2000 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
4 UNIT (ADDITIONAL) MATHEMATICS

QUESTION 1

(a) Let $u = \sin x$,

$\therefore \dfrac{du}{dx} = \cos x \Rightarrow du = \cos x\, dx.$

$\therefore \displaystyle\int \frac{\cos x}{\sin^4 x}\, dx = \int u^{-4}\, du$

$= \dfrac{u^{-3}}{-3} + C$

$= -\dfrac{1}{3\sin^3 x} + C.$

(b) $\displaystyle\int \frac{4}{x^2+6x+10}\, dx = 4\int \frac{dx}{(x+3)^2+1}$

$= 4\tan^{-1}(x+3) + C.$

(c) (i) $\dfrac{9}{x^2(3-x)} \equiv \dfrac{ax+b}{x^2} + \dfrac{c}{3-x}.$

$\therefore \quad 9 \equiv (3-x)(ax+b) + cx^2.$

Substitute $x = 3$: $9 = 9c$

$c = 1.$

Substitute $x = 0$ (constant term): $9 = 3b$

$b = 3.$

Coefficient of x^2: $0 = -a + c$

$a = c$

$a = 1.$

$\therefore a = 1, b = 3, c = 1.$

(ii) $\displaystyle\int \frac{9}{x^2(3-x)}\, dx = \int \frac{x+3}{x^2} + \frac{1}{3-x}\, dx$

$= \displaystyle\int \frac{1}{x} + 3x^{-2} + \frac{1}{3-x}\, dx$

$= \ln|x| + \dfrac{3x^{-1}}{-1} - \ln|3-x| + C$

$= \ln\left|\dfrac{x}{3-x}\right| - \dfrac{3}{x} + C.$

(d) $\displaystyle\int \sqrt{x}\ln x\, dx = \int \frac{d}{dx}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\ln x\, dx$

$= \dfrac{2}{3}x^{\frac{3}{2}}\ln x - \dfrac{2}{3}\displaystyle\int x^{\frac{3}{2}}\cdot\frac{1}{x}\, dx$

$= \dfrac{2}{3}x\sqrt{x}\ln x - \dfrac{2}{3}\cdot\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \dfrac{2}{3}x\sqrt{x}\ln x - \dfrac{4}{9}x\sqrt{x} + C$

$= \dfrac{2x\sqrt{x}}{9}(3\ln x - 2) + C.$

(e) $t = \tan\dfrac{\theta}{2}$

$\theta = 2\tan^{-1}t$

$d\theta = \dfrac{2\, dt}{1+t^2}.$

$\displaystyle\int \frac{d\theta}{1+\sin\theta+\cos\theta} = \int \frac{1}{\left(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2\, dt}{(1+t^2)}$

$= \displaystyle\int \frac{2\, dt}{1+t^2+2t+1-t^2}$

$= \displaystyle\int \frac{2\, dt}{2+2t}$

$= \displaystyle\int \frac{dt}{1+t}$

$= \ln|1+t| + C$

$= \ln\left|1+\tan\dfrac{\theta}{2}\right| + C.$

QUESTION 2

(a) $(x+iy)^2 = 24 + 10i$

$x^2 - y^2 + 2xyi = 24 + 10i$

$\therefore \quad x^2 - y^2 = 24 \quad ①$

$xy = 5 \quad ②$

Method 1:

x, y must be integers, so by inspection,
$x = 5, y = 1$ or $x = -5, y = -1.$

Method 2:

Substituting $y = \dfrac{5}{x}$ from ② into ①:

$x^2 - \dfrac{25}{x^2} - 24 = 0$

$x^4 - 24x^2 - 25 = 0$

$(x^2-25)(x^2+1) = 0$

$x = \pm 5$ (x is real).

$\therefore\ x = 5, y = 1$ or $x = -5, y = -1$.

(b) $z^2+az+(1+i) = 0$.

Substitute $z = i$, since i is a root.

$i^2+ai+1+i = 0$

$-1+ai+1+i = 0$

$i(a+1) = 0$

$a = -1$.

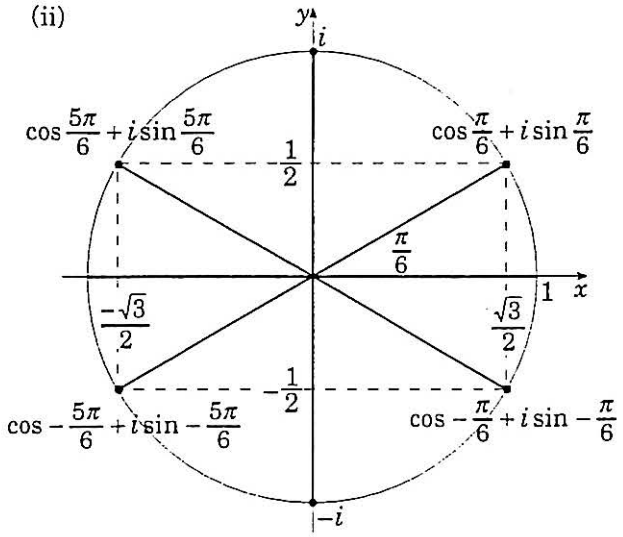
(c) (i) $z = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

$z^6 = \cos\left(6\times\frac{\pi}{6}\right)+i\sin\left(6\times\frac{\pi}{6}\right)$

$= \cos\pi+i\sin\pi$

$= -1$.

(ii)



The six solutions are equally spaced around the unit circle, separated by an angle of $\frac{\pi}{3}$.

(d) **Method 1:**

$z\bar{z} = |z|^2 = x^2+y^2$.

$z+\bar{z} = 2\text{Re}(z) = 2x$.

$z\bar{z}+2(z+\bar{z}) \le 0$

$x^2+y^2+4x \le 0$

$(x+2)^2+y^2 \le 2^2$.

This is a circle, centre (–2, 0) and radius 2.

Method 2:

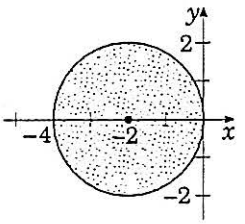
$z\bar{z}+2(z+\bar{z}) \le 0$

$(z+2)(\bar{z}+2) \le 4$

$(z+2)\overline{(z+2)} \le 4$

$|z+2|^2 \le 4$

$|z+2| \le 2$.



(e) (i) $OA = \omega$, $\therefore\ OC = 2i\omega$.

(ii) **Method 1:**

D is the midpoint of AC and therefore OD is the average of OA and OC.

$\therefore\quad OD = \frac{\omega+2i\omega}{2}$

$= \frac{\omega}{2}+i\omega$.

Method 2:

$OB = OA+OC$

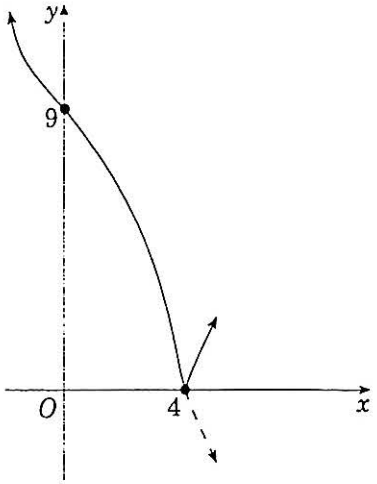
$= \omega+2i\omega$.

$OD = \frac{OB}{2}$

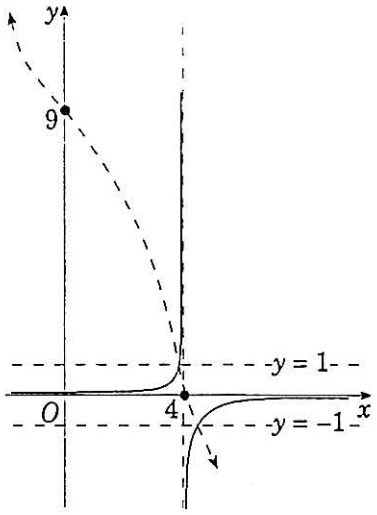
$= \frac{\omega}{2}+i\omega$.

QUESTION 3

(a) (i) $y = |f(x)|$

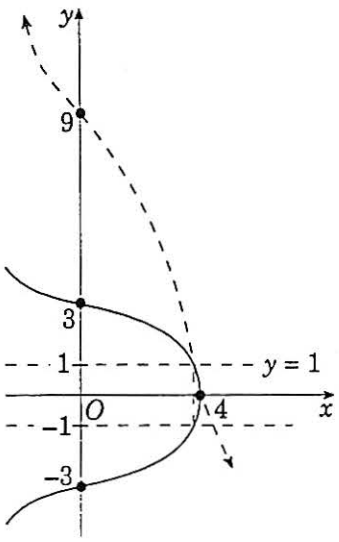


(ii) $y = \frac{1}{f(x)}$

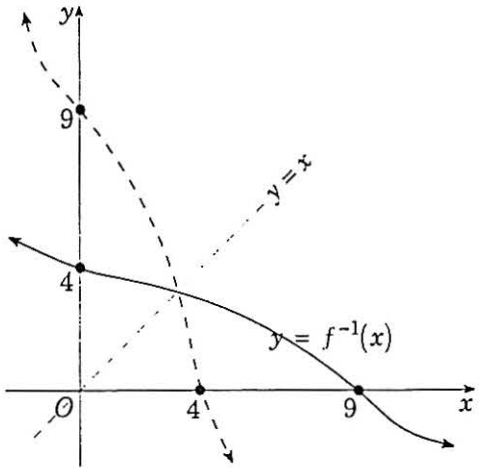


(iii) $y^2 = f(x)$

$y = \pm\sqrt{f(x)}$



(iv)



(b) (i) Area of ellipse $= \pi ab$.

$a = y, \quad b = \frac{y}{2}$.

$\therefore$ Area of semi-ellipse $= \frac{1}{2} \times \pi \times y \times \frac{y}{2}$

$= \frac{\pi y^2}{4}$

$= \frac{\pi \times 4x}{4}$

$= \pi x$.

$\therefore$ When $x = h$, the area $= \pi h$.

(ii) $V = \int_0^4 A \, dx$

$= \int_0^4 \pi x \, dx$

$= \pi\left[\frac{x^2}{2}\right]_0^4$

$= 8\pi$.

(iii) **Method 1:**

The cross-sections for $\mathcal{T}$ will be circles. The semi-ellipse cross-section is half the area of the semicircle, and therefore one-quarter the area of the circle.

This applies equally to all cross-sections. Therefore $\mathcal{S}$ has one-quarter the volume of $\mathcal{T}$.

Method 2: $V_{\mathcal{T}} = \int_0^4 \pi y^2 \, dx$

$= \pi \int_0^4 4x \, dx$

$= \pi\left[2x^2\right]_0^4$

$= 32\pi$

$= 4V_{\mathcal{S}}$.

(c) $n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$

$100! = \sqrt{2\pi}\, 100^{100+\frac{1}{2}} e^{-100}$

$= \sqrt{2\pi}\, e^{-100} \times 10^{201}$.

Time $= \frac{100!}{10^{12}}$ seconds

$= \frac{100!}{10^{12} \times 60^2 \times 24 \times 365\frac{1}{4}}$ years

$= \frac{100!}{6^2 \times 24 \times 365\frac{1}{4} \times 10^{14}}$

$= \frac{\sqrt{2\pi}\, e^{-100}}{6^2 \times 24 \times 365\frac{1}{4}} \times 10^{201-14}$

$= 2{\cdot}9549 \times 10^{-49} \times 10^{187}$

$= 3{\cdot}0 \times 10^{138}$ years (2 sig. figs).

QUESTION 4

(a) (i) $\frac{dy}{dx} = \frac{\frac{dy}{dp}}{\frac{dx}{dp}} = \frac{-\frac{c}{p^2}}{c} = -\frac{1}{p^2}$.

$\therefore$ The normal has gradient p^2.

The equation of the normal is

$y - \frac{c}{p} = p^2(x - cp)$

$py - c = p^3(x - cp)$.

(ii) The x coordinates of P and Q are the simultaneous solutions of

$py - c = p^3(x - cp)$ ①

$xy = c^2$ ②

① $\times x$:

$pxy - cx = p^3x(x - cp)$

$pc^2 - cx = p^3x(x - cp)$, using ②

$\frac{c^2}{p^2} - \frac{c}{p^3}x = x^2 - cpx$.

$\therefore \; x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0.$

(iii) One root of the equation in (ii) is cp ($= x_P$). Using product of roots, the other root, x_Q, satisfies $cp \times x_Q = -\frac{c^2}{p^2}$.

$\therefore \quad x_Q = -\frac{c}{p^3}$.

$y_Q = \frac{c^2}{x_Q} = -\frac{c^2 p^3}{c} = -cp^3$.

$\therefore\ Q = \left(-\frac{c}{p^3}, -cp^3\right)$.

$\angle QRP$ will be a right angle if $m_{QR} \times m_{RP} = -1$.

$m_{RP} = \frac{\frac{c}{p}}{cp} = \frac{1}{p^2}$.

$R = \left(-cp, -\frac{c}{p}\right)$.

$\therefore \quad m_{QR} = \frac{-cp^3 - \left(-\frac{c}{p}\right)}{-\frac{c}{p^3} - (-cp)}$

$= \frac{-cp^6 + cp^2}{-c + cp^4}$

$= \frac{cp^2(1-p^4)}{-c(1-p^4)}$

$= -p^2$.

$\therefore\ m_{QR} \times m_{RP} = \frac{1}{p^2} \times (-p^2)$

$= -1$ as required.

(b) $\frac{dT_1}{dt} = -k(T_1 - T_2)$ ①

$\frac{dT_2}{dt} = \frac{3}{4}k(T_1 - T_2)$. ②

(i) Consider $\frac{3}{4}T_1 + T_2$:

$\frac{d}{dt}\left(\frac{3}{4}T_1 + T_2\right) = \frac{3}{4}\frac{dT_1}{dt} + \frac{dT_2}{dt}$

$= -\frac{3}{4}k(T_1 - T_2) + \frac{3}{4}k(T_1 - T_2)$

$= 0$.

$\therefore\ \frac{3}{4}T_1 + T_2 = C$ for some constant C. ③

(ii) Using $T_2 = C - \frac{3}{4}T_1$ from ③, equation ① becomes $\frac{dT_1}{dt} = -k\left(T_1 - C + \frac{3}{4}T_1\right)$

$= kC - \frac{7}{4}kT_1$. ④

To show that $T_1 = \frac{4}{7}C + Be^{-\frac{7}{4}kt}$ satisfies ④, we substitute it in:

LHS $= \frac{d}{dt}\left(\frac{4}{7}C + Be^{-\frac{7}{4}kt}\right)$

$= 0 - \frac{7}{4}kBe^{-\frac{7}{4}kt}$

$= -\frac{7}{4}kBe^{-\frac{7}{4}kt}$.

RHS $= kC - \frac{7}{4}k\left(\frac{4}{7}C + Be^{-\frac{7}{4}kt}\right)$

$= kC - kC - \frac{7}{4}kBe^{-\frac{7}{4}kt}$

$= -\frac{7}{4}kBe^{-\frac{7}{4}kt}$ = LHS.

(iii) At $t = 0$, $T_1 = 120$ and $T_2 = 22$.

$\therefore \quad C = \frac{3}{4}T_1 + T_2$

$= \frac{3}{4} \times 120 + 22$

$= 112$.

Also, $T_1 = \frac{4}{7}C + Be^{-\frac{7}{4}kt}$ becomes (at $t = 0$)

$120 = \frac{4}{7} \times 112 + B$

$B = 56$.

$\therefore \quad T_1 = 64 + 56e^{-\frac{7}{4}kt}$.

At $t = 10$, $T_1 = 90$.

$90 = 64 + 56e^{-\frac{7}{4}k \times 10}$

$e^{-\frac{7}{4}k \times 10} = \frac{13}{28}$

$k = -\frac{2}{35}\ln\left(\frac{13}{28}\right)$

$= 0{\cdot}438\,431$,

but note that it is not necessary to find k.

At $t = 20$, $T_1 = 64 + 56e^{-\frac{7}{4}k \times 20}$

$= 64 + 56\left(e^{-\frac{7}{4}k \times 10}\right)^2$

$= 64 + 56 \times \left(\frac{13}{28}\right)^2$

$= 76\frac{1}{14}^\circ$ ($= 76{\cdot}07^\circ$ to 2 d.p.).

QUESTION 5

(a) (i) $p(\alpha) = 0$.

$a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + e = 0$

$e = -(a\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha)$

$= -\alpha(a\alpha^3 + b\alpha^2 + c\alpha + d)$.

a, b, c, d and α are integers, so the expression in the brackets is an integer. Therefore α divides e.

(ii) $q(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$.

From (i), an integer zero (root) must divide -3. The only possibilities are ± 1 and ± 3. We test all four.

$q(1) = 4 - 1 + 3 + 2 - 3 = 5 \neq 0$.

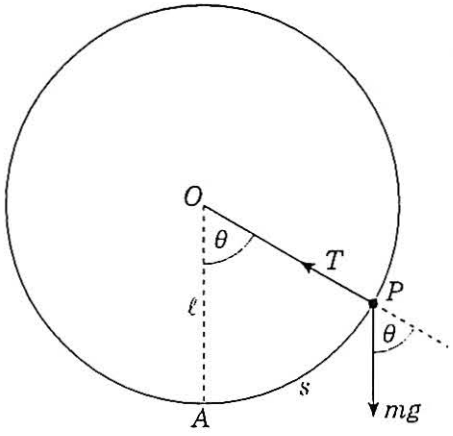
$q(-1) = 4 + 1 + 3 - 2 - 3 = 3 \neq 0$.

$q(3) = 324 - 27 + 27 + 6 - 3 = 327 \neq 0$.

$q(-3) = 324 + 27 + 27 - 6 - 3 = 369 \neq 0$.

Since none of these is a zero, $q(x)$ does not have an integer zero.

(b)



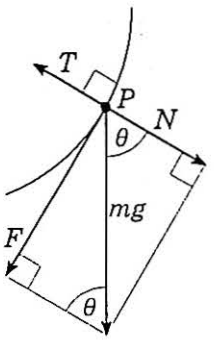
(i) **Method 1:** $s = \ell\theta$, $v = \dfrac{ds}{dt} = \ell\dfrac{d\theta}{dt}$.

$$\begin{aligned}\frac{d^2s}{dt^2} &= \frac{dv}{dt}\\ &= \frac{dv}{d\theta}\cdot\frac{d\theta}{dt}\\ &= \frac{dv}{d\theta}\cdot\frac{v}{\ell}\\ &= \frac{1}{\ell}\frac{dv}{d\theta}\frac{d\left(\frac{1}{2}v^2\right)}{dv}\\ &= \frac{1}{\ell}\frac{d}{d\theta}\left(\frac{1}{2}v^2\right).\end{aligned}$$

Method 2: $s = \ell\theta$, $\dfrac{ds}{d\theta} = \ell$.

$$\begin{aligned}\text{RHS} &= \frac{1}{\ell}\cdot\frac{d\left(\frac{1}{2}v^2\right)}{d\theta}\\ &= \frac{1}{\ell}\cdot\frac{d\left(\frac{1}{2}v^2\right)}{dv}\cdot\frac{dv}{d\theta}\\ &= \frac{1}{\ell}\cdot v\cdot\frac{dv}{d\theta}\\ &= \frac{1}{\ell}\cdot\frac{ds}{dt}\cdot\frac{dv}{d\theta}\\ &= \frac{1}{\ell}\cdot\frac{ds}{d\theta}\cdot\frac{dv}{dt}\\ &= \frac{1}{\ell}\cdot\ell\cdot\frac{d^2s}{dt^2}\\ &= \frac{d^2s}{dt^2}.\end{aligned}$$

(ii)



From the diagram $F = mg\sin\theta$ is the only tangential force, but this is in the opposite direction to the one in which s increases.

Thus $\Sigma F = ma$ (Newton's law) becomes

$$-mg\sin\theta = m\left[\frac{1}{\ell}\cdot\frac{d}{d\theta}\left(\frac{1}{2}v^2\right)\right] \text{ from (i)}$$

$$\frac{1}{\ell}\cdot\frac{d}{d\theta}\left(\frac{1}{2}v^2\right) = -g\sin\theta.$$

(iii) From (ii), we have $\dfrac{d}{d\theta}\left(\frac{1}{2}v^2\right) = -g\ell\sin\theta$

$$\frac{1}{2}v^2 = g\ell\cos\theta + C.$$

At $\theta = 0$, $v = V$, so $\dfrac{1}{2}V^2 = g\ell + C$

$$C = \frac{1}{2}V^2 - g\ell.$$

So $\dfrac{1}{2}v^2 = g\ell\cos\theta + \dfrac{1}{2}V^2 - g\ell$

$$\begin{aligned}V^2 &= v^2 + 2g\ell - 2g\ell\cos\theta\\ &= v^2 + 2g\ell(1-\cos\theta).\end{aligned}$$

(iv) From the diagram in (ii), we have $N = mg\cos\theta$ and so the normal forces on P directed towards the centre O are $\Sigma F = T - N$

$$= T - mg\cos\theta.$$

The force required to keep P moving in a circle is $\dfrac{1}{\ell}mv^2$, so $T - mg\cos\theta = \dfrac{1}{\ell}mv^2$.

(v) Substituting $V^2 = 3g\ell$ in (iii), we have

$$\begin{aligned}3g\ell &= v^2 + 2g\ell(1-\cos\theta)\\ g\ell &= v^2 - 2g\ell\cos\theta\\ v^2 &= g\ell(1+2\cos\theta).\end{aligned}$$

Substituting this into (iv) gives

$$\begin{aligned}T - mg\cos\theta &= \frac{1}{\ell}mg\ell(1+2\cos\theta)\\ T &= mg\cos\theta + mg(1+2\cos\theta)\\ &= mg(1+3\cos\theta).\end{aligned}$$

When $T = 0$, $\cos\theta = -\dfrac{1}{3}$,

$\therefore$ $\theta = 1{\cdot}911$ radians $(109° 28')$.

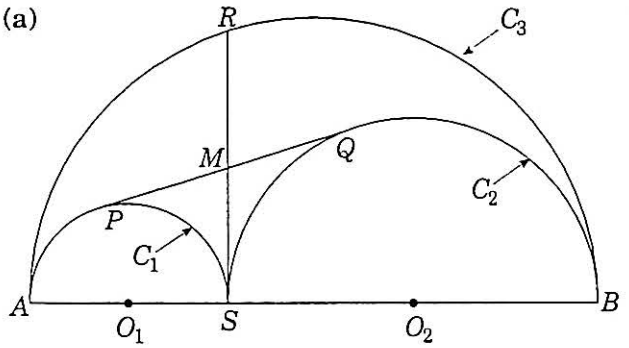
(vi) From the point at which $T = 0$, P will follow a parabolic arc (projectile motion) within the circle. Its 'initial' velocity will be tangential to the circle, with a magnitude given by

$$v^2 = g\ell\left(1-\frac{2}{3}\right)$$

$$v = \sqrt{\frac{g\ell}{3}}.$$

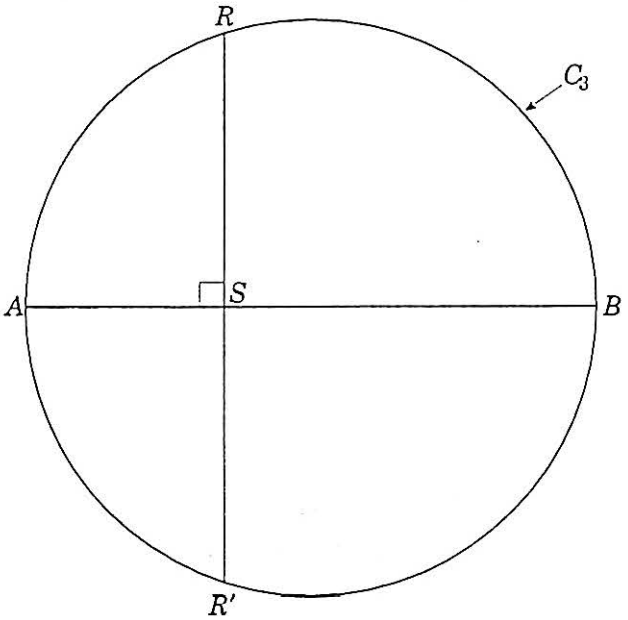
QUESTION 6

(a)



(i) $MP = MS$ (tangents to C_1 from common pt. M)
$MQ = MS$ (tangents to C_2 from common pt. M)
$\therefore\ MP = MS = MQ.$

(ii) Consider the whole of circle C_3.



Let R' be the point where RS produced meets C_3. AB is a diameter, and $AB \perp RR'$, therefore AB bisects RR'.
Hence $RS = SR'$.
By the intersecting chords theorem, we have $RS \cdot SR' = AS \cdot SB$.
that is, $RS^2 = (2r_1)\cdot(2r_2)$
$= 4r_1r_2.$

(iii) O_1M bisects $\angle PMS$.
O_2M bisects $\angle SMQ$.
So $\angle PMQ = 2\angle O_1MO_2$,
but $\angle PMQ = 180°$ (straight $\angle$),
so $\angle O_1MO_2 = 90°$.

Method 1:
Hence there is a circle with O_1O_2 as the diameter and passing through M.
By a similar argument to that used in (ii), using the intersecting chords theorem,

$$MS^2 = O_1S \cdot O_2S$$
$$= r_1r_2.$$

Method 2:
$\Delta O_1MS \parallel\!\parallel \Delta O_1O_2M$ (right-angled, $\angle MO_1S$ common)
$\Delta O_2MS \parallel\!\parallel \Delta O_2O_1M$ (right-angled, $\angle MO_2S$ common)
$\therefore\ \Delta O_1MS \parallel\!\parallel \Delta MO_2S.$

$\frac{MS}{O_1S} = \frac{O_2S}{MS}$ (corres. ratios in similar Δs),
$\therefore\ MS^2 = O_1S \cdot O_2S = r_1r_2.$

(iv) From (ii) and (iii) we have $MS^2 = \frac{1}{4}RS^2$

$$MS = \frac{1}{2}RS.$$

Therefore M is the midpoint of RS and $MS = MR$. Combining this with (i), we have $MP = MS = MQ = MR$.
Thus $PSQR$ is a quadrilateral with equal diagonals that bisect each other.
Hence it is a rectangle.

(b) (i) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}x\right]_0^{\frac{1}{2}}$
$= \sin^{-1}\frac{1}{2}$
$= \frac{\pi}{6}.$

(ii) For $n \geq 2$ and $0 \leq x \leq \frac{1}{2}$,
$0 \leq x^n \leq x^2$
$1 \geq 1-x^n \geq 1-x^2$
$1 \geq \sqrt{1-x^n} \geq \sqrt{1-x^2}$
$1 \leq \frac{1}{\sqrt{1-x^n}} \leq \frac{1}{\sqrt{1-x^2}}.$

$$\int_0^{\frac{1}{2}} 1\,dx \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$$

$$\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \frac{\pi}{6}.$$

QUESTION 7

(a) (i) **Method 1:**

Let $y = \ln x$.

$$y' = \frac{1}{x}$$

$$y'' = -\frac{1}{x^2}.$$

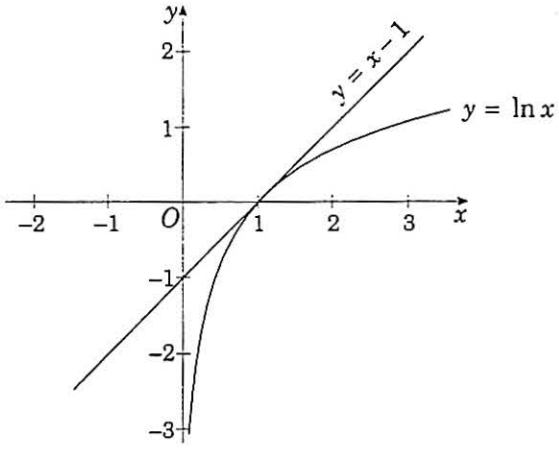
The tangent at $x = 1$ has gradient $m = \frac{1}{1} = 1$, and so equation is $y - 0 = 1(x - 1)$

$$y = x - 1.$$

Since $y'' < 0$ $(x > 0)$, the curve is concave down.

Since it is continuous and differentiable on $x > 0$ it follows that the curve is below any tangent except at the point of contact, which in this case is $x = 1$.

Therefore $\ln x \le x - 1$ for all $x > 0$, with equality at $x = 1$ only.



Method 2:

Let $y = \ln x - x + 1$.

$$y' = \frac{1}{x} - 1$$

$$y'' = -\frac{1}{x^2}.$$

Since $y'' < 0$, the curve is concave down and any stationary point must be a maximum.

Stationary point where $y' = 0$:

$$\frac{1}{x} - 1 = 0$$

$$x = 1.$$

$\therefore\ y = \ln 1 - 1 + 1 = 0.$

Therefore the curve has a maximum value of 0 at $x = 1$.

$\therefore\ \ln x - x + 1 \le 0$

$$\ln x \le x - 1,$$

with equality at $x = 1$ only.

(ii) When $x_i > 0$, $y_i > 0$ for $i = 1, 2, \ldots, n$

$$\frac{y_i}{x_i} > 0 \text{ for } i = 1, 2, \ldots, n.$$

$\therefore$ From (i) we deduce that

$$\ln\left(\frac{y_i}{x_i}\right) \le \frac{y_i}{x_i} - 1 \text{ for } i = 1, 2, \ldots, n,$$

with equality holding only when $\frac{y_i}{x_i} = 1$, that is, $y_i = x_i$.

$$\therefore\ \sum_{i=1}^{n} x_i \ln\left(\frac{y_i}{x_i}\right) \le \sum_{i=1}^{n} x_i\left(\frac{y_i}{x_i} - 1\right)$$

$$= \sum_{i=1}^{n}(y_i - x_i)$$

$$= \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i$$

$$= 1 - 1, \text{ since } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1,$$

$$= 0.$$

Equality in the summation requires equality for each term, and hence $y_i = x_i$ for all i.

(iii) Let $y_i = \frac{1}{n}$ for $i = 1, 2, \ldots, n$.

Then $\frac{y_i}{x_i} = \frac{1}{nx_i}$.

From (ii), we get $\sum_{i=1}^{n} x_i \ln\left(\frac{1}{nx_i}\right) \le 0$

$$\sum_{i=1}^{n} x_i\left[\ln\left(\frac{1}{x_i}\right) + \ln\left(\frac{1}{n}\right)\right] \le 0$$

$$\sum_{i=1}^{n} x_i \ln\left(\frac{1}{x_i}\right) - \sum_{i=1}^{n} x_i \ln n \le 0$$

$$\sum_{i=1}^{n} x_i \ln\left(\frac{1}{x_i}\right) \le \ln n \sum_{i=1}^{n} x_i$$

$$= \ln n \times 1.$$

Since equality is possible if $x_i = y_i = \frac{1}{n}$ for all i, it follows that $\sum_{i=1}^{n} x_i \ln\left(\frac{1}{x_i}\right)$ has a maximum value of $\ln n$.

(iv) By the change of base rule,

$$\sum_{i=1}^{n} x_i \log_2\left(\frac{1}{x_i}\right) = \sum_{i=1}^{n} x_i \frac{\ln\left(\frac{1}{x_i}\right)}{\ln 2}$$

$$= \frac{1}{\ln 2}\sum_{i=1}^{n} x_i \ln\left(\frac{1}{x_i}\right).$$

From (iii), this has a maximum value of

$$\frac{1}{\ln 2}\ln n = \log_2 n.$$

Therefore the result in (iii) still holds if ln is replaced by $\log_2$.

(b) (i) $QS + QR > RS$ (triangle inequality)

$QS + QR + RS' > RS + RS'$.

But $QR + RS' = S'Q$

$QS + S'Q > S'R + RS$.

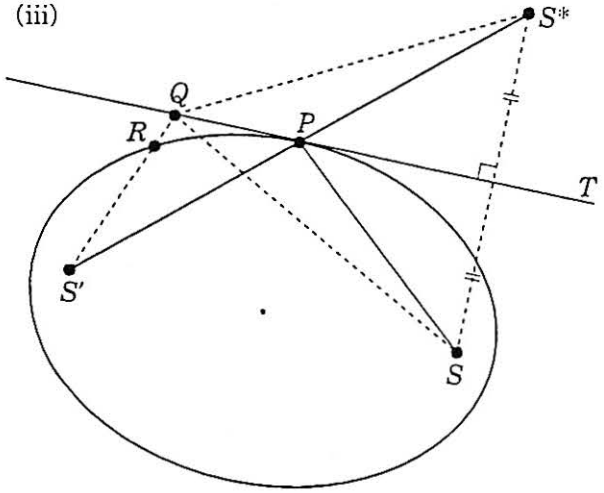
(ii) $S'P + PS = S'R + RS$ (sum of distances from point on an ellipse to foci is constant)

$< S'Q + QS$, from (i).

This applies to all $Q \neq P$ on the tangent.

So the shortest path is through P.

(iii)



QPT is the perpendicular bisector of SS^*, so $SP = SP^*$ and $SQ = SQ^*$.

$S'P + PS^* = S'P + PS$ $(SP^* = SP)$

$< S'Q + QS$, from (iii), for all $Q \neq P$,

$= S'Q + QS^*$.

A path from S' to S^* must pass through the tangent, so $S'PS^*$ is the shortest path from S' to S^*.

Hence $S'PS^*$ is a straight line. Therefore $\angle QPS' = \angle TPS^*$ (vertically opposite $\angle$s).

But $\angle TPS^* = \angle TPS$ (altitude PT bisects $\angle S^*PS$ in isosceles ΔS^*PS)

$\therefore \quad \angle QPS' = \angle TPS$.

QUESTION 8

(a) (i) $\displaystyle\sum_{k=1}^{n}\left(z + z^2 + \cdots + z^k\right)$

$\displaystyle= \sum_{k=1}^{n} \frac{z\left(1 - z^k\right)}{1 - z}, \qquad z \neq 1$

$\displaystyle= \frac{z}{1-z}\sum_{k=1}^{n}\left(1 - z^k\right)$

$\displaystyle= \frac{z}{1-z}\left(\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} z^k\right)$

$\displaystyle= \frac{z}{1-z}\left[n - \left(z + z^2 + \cdots + z^n\right)\right]$

$\displaystyle= \frac{z}{1-z}\left[n - \frac{z\left(1 - z^n\right)}{1-z}\right]$

$\displaystyle= \frac{nz}{1-z} - \frac{z^2\left(1 - z^n\right)}{(1-z)^2}$.

(ii) When $z = \cos\theta + i\sin\theta$,

$z^k = (\cos\theta + i\sin\theta)^k$

$= \cos k\theta + i\sin k\theta$,

from de Moivre's theorem.

Therefore $\mathrm{Im}\left(z^k\right) = \sin k\theta$.

Now using (i):

$\mathrm{Im}(\mathrm{LHS}) = \mathrm{Im}\displaystyle\sum_{k=1}^{n}\left(z + z^2 + \cdots + z^k\right)$

$\displaystyle= \sum_{k=1}^{n}\left[\mathrm{Im}(z) + \mathrm{Im}\left(z^2\right) + \cdots + \mathrm{Im}\left(z^k\right)\right]$

$\displaystyle= \sum_{k=1}^{n}(\sin\theta + \sin 2\theta + \cdots + \sin k\theta)$.

Now $\dfrac{z}{1-z} = \dfrac{i}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$, (given in question)

$\dfrac{z^2}{(1-z)^2} = \dfrac{i^2}{4\sin^2\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^2$

$= \dfrac{-1}{4\sin^2\frac{\theta}{2}}(\cos\theta + i\sin\theta)$.

$\dfrac{z^2}{(1-z)^2}\cdot z^n$

$= \dfrac{-1}{4\sin^2\frac{\theta}{2}}(\cos\theta + i\sin\theta)(\cos n\theta + i\sin n\theta)$

$= \dfrac{-\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]}{4\sin^2\frac{\theta}{2}}$.

$\therefore \quad \mathrm{Im}\left(\dfrac{nz}{1-z}\right) = \dfrac{n\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$

$\mathrm{Im}\left[\dfrac{z^2}{(1-z)^2}\right] = \dfrac{-\sin\theta}{4\sin^2\frac{\theta}{2}}$

$\mathrm{Im}\left[\dfrac{z^2}{(1-z)^2}\cdot z^n\right] = \dfrac{-\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$.

Again using (i):

$\mathrm{Im}(\mathrm{RHS})$

$= \mathrm{Im}\left(\dfrac{nz}{1-z}\right) - \mathrm{Im}\left(\dfrac{z^2}{1-z^2}\right) + \mathrm{Im}\left(\dfrac{z^2 z^n}{1-z^2}\right)$

$= \dfrac{n\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \dfrac{\sin\theta}{4\sin^2\frac{\theta}{2}} - \dfrac{\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$

$$= \frac{n\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\theta - \sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{n\sin\theta + \sin\theta - \sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}.$$

Hence $\sum_{k=1}^{n}(\sin\theta + \sin 2\theta + \cdots + \sin k\theta)$

$$= \frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}.$$

(b) (i) For $P(r, s)$ we have to obtain r heads before s tails. Consider the situation after the first coin is tossed. If it was a head, we now have to obtain only $r-1$ heads before s tails, $(P(r-1, s))$.

However if it was a tail we still have to obtain r heads, but now it is before $s-1$ tails, $(P(r, s-1))$.

$\therefore\ P(r, s)$

$= P(H)\cdot P(r-1, s) + P(T)\cdot P(r, s-1)$

$= \frac{1}{2}P(r-1, s) + \frac{1}{2}P(r, s-1).$

(ii) **Method 1:**

$P(1,3) = P(H \text{ or } TH \text{ or } TTH)$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$= \frac{7}{8}.$

$P(2,2) = P(HH \text{ or } HTH \text{ or } THH)$

$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$

$= \frac{1}{2}.$

$P(2,3) = \frac{1}{2}P(1,3) + \frac{1}{2}P(2,2)$, from (i)

$= \frac{1}{2}\times\frac{7}{8} + \frac{1}{2}\times\frac{1}{2}$

$= \frac{11}{16}.$

Method 2:

$P(0, n) = 1$ and $P(n, 0) = 0$ for all positive integers n. Therefore, using (i) repeatedly:

$P(1,1) = \frac{1}{2}P(1,0) + \frac{1}{2}P(0,1)$

$= \frac{1}{2}\times 0 + \frac{1}{2}\times 1$

$= \frac{1}{2}.$

$P(1,2) = \frac{1}{2}P(1,1) + \frac{1}{2}P(0,2)$

$= \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 1$

$= \frac{3}{4}.$

$P(2,1) = \frac{1}{2}P(2,0) + \frac{1}{2}P(1,1)$

$= \frac{1}{2}\times 0 + \frac{1}{2}\times\frac{1}{2}$

$= \frac{1}{4}.$

$P(2,2) = \frac{1}{2}P(2,1) + \frac{1}{2}P(1,2)$

$= \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times\frac{3}{4}$

$= \frac{1}{2}.$

$P(1,3) = \frac{1}{2}P(1,2) + \frac{1}{2}P(0,3)$

$= \frac{1}{2}\times\frac{3}{4} + \frac{1}{2}\times 1$

$= \frac{7}{8}.$

$P(2,3) = \frac{1}{2}P(2,2) + \frac{1}{2}P(1,3)$

$= \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{7}{8}$

$= \frac{11}{16}.$

These calculations can be summarised in the following table. Each entry is the average of the entries above and to the left.

		s				
		0	1	2	3	4
r	0	×	1	1	1	1
	1	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$
	2	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{11}{16}$	$\frac{13}{16}$
	3	0	$\frac{1}{8}$	$\frac{5}{16}$	$\frac{1}{2}$	$\frac{21}{32}$

(iii) It is required to prove that for $n = r+s-1$,

$$P(r, s) = \frac{1}{2^n}\left\{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s-1}\right\}$$

for $s \geq 1$.

Note that this inductive proof is based on the diagonals of the table given in (ii).

At each step in the proof, we must consider all the values of r and s for which

$$r + s - 1 = n.$$

The proof can be grounded at $n = 0$. For $n = 0$ there is only one case to be considered, namely $r = 0, s = 1$.

$$\text{LHS} = P(0, 1) = 1.$$

$$\text{RHS} = \frac{1}{2^0}\binom{0}{0}$$
$$= \frac{1}{1} \times 1$$
$$= 1.$$

Thus the formula is correct for $n = 0$.

If you choose to ground the proof at $n = 1$ you need to consider two cases, namely $r = 0, s = 2$ and $r = 1, s = 1$.

Case 1: $r = 0$ and $s = 2$

$$\text{LHS} = P(0, 2) = 1.$$

$$\text{RHS} = \frac{1}{2^1}\left\{\binom{1}{0} + \binom{1}{1}\right\}$$
$$= \frac{1}{2}(1 + 1)$$
$$= 1.$$

Case 2: $r = 1$ and $s = 1$

$$\text{LHS} = P(1, 1) = \frac{1}{2}, \quad \text{from (ii).}$$

$$\text{RHS} = \frac{1}{2^1}\left\{\binom{1}{0}\right\}$$
$$= \frac{1}{2} \times 1$$
$$= \frac{1}{2}.$$

Thus the formula is correct for all cases for which $n = 1$.

Assume the formula is true for $n = k$ (a positive integer). That is, assume

$$P(r, s) = \frac{1}{2^k}\left\{\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{s-1}\right\}$$

for $r + s - 1 = k, \ s \geq 1$.

This assumption includes all the values of r and s in this table.

s	1	2	3	$\cdots$	$k+1$
r	k	$k-1$	$k-2$	$\cdots$	0

We need to show that the formula is true for all the values of r and s for which $n = r + s - 1 = k + 1$. We consider two cases, namely $r = 0$ and $r \geq 1$.

Case 1: $r = 0, \ s = k + 2$

$$\text{LHS} = P(0, k+2) = 1.$$

$$\text{RHS} = \frac{1}{2^{k+1}}\left\{\binom{k+1}{0} + \binom{k+1}{1} + \cdots + \binom{k+1}{k+1}\right\},$$
$$= \frac{2^{k+1}}{2^{k+1}} \qquad (\text{since } s - 1 = k + 1),$$
$$= 1.$$

Case 2: $r \geq 1, \ s = k + 2 - r$

Since $r, s \geq 1$, we can use the formula from (i), noting that the assumption applies to both terms on the RHS.

$$P(r, s)$$
$$= \frac{1}{2}P(r-1, s) + \frac{1}{2}P(r, s-1)$$
$$= \frac{1}{2} \times \frac{1}{2^k}\left\{\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{s-1}\right\}$$
$$+ \frac{1}{2} \times \frac{1}{2^k}\left\{\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{s-2}\right\},$$
by assumption,
$$= \frac{1}{2^{k+1}}\left\{\binom{k}{0} + \left[\binom{k}{0} + \binom{k}{1}\right] + \left[\binom{k}{1} + \binom{k}{2}\right]\right.$$
$$\left. + \cdots + \left[\binom{k}{s-2} + \binom{k}{s-1}\right]\right\}$$
$$= \frac{1}{2^{k+1}}\left\{\binom{k}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \cdots + \binom{k+1}{s-1}\right\},$$
since $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$,
$$= \frac{1}{2^{k+1}}\left\{\binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2}\right.$$
$$\left. + \cdots + \binom{k+1}{s-1}\right\},$$
since $\binom{k}{0} = \binom{k+1}{0} = 1$.

This is the given formula.

We have shown that if the formula is true for all values of r and s for which $n = k$, then it is also true for all values of r and s for which $n = k + 1$.

We have also shown that it is true for the only case when $n = 0$. Therefore it is true for both cases when $n = 1$, and hence for all three cases when $n = 2$, and so on.

By the principle of mathematical induction, the formula is true for all cases for which n is a non-negative integer, and hence for all non-negative integers r and positive integers s.

END OF 4 UNIT (ADDITIONAL) SOLUTIONS

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$