frac{1}{3} + 2\right)\right]$$

$$= \frac{2}{3}\left(8\sqrt{2} - 7\right).$$

(e) (i) $\dfrac{5x^2-3x+1}{(x^2+1)(x-2)} \equiv \dfrac{ax+1}{x^2+1} + \dfrac{b}{x-2}$

$$\equiv \frac{(ax+1)(x-2) + b(x^2+1)}{(x^2+1)(x-2)}.$$

Therefore $(ax+1)(x-2) + b(x^2+1)$

$$\equiv 5x^2 - 3x + 1.$$

Substitute $x = 2$: $\quad 5b = 15$

$\therefore \quad b = 3.$

Coefficient of x^2: $\quad a + 3 = 5$

$\therefore \quad a = 2.$

(ii) $\displaystyle\int \frac{5x^2-3x+1}{(x^2+1)(x-2)}\ dx$

$$= \int\left(\frac{2x+1}{x^2+1} + \frac{3}{x-2}\right) dx, \quad [\text{by (i)}]$$

$$= \int\left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{3}{x-2}\right) dx$$

$$= \ln\left(x^2+1\right) + \tan^{-1} x + 3\ln(x-2) + c, \quad x > 2$$

$$= \ln\left(x^2+1\right)(x-2)^3 + \tan^{-1} x + c, \quad x > 2.$$

Note that for $x < 2$,

$$\int \frac{5x^2-3x+1}{(x^2+1)(x-2)}\ dx$$

$$= \ln\left(x^2+1\right)(2-x)^3 + \tan^{-1} x + c.$$

QUESTION 2

(a) $zw = (2+3i)(1+i)$

$$= 2 + 2i + 3i - 3$$

$$= -1 + 5i.$$

$$\frac{1}{w} = \frac{1}{1+i}\cdot\frac{1-i}{1-i}$$

$$= \frac{1-i}{1+i}$$

$$= \frac{1}{2} - \frac{1}{2}i.$$

(b) (i) $\left|1+\sqrt{3}i\right| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2.$

$$\arg\left(1+\sqrt{3}i\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

$$\therefore \quad 1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + 2\sin\frac{\pi}{3}\right)$$

$$= 2\,\text{cis}\,\frac{\pi}{3}.$$

(ii) $\left(1+\sqrt{3}i\right)^{10} = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{10}$

$$= 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right),$$

using de Moivre's theorem,

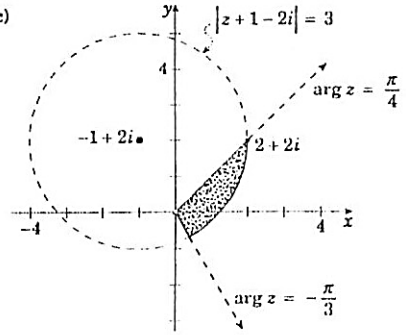
$$= 2^{10}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= -2^{10}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= -2^9\left(1+\sqrt{3}i\right)$$

$$= -512 - 512\sqrt{3}i.$$

(c)



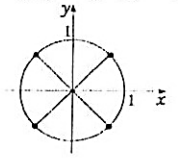
(d) $z^4 = -1$

$$= \cos\pi + i\sin\pi$$

$$= \text{cis}\,\pi.$$

$$z = \text{cis}\left(\frac{\pi}{4} + \frac{2\pi n}{4}\right), \quad n \text{ an integer}$$

$$= \text{cis}\left(\frac{\pi}{4}\right), \text{cis}\left(-\frac{\pi}{4}\right), \text{cis}\left(\frac{3\pi}{4}\right), \text{cis}\left(-\frac{3\pi}{4}\right).$$



(e) (i) $\overrightarrow{BA} = z_1 - z_2$ and $\overrightarrow{BC} = z_3 - z_2$.

Rotating $\overrightarrow{BC}$ anticlockwise by 90° gives $\overrightarrow{BA}$. Hence, $z_1 - z_2 = i(z_3 - z_2)$.

Squaring both sides gives

$$(z_1 - z_2)^2 = -(z_3 - z_2)^2.$$

(ii) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$. But $\overrightarrow{AD} = \overrightarrow{BC}$.

Therefore $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$$= z_1 + (z_3 - z_2).$$

So D is represented by the complex number $z_1 - z_2 + z_3$.

QUESTION 3

(a) $\mathcal{H}$ has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

(i) x intercepts:

When $y = 0$, $x^2 = 9$. Therefore $\mathcal{H}$ intersects the x axis at (3, 0) and (-3, 0).

Eccentricity:

$b^2 = a^2(e^2 - 1)$ where $a^2 = 9$, $b^2 = 16$.

Therefore $16 = 9(e^2 - 1)$.

Hence $e^2 - 1 = \frac{16}{9}$

$$e^2 = \frac{25}{9}$$

$$e = \frac{5}{3}.$$

Foci of $\mathcal{H}$ are $(\pm ae, 0)$,

that is, (-5, 0) and (5, 0).

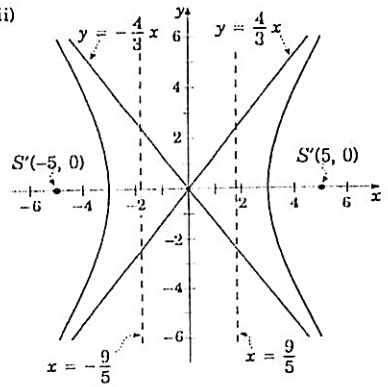
(ii) Directrices are $x = \pm\frac{a}{e}$,

that is, $x = -\frac{9}{5}$ and $x = \frac{9}{5}$.

Asymptotes are $y = \pm\frac{b}{a}x$,

that is, $y = -\frac{4}{3}x$ and $y = \frac{4}{3}x$.

(iii)



(b) We have $\alpha + \beta + \gamma = 3$ —①

$\alpha^2 + \beta^2 + \gamma^2 = 1$ —②

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2.$ —③

(i) $\alpha^2 + \beta^2 + \gamma^2$
$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha).$

That is, $1 = (3)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha).$

Therefore $\alpha\beta + \beta\gamma + \gamma\alpha = 4.$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}.$

That is, $2 = \frac{4}{\alpha\beta\gamma}.$

Therefore $\alpha\beta\gamma = 2.$

If α, β, γ are the roots of
$x^3 - 3x^2 + 4x - 2 = 0,$
then $\alpha + \beta + \gamma = -\frac{-3}{1} = 3,$
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{4}{1} = 4,$
$\alpha\beta\gamma = -\frac{-2}{1} = 2.$

This is substantiated by the above working, and thus α, β, γ are indeed the roots of the given cubic.

Alternatively, a polynomial with roots α, β, γ is $(x - \alpha)(x - \beta)(x - \gamma)$
$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$
$= x^3 - 3x^2 + 4x - 2.$

(ii) Let $P(x) = x^3 - 3x^2 + 4x - 2.$
$P(1) = 0.$
$\therefore$ $x = 1$ is a root of $x^3 - 3x^2 + 4x - 2 = 0.$

METHOD 1

$\therefore$ $(x - 1)$ is a factor of $P(x)$.

The quadratic factor can be found by inspection:

$P(x) = (x - 1)\left(x^2 - 2x + 2\right)$
$= (x - 1)\left[(x - 1)^2 + 1\right]$
$= (x - 1)(x - 1 - i)(x - 1 + i).$

$\therefore$ $\alpha, \beta, \gamma = 1,\ 1 + i,\ 1 - i.$

METHOD 2

Put $\alpha = 1$ into ① and ②:

$\therefore$ $\beta + \gamma = 2$ —④

$\beta^2 + \gamma^2 = 0.$ —⑤

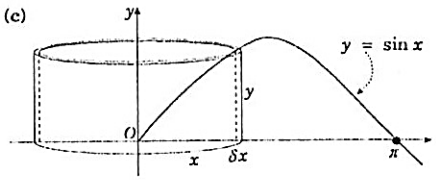
From ④, $\gamma = 2 - \beta.$

Substitute into ⑤:

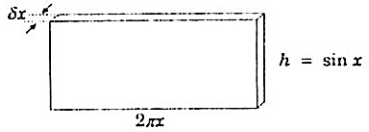
$\beta^2 + (2 - \beta)^2 = 0$
$2\beta^2 - 4\beta + 4 = 0$
$\beta^2 - 2\beta + 2 = 0$
$\beta = \frac{-2 \pm \sqrt{4 - 8}}{2}$
$= 1 \pm i.$

$\therefore$ $\alpha, \beta, \gamma = 1,\ 1 + i,\ 1 - i.$

(c)



Volume of shell $= \delta V = 2\pi xy\ \delta x$
$= 2\pi x \sin x\ \delta x.$



Volume of solid:

$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} 2\pi x \sin x\ \delta x$

$= 2\pi \int_0^{\pi} x \sin x\ dx$

$= 2\pi \left(\left[-x \cos x \right]_0^{\pi} - \int_0^{\pi} -\cos x\ dx \right),$ by parts

$= 2\pi \left(\left[x \cos x \right]_{\pi}^{0} + \left[\sin x \right]_0^{\pi} \right)$

$= 2\pi \left[(0 + \pi) + (0 - 0) \right]$

$= 2\pi^2$ cubic units.

QUESTION 4

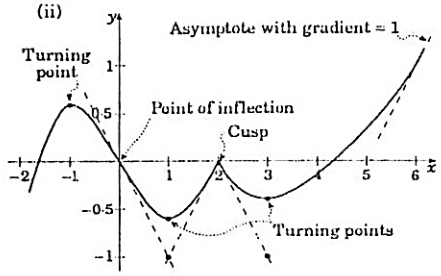
(a) (i) The turning points are at $x = -1,\ 1,\ 3,$ where $f'(x) = 0.$

At $x = -1$, $f'(x)$ is decreasing, so this is a maximum.

At $x = 1$, $f'(x)$ is increasing, so this is a minimum.

At $x = 3$, $f'(x)$ is increasing, so this is a minimum.

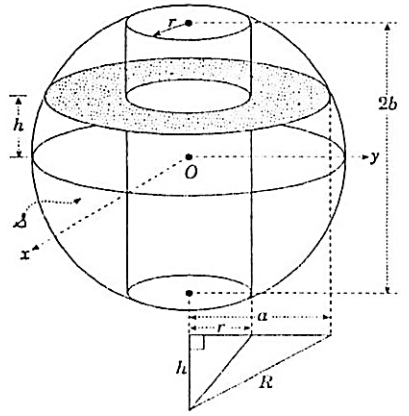
(ii)



Note:

1. Gradients are ± 1 at the point of inflection, both sides of the cusp, and on the asymptote.
2. The turning points are $(-1,\ 0{\cdot}6)$, $(1,\ -0{\cdot}6)$ and $(3,\ -0{\cdot}4)$, where the stated x values are exact and the y values are approximate.

(b) (i) Let a be the outer radius of the shaded annulus.



Area $= \pi a^2 - \pi r^2$
$= \pi\left(R^2 - h^2\right) - \pi r^2$
$= \pi\left(R^2 - h^2 - r^2\right).$

(ii) R, r and b are related constants, so we can eliminate R and r from this formula and write it in terms of b and the variable h.



$R^2 - r^2 = b^2$

$\therefore$ Area $= \pi\left(b^2 - h^2\right)$

Volume $= 2\int_0^b \pi\left(b^2 - h^2\right)\ dh$

$= 2\pi \left[b^2 h - \frac{h^3}{3} \right]_0^b$

$= 2\pi \left(b^3 - \frac{b^3}{3} \right)$

$= \frac{4}{3}\pi b^3.$

Comment:

Since this volume can be written in terms of b it does *not* depend on the values of r and R. This figure can be a small sphere with a skinny hole, or a large sphere with a fat hole. If you notice this, then there is a simple check on the answer.

Take the limit as $r \to 0$.
The volume doesn't change.
The diagram becomes a sphere of radius b, with a zero volume hole and the correct volume of $\frac{4}{3}\pi b^3$.

(c) $\frac{d}{dx}\left(\tan^{-1}\frac{x}{x+1} + \tan^{-1}\frac{1}{2x+1} \right)$

$= \frac{1}{1 + \left(\frac{x}{x+1}\right)^2} \times \frac{(x+1) - x}{(x+1)^2}$
$+ \frac{1}{1 + \left(\frac{1}{2x+1}\right)^2} \times \frac{-2}{(2x+1)^2}$

$= \frac{1}{(x+1)^2 + x^2} - \frac{2}{(2x+1)^2 + 1}$

$= \frac{1}{2x^2 + 2x + 1} - \frac{2}{4x^2 + 4x + 2}$

$= 0.$

Since the derivative is zero, the original function is a constant. However it is discontinuous at $x = -1$, $(x + 1 = 0)$ and $x = -\frac{1}{2}$, $(2x + 1 = 0)$, and so *may* have different values on the domains $x < -1$, $-1 < x < -\frac{1}{2}$ and $x > -\frac{1}{2}$.

For $2x+1>0$ or $x>-\frac{1}{2}$, we can find the constant by substituting a suitable value.

For example, at $x=0$:

$$\begin{aligned}&\tan^{-1}\frac{x}{x+1}+\tan^{-1}\frac{1}{2x+1}\\&=\tan^{-1}0+\tan^{-1}1\\&=0+\frac{\pi}{4}\\&=\frac{\pi}{4}.\end{aligned}$$

Thus the constant is $\frac{\pi}{4}$.

Although it is not required, it can be shown that $\tan^{-1}\frac{x}{x+1}+\tan^{-1}\frac{1}{2x+1}$

$$=\begin{cases}\frac{\pi}{4}, & x>-\frac{1}{2}\\ -\frac{3\pi}{4}, & -1<x<-\frac{1}{2}\\ \frac{\pi}{4}, & x<-1.\end{cases}$$

QUESTION 5

(a) (i) $'E$ can be represented parametrically by $x=a\cos\theta$ and $y=b\sin\theta$.

Then $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{b\cos\theta}{-a\sin\theta}$. (*)

Equation of tangent at P:

$$\begin{aligned}y-b\sin\theta&=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)\\ya\sin\theta-ab\sin^2\theta&=-xb\cos\theta+ab\cos^2\theta\\xb\cos\theta+ya\sin\theta&=ab\left(\cos^2\theta+\sin^2\theta\right)\\xb\cos\theta+ya\sin\theta&=ab.\end{aligned}$$

Therefore $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

(ii)

$$\begin{aligned}&m_{QR}\\&=\frac{b\sin(\theta-\phi)-b\sin(\theta+\phi)}{a\cos(\theta-\phi)-a\cos(\theta+\phi)}\\&=\frac{b(\sin\theta\cos\phi-\cos\theta\sin\phi-\sin\theta\cos\phi-\cos\theta\sin\phi)}{a(\cos\theta\cos\phi+\sin\theta\sin\phi-\cos\theta\cos\phi+\sin\theta\sin\phi)}\\&=\frac{-2b\cos\theta\sin\phi}{2a\sin\theta\sin\phi}\\&=\frac{-b\cos\theta}{a\sin\theta}\\&=m_{\text{tangent at }P}\quad[\text{by (*)}].\end{aligned}$$

Therefore chord QR is parallel to the tangent at P.

(iii) Let $M=(x_M,y_M)$ be the midpoint of QR.

$$\begin{aligned}\text{Then, }x_M&=\frac{1}{2}\left[a\cos(\theta+\phi)+a\cos(\theta-\phi)\right]\\&=\frac{a}{2}(\cos\theta\cos\phi-\sin\theta\sin\phi\\&\quad+\cos\theta\cos\phi+\sin\theta\sin\phi)\\&=a\cos\theta\cos\phi,\\\text{and }y_M&=\frac{1}{2}\left[b\sin(\theta+\phi)+b\sin(\theta-\phi)\right]\\&=\frac{b}{2}(\sin\theta\cos\phi+\cos\theta\sin\phi\\&\quad+\sin\theta\cos\phi-\cos\theta\sin\phi)\\&=b\sin\theta\cos\phi.\end{aligned}$$

The equation of OP is $y=\frac{b\sin\theta}{a\cos\theta}x$ and

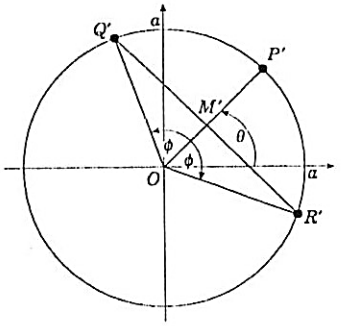
$$\begin{aligned}\text{since }\frac{b\sin\theta}{a\cos\theta}x_M&=\frac{b\sin\theta}{a\cos\theta}a\cos\theta\cos\phi\\&=b\sin\theta\cos\phi\\&=y_M,\end{aligned}$$

M lies on OP and therefore OP bisects QR.

Note that parts (ii) and (iii) can be done using auxiliary circle methods. In the transformation from ellipse to auxiliary circle, x values and changes in x values are preserved. y values and changes in y values are both multiplied by $\frac{a}{b}$.

Hence gradients are multiplied by $\frac{a}{b}$, and so parallelism is preserved.

Also, ratios of intervals on the same line are preserved.



OP' bisects $\angle Q'OR'$, therefore OP' bisects $Q'R'$ and hence OP bisects QR, since ratios are preserved (part iii).

Also since OP' bisects $\angle Q'OR'$, OP' is perpendicular to $Q'R'$. Hence $Q'R'$ is parallel to the tangent at P', since it is also perpendicular to OP'.

Hence QR is parallel to the tangent at P, since parallelism is preserved (part ii).

(b) (i) Net force $=F_{\text{engine}}-F_{\text{resistive}}$

$=F-kv^2\quad(k>0)$.

Therefore

$$\begin{aligned}m\frac{dv}{dt}&=F-kv^2\quad\text{(Newton's law)}\\\frac{dv}{dt}&=\frac{1}{m}\left(F-kv^2\right).\end{aligned}$$

(ii)

$$\begin{aligned}v\frac{dv}{dx}&=\frac{1}{m}\left(F-kv^2\right).\\\frac{dv}{dx}&=\frac{F-kv^2}{mv}\\\frac{dx}{dv}&=\frac{mv}{F-kv^2}.\\\int_{x_1}^{x_2}dx&=\int_{v_1}^{v_2}\frac{mv}{F-kv^2}\,dv,\end{aligned}$$

where $x=x_1$ when $v=v_1$ and $x=x_2$ when $v=v_2$.

$$x_2-x_1=\frac{-m}{2k}\int_{v_1}^{v_2}\frac{-2kv}{F-kv^2}\,dv.$$

$\therefore$ Distance travelled

$$\begin{aligned}&=\frac{-m}{2k}\left[\log_e\left(F-kv^2\right)\right]_{v_1}^{v_2}\text{ assuming }F>kv^2.\\&=\frac{m}{2k}\left[\log_e\left(F-kv^2\right)\right]_{v_2}^{v_1}\\&=\frac{m}{2k}\left[\log_e\left(F-kv_1^2\right)-\log_e\left(F-kv_2^2\right)\right]\\&=\frac{m}{2k}\log_e\left(\frac{F-kv_1^2}{F-kv_2^2}\right).\end{aligned}$$

Note:

The assumption that $F>kv^2$ is reasonable since it is likely that $\frac{dv}{dt}>0$ under full power.

However, if by some chance the submarine is going faster than its normal full speed, then this assumption might be wrong.

A similar working, with $\log_e\left(kv^2-F\right)$ as the integral, would lead to the same result.

(c) (i) *METHOD 1*

There are $\binom{22}{4}$ ways to choose the group of 4.

Then there are $\binom{18}{5}$ ways to choose the group of 5.

Then there are $\binom{13}{6}$ ways to choose the group of 6.

Then there are $\binom{7}{7}=1$ ways to choose the final group of 7.

Therefore the number of ways of choosing the four groups is

$$\begin{aligned}\binom{22}{4}\binom{18}{5}\binom{13}{6}&=\frac{22!}{4!\,18!}\times\frac{18!}{5!\,13!}\times\frac{13!}{6!\,7!}\\&=\frac{22!}{4!\,5!\,6!\,7!}.\end{aligned}$$

METHOD 2

There are 22! ways to arrange the students. But those in the group of 4 can be rearranged in 4! ways, those in the group of 5 can be rearranged in 5! ways, those in the group of 6 can be rearranged in 6! ways, and those in the group of 7 can be rearranged in 7! ways.

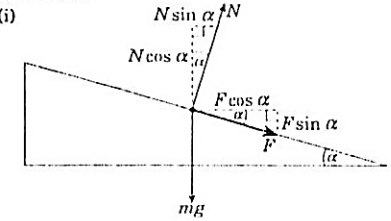
Therefore there are $\frac{22!}{4!\,5!\,6!\,7!}$ ways.

(ii) The students in the group of 4 can be arranged in 4! ways, the students in the group of 5 can be arranged in 5! ways, the students in the group of 6 can be arranged in 6! ways, the students in the group of 7 can be arranged in 7! ways, while the groups can be arranged in $(4-1)!=3!$ ways.

Therefore, there are $3!\,4!\,5!\,6!\,7!$ ways.

QUESTION 6

(a) (i)



Using $\Sigma F=ma$ (Newton's law), we have:

Horizontal components:

$$N\sin\alpha+F\cos\alpha=\frac{mv^2}{r}.$$

Vertical components:

$$N\cos\alpha-F\sin\alpha-mg=0.$$

Rearranging:

$$F\sin\alpha=N\cos\alpha-mg\qquad\text{---①}$$

$$F\cos\alpha=\frac{mv^2}{r}-N\sin\alpha.\qquad\text{---②}$$

(ii) ① $\times\sin\alpha$:

$$F\sin^2\alpha=N\sin\alpha\cos\alpha-mg\sin\alpha.$$

② $\times\cos\alpha$:

$$F\cos^2\alpha=\frac{mv^2}{r}\cos\alpha-N\sin\alpha\cos\alpha.$$

Adding gives:

$$\begin{aligned}F&=\frac{mv^2}{r}\cos\alpha-mg\sin\alpha\\&=m\left(v^2-gr\tan\alpha\right)\frac{\cos\alpha}{r}.\end{aligned}$$

(iii) For no sideways friction, set $F = 0$ at

$$v = 100 \text{ km h}^{-1} = \frac{100}{3{\cdot}6} \text{ m s}^{-1}.$$

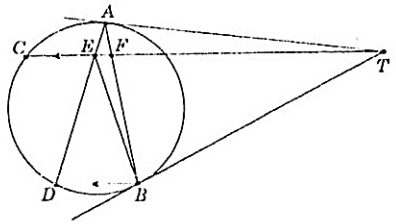
$$v^2 - gr\tan\alpha = 0.$$

$\left(\text{N.B. } \cos\alpha \neq 0 \text{ since } \alpha < \frac{\pi}{2}.\right)$

$$\tan\alpha = \frac{\left(\frac{100}{3{\cdot}6}\right)^2}{200\times 9{\cdot}8}$$

$$\alpha \doteqdot 21{\cdot}49^\circ \quad \left(= 21^\circ 29'\right).$$

(b)



(i) $\angle AET = \angle ADB$ (Corresponding, $CT \parallel DB$)

$\angle ADB = \angle FAT$ ($\angle$ between chord and tangent equals $\angle$ in alternate segment).

$\therefore \angle AET = \angle FAT$ (Both $= \angle ADB$).

Also, $\angle ATF = \angle ETA$ (Common).

So $\Delta TFA \parallel\!\parallel \Delta TAE$ (Same angles).

(ii) $\dfrac{TA}{TE} = \dfrac{TF}{TA}$ [Scale ratios of similar Δs in (i)].

$\therefore TE\cdot TF = TA^2$.

But $TA = TB$ (Tangents from an external point are equal).

$\therefore TE\cdot TF = TB^2$.

(iii) $\angle BTE = \angle FTB$ (Common)

$\dfrac{TE}{TB} = \dfrac{TB}{TF}$, from (ii).

$\therefore \Delta EBT \parallel\!\parallel \Delta BFT$ (2 pairs of corresponding sides are in proportion and their included $\angle$s are equal).

(iv) $\angle EDB = \angle FBT$ ($\angle$ between chord and tangent equals $\angle$ in alternate segment).

$\angle FBT = \angle BET$ (Corresponding, $\Delta EBT \parallel\!\parallel \Delta BFT$)

$\angle BET = \angle EBD$ (Alternate, $CT \parallel DB$)

$\therefore \angle EDB = \angle EBD$

$\therefore \Delta DEB$ is isosceles (Base $\angle$s equal).

QUESTION 7

(a) $z = \frac{1}{2}\left(\cos\theta + i\sin\theta\right)$ where θ is real.

(i) $|z| = \frac{1}{2}$ since $\left|\cos\theta + i\sin\theta\right| = 1$.

(ii)
$$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$$
$$= \frac{2}{2-\cos\theta - i\sin\theta} \times \frac{2-\cos\theta + i\sin\theta}{2-\cos\theta + i\sin\theta}$$
$$= \frac{2(2-\cos\theta + i\sin\theta)}{(2-\cos\theta)^2 + \sin^2\theta}$$
$$= \frac{2(2-\cos\theta) + 2i\sin\theta}{4 - 4\cos\theta + \cos^2\theta + \sin^2\theta}$$
$$= \frac{2(2-\cos\theta) + 2i\sin\theta}{5 - 4\cos\theta}. \qquad (\dagger)$$

Therefore, $\text{Im}\left(\dfrac{1}{1-z}\right) = \dfrac{2\sin\theta}{5-4\cos\theta}$.

(iii)
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$
$$= 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{2^2}(\cos 2\theta + i\sin 2\theta) + \frac{1}{2^3}(\cos 3\theta + i\sin 3\theta) + \cdots$$
(de Moivre's theorem)
$$= \left(1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \cdots\right) + i\left(\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^3}\sin 3\theta + \cdots\right).$$

$\therefore 1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \cdots$
$$= \text{Re}\left(\frac{1}{1-z}\right)$$
$$= \frac{2(2-\cos\theta)}{5-4\cos\theta}, \quad [\text{from } \dagger \text{ in (ii)}].$$

(b) (i) $x^3 - 3x - 1 = 0$. (*)

If $x = \frac{p}{q}$ is a root of $ax^3 - 3x + b = 0$,

then $a\left(\frac{p}{q}\right)^3 - 3\left(\frac{p}{q}\right) + b = 0$

and $ap^3 - 3pq^2 + bq^3 = 0$.

This can be written as either

$$ap^3 = q\left(3pq - bq^2\right). \qquad \text{—①}$$

or $$bq^3 = p\left(3q^2 - ap^2\right). \qquad \text{—②}$$

In ①, $3pq - bq^2$ is an integer and q has no factors in common with p, so q divides a.

In ②, $3q^2 - ap^2$ is an integer and p has no factors in common with q, so p divides b.

If $x = \frac{p}{q}$ is a rational root of (*), then p divides -1 and q divides 1.

The only possibilities are $x = \pm 1$.

Substituting into (*) gives

$x = 1$: LHS $= 1 - 3 - 1 \neq 0$

$x = -1$: LHS $= -1 + 3 - 1 \neq 0$.

Since neither possibility works, (*) does not have a rational root.

(ii) If $r + s\sqrt{d}$ is a root of (*),

then $\left(r + s\sqrt{d}\right)^3 - 3\left(r + s\sqrt{d}\right) - 1 = 0$.

$r^3 + 3r^2s\sqrt{d} + 3rs^2d + s^3d\sqrt{d} - 3r - 3s\sqrt{d} - 1 = 0$.

$\left(r^3 + 3rs^2d - 3r - 1\right) + \left(3r^2s + s^3d - 3s\right)\sqrt{d} = 0$.

The first term is rational so the second term must be too. Since $\sqrt{d}$ is irrational, the only way this can happen is for the second term to be zero.

Now $d \neq 0$, since $\sqrt{d}$ is irrational.

Hence $3r^2s + s^3d - 3s = 0$. —①

Now since the second term is zero, the first term must also be zero.

$r^3 + 3rs^2d - 3r - 1 = 0$. —②

Now substitute $r - s\sqrt{d}$ into (*).

$$\text{LHS} = \left(r - s\sqrt{d}\right)^3 - 3\left(r - s\sqrt{d}\right) - 1$$
$$= r^3 - 3r^2s\sqrt{d} + 3rs^2d - s^3d\sqrt{d} - 3r + 3s\sqrt{d} - 1$$
$$= \left(r^3 + 3rs^2d - 3r - 1\right) - \left(3r^2s + s^3d - 3s\right)\sqrt{d}$$
$$= 0 - 0\times\sqrt{d}, \text{ using ① and ②}$$
$$= 0 = \text{RHS}.$$

Therefore $r - s\sqrt{d}$ is also a root of (*).

Now consider $r + s\sqrt{d}$ where r, s, d are rational and $\sqrt{d}$ may or may not be rational.

If $\sqrt{d}$ is rational, then $r + s\sqrt{d}$ is rational. By (i) this is not possible.

If $\sqrt{d}$ is irrational, then $r - s\sqrt{d}$ is also a root. Let the third root be α and consider the sum of the roots.

Since there is no x^2 term in (*),

$\left(r - s\sqrt{d}\right) + \left(r - s\sqrt{d}\right) + \alpha = 0$

$\therefore \alpha = -2r$.

In this case the third root α is rational.

By (i) this is not possible.

Hence (*) cannot have any root of the form $r + s\sqrt{d}$ where r, s and d are rational.

(iii) Substitute $x = 2\cos\frac{\pi}{9}$ into (*).

$$\text{LHS} = \left(2\cos\frac{\pi}{9}\right)^3 - 3\left(2\cos\frac{\pi}{9}\right) - 1$$
$$= 8\cos^3\frac{\pi}{9} - 6\cos\frac{\pi}{9} - 1$$
$$= 2\left(4\cos^3\frac{\pi}{9} - 3\cos\frac{\pi}{9}\right) - 1$$
$$= 2\cos\frac{3\pi}{9} - 1, \text{ using the given identity,}$$
$$= 2\cos\frac{\pi}{3} - 1$$
$$= 2\times\frac{1}{2} - 1$$
$$= 0 = \text{RHS}.$$

Therefore $2\cos\frac{\pi}{9}$ is a root of (*).

QUESTION 8

(a) (i) $a^2 - 2ab + b^2 = (a-b)^2$

≥ 0, since $a - b$ is real.

$\therefore \quad a^2 + b^2 \geq 2ab$

$2ab \leq a^2 + b^2$.

$$(a+b+c)^2 - 3(ab+bc+ca)$$
$$= a^2 + b^2 + c^2 + 2(ab+bc+ca) - 3(ab+bc+ca)$$
$$= a^2 + b^2 + c^2 - (ab+bc+ca)$$
$$= \frac{1}{2}\left(a^2+b^2\right) + \frac{1}{2}\left(b^2+c^2\right) + \frac{1}{2}\left(c^2+a^2\right) - (ab+bc+ca)$$
$$\geq \frac{1}{2}(2ab) + \frac{1}{2}(2bc) + \frac{1}{2}(2ac) - (ab+bc+ca)$$
$$= 0.$$

That is, $(a+b+c)^2 \geq 3(ab+bc+ca)$

or $3(ab+bc+ca) \leq (a+b+c)^2$.

(ii) *METHOD 1*

If a, b and c are the three sides of a triangle, then $b \leq a + c$ (since the length of any one side is not greater than the sum of the other two).

Therefore, $b - c \leq a$. —①

Similarly, $c \leq a + b$

and therefore $b - c \geq -a$. —②

Combining ① and ② gives

$$-a \leq b - c \leq a$$

$\therefore \quad |b - c| \leq a$.

Squaring both sides gives $(b-c)^2 \leq a^2$.

METHOD 2

In the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$.

Now $2bc \geq 2bc\cos A$ (since $b > 0$, $c > 0$ and $\cos A \leq 1$).

Therefore, $-2bc \leq -2bc\cos A$.

Thus $b^2 + c^2 - 2bc \leq b^2 + c^2 - 2bc\cos A$

$\therefore \quad b^2 + c^2 - 2bc \leq a^2$

$(b-c)^2 \leq a^2$.

Similarly, $(a-c)^2 \leq b^2$

$(a-b)^2 \leq c^2$.

Therefore

$b^2 - 2bc + c^2 \leq a^2$

$a^2 - 2ca + c^2 \leq b^2$

$a^2 - 2ab + b^2 \leq c^2$.

Adding gives

$2\left(a^2 + b^2 + c^2\right) - 2(ab + bc + ca) \leq a^2 + b^2 + c^2$,

and thus $a^2 + b^2 + c^2 \leq 2(ab + bc + ca)$.

But $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$.

Therefore $(a + b + c)^2 - 2(ab + bc + ca) \leq 2(ab + bc + ca)$,

or $(a + b + c)^2 \leq 4(ab + bc + ca)$.

(b) (i) e^x is monotonic increasing, so when $0 \leq x \leq 1$, $e^x \leq e^1 < 3$.

So $\displaystyle\int_0^1 x^\alpha e^x\,dx < \int_0^1 x^\alpha 3\,dx$

$\displaystyle= 3\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1$

$\displaystyle= \frac{3}{\alpha+1}$.

That is, $\displaystyle\int_0^1 x^\alpha e^x\,dx < \frac{3}{\alpha+1}$.

(ii) Consider $n = 0$:

$\displaystyle\int_0^1 x^0 e^x\,dx = \int_0^1 e^x\,dx$

$\displaystyle= \left[e^x\right]_0^1$

$= e - 1$.

So $a_0 = -1$ and $b_0 = 1$, both integers, as required.

Assume the property is true for $n = k$, $k = 0, 1, 2, \ldots$

Then $\displaystyle\int_0^1 x^{k+1}e^x\,dx$

$\displaystyle= \int_0^1 x^{k+1}\left(\frac{d}{dx}e^x\right)dx$

$\displaystyle= \left[x^{k+1}e^x\right]_0^1 - \int_0^1 (k+1)\,x^k e^x\,dx$ (N.B. $k + 1 > 0$.)

$= (e - 0) - (k+1)\left(a_k + b_k e\right)$, by assumption,

$= -(k+1)\,a_k + \left[1 - (k+1)\,b_k\right]e$.

Since k, a_k and b_k are integers, so are $a_{k+1} = -(k+1)\,a_k$ and $b_{k+1} = 1 - (k+1)\,b_k$.

Hence by mathematical induction the property is proven.

(iii) $\left|a + br\right| = \left|a + b\dfrac{p}{q}\right|$

$\displaystyle= \frac{1}{q}\left|qa + bp\right|$, since $q > 0$.

But $qa + bp$ is an integer and thus $\left|qa + bp\right| = 0$ or $\left|qa + bp\right| \geq 1$.

Hence, $\dfrac{1}{q}\left|qa + bp\right| = 0$ or $\dfrac{1}{q}\left|qa + bp\right| \geq \dfrac{1}{q}$.

That is, $\left|a + br\right| = 0$ or $\left|a + br\right| \geq \dfrac{1}{q}$.

(iv) Suppose, by way of contradiction, that e is rational so that $e = \dfrac{p}{q}$, where p and q are positive integers.

$\displaystyle\int_0^1 x^n e^x\,dx = a_n + b_n e$ [a_n, b_n integers, from (ii)]

$= \left|a_n + b_n e\right|$, since the integral is positive, since $x^n e^x > 0$ on $0 < x < 1$

$\displaystyle= \left|a_n + b_n\frac{p}{q}\right|$, by assumption of rationality

$\geq \dfrac{1}{q}$, from (iii).

Now consider the case when $n = 3q$ (or larger).

$\displaystyle\int_0^1 x^n e^x\,dx < \frac{3}{3q+1} < \frac{3}{3q}$

$\displaystyle= \frac{1}{q}$.

This is a contradiction, so the assumption is false. Therefore e is irrational.

END OF MATHEMATICS EXTENSION 2 SOLUTIONS

STANDARD INTEGRALS

$\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}$, $n \neq -1$; $x \neq 0$, if $n < 0$

$\displaystyle\int \frac{1}{x}\,dx = \ln x$, $x > 0$

$\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax}$, $a \neq 0$

$\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax$, $a \neq 0$

$\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax$, $a \neq 0$

$\displaystyle\int \sec^2 ax\,dx = \frac{1}{a}\tan ax$, $a \neq 0$

$\displaystyle\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax$, $a \neq 0$

$\displaystyle\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$, $a \neq 0$

$\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}$, $a > 0$, $-a < x < a$

$\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right)$, $x > a > 0$

$\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x$, $x > 0$