1999 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
4 UNIT (ADDITIONAL) MATHEMATICS

QUESTION 1

(a) $\displaystyle\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$

$\displaystyle = -\frac{1}{2}\Big[e^{-x^2}\Big]_0^1$

$\displaystyle = -\frac{1}{2}\left(e^{-1}-e^{0}\right)$

$\displaystyle = \frac{1}{2}\left(1-\frac{1}{e}\right)$

$\displaystyle = \frac{e-1}{2e}.$

(b) $u = e^x,\ du = e^x\,dx$

$\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \sin^{-1}u + C$

$= \sin^{-1}\left(e^x\right) + C.$

(c) $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx = \int \frac{2x^2(2x-1)+1}{2x-1}$

$\displaystyle = \int 2x^2 + \frac{1}{2x-1}\,dx$

$\displaystyle = \frac{2x^3}{3} + \frac{1}{2}\ln|2x-1| + C.$

(d) (i) $\displaystyle\frac{x^2+2x}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}$

$\Rightarrow\ x^2+2x = (ax+b)(x-2) + c(x^2+4)$ —*

$= ax^2 - 2ax + bx - 2b + cx^2 + 4c$

$= (a+c)x^2 + x(b-2a) + 4c - 2b.$

Equating coefficients:

$1 = a+c \ \Rightarrow\ c = 1-a$ —①

$2 = b-2a \ \Rightarrow\ b = 2+2a$ —②

$0 = 4c-2b \ \Rightarrow\ 2b = 4c$

$b = 2c$ —③

Substituting ③ into ②:

$2c = 2+2a$

$c = 1+a$ —④

But $c = 1-a$, from ①

$\therefore\ a = 0,\ b = 2 \Rightarrow c = 1.$

Alternatively, substituting $x = 2$ into *

gives $8 = 8c$

$\therefore\ c = 1,$

and $a = 0,\ b = 2$ follows from ① and ②.

(ii) $\displaystyle\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$

$\displaystyle = \int \frac{2}{x^2+4} + \frac{1}{x-2}\,dx$

$\displaystyle = 2\times\frac{1}{2}\tan^{-1}\frac{x}{2} + \ln|x-2| + C$

$\displaystyle = \tan^{-1}\frac{x}{2} + \ln|x-2| + C.$

(e) $\displaystyle\int_0^{\frac{\pi}{2}} x^2\sin x\,dx$

$\displaystyle = \int_0^{\frac{\pi}{2}} x^2\cdot\frac{d}{dx}(-\cos x)\,dx$

$\displaystyle = \Big[-x^2\cos x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x(-\cos x)\,dx$

$\displaystyle = 0 + \int_0^{\frac{\pi}{2}} 2x\frac{d}{dx}(\sin x)\,dx$

$\displaystyle = \Big[2x\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin x\,dx$

$\displaystyle = \left(2\cdot\frac{\pi}{2}\cdot\sin\frac{\pi}{2} - 0\right) + 2\Big[\cos x\Big]_0^{\frac{\pi}{2}}$

$\displaystyle = \pi + 2\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= \pi - 2.$

QUESTION 2

(a) $z = 3+2i,\quad w = -1+i$

(i) $zw = (3+2i)(-1+i)$

$= -3+3i-2i+2i^2$

$= -5+i.$

(ii) $\displaystyle\frac{2}{iw} = \frac{2}{-i+i^2} = \frac{2}{-1-i}\times\frac{-1+i}{-1+i}$

$\displaystyle = \frac{-2+2i}{(-1)^2-i^2}$

$\displaystyle = \frac{-2+2i}{2}$

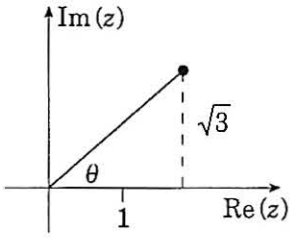
$= -1+i.$

(b) $\alpha = 1+i\sqrt{3}$

(i) $|\alpha| = \sqrt{1^2+\left(\sqrt{3}\right)^2}$

$= \sqrt{4}$

$= 2.$



Let $\theta = \arg\alpha$

$\tan\theta = \sqrt{3}$

$\arg\alpha = \frac{\pi}{3}.$

(ii) $\alpha^{11} = 2^{11}\left(\cos\frac{11\pi}{3}+i\sin\frac{11\pi}{3}\right)$

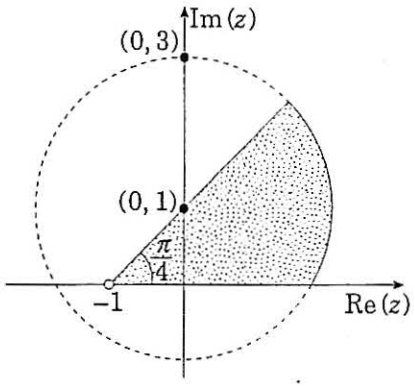
$= 2^{11}\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$

$= 2^{11}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

$= 2^{10}-i2^{10}\sqrt{3}.$

$\therefore \quad a = 2^{10},\ b = -2^{10}\sqrt{3}.$

(c)



(d) $2z^3-3z^2+18z+10 = 0$

(i) Given that $1-3i$ is a root

$\Rightarrow 1+3i$ is a root, since coefficients of the equation are real.

$\overline{2z^3-3z^2+18z+10} = \overline{0} \quad (\overline{0}=0)$

$\overline{2z^3}-\overline{3z^2}+\overline{18z}+\overline{10} = 0$

$\left(\overline{a\pm b} = \overline{a}\pm\overline{b} \ \forall\ a,b\in\mathbb{C}\right)$

$2\overline{z}^3-3\overline{z}^2+18\overline{z}+10 = 0$

$\left(\overline{abc} = \overline{a}\,\overline{b}\,\overline{c} \ \forall\ a,b,c\in\mathbb{C}\right)$

$2\overline{z}^3-3\overline{z}^2+18\overline{z}+10 = 0$

$(\overline{a} = a \text{ for } a\in\mathbb{R}).$

That is, $\overline{z}$ is a root of the equation.

(ii) Let β be the 3rd root:

$1+3i+1-3i+\beta = \frac{3}{2}$

$\left(\text{since sum of roots} = -\frac{b}{a}\right)$

$2+\beta = \frac{3}{2}$

$\beta = -\frac{1}{2}.$

Hence the three roots are $1-3i,\ 1+3i,\ -\frac{1}{2}.$

OR Using the factor theorem,

$2z^3-3z^2+18z+10$

$= \left[z-(1-3i)\right]\left[z-(1+3i)\right](az+b)$

$= \left(z^2-2z+10\right)(az+b)$

$\Rightarrow b = 1,\ a = 2.$

So the third root is $z = -\frac{1}{2}.$

(e) (i) **Method 1**

$\overrightarrow{OA} = z_1$

$\overrightarrow{AP} = iz_1$

$\therefore \quad \overrightarrow{OP} = \overrightarrow{OA}+\overrightarrow{AP}$

$= z_1+iz_1$

$= (1+i)z_1.$

Method 2

$|OP| = |OA|\times\sqrt{2}$

$\arg(OP) = \arg(OA)+\frac{\pi}{4}$

} since OAP is an isosceles right Δ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \quad \overrightarrow{OP} = OA\times\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$= z_1\times\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$

$= z_1(1+i).$

(ii) $\overrightarrow{OQ} = z_2(1-i)$

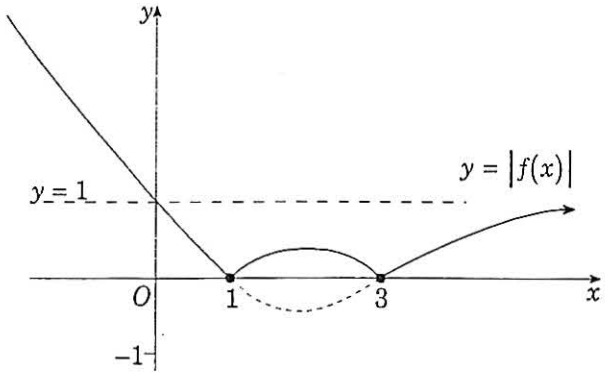
Since M is the midpoint of PQ,

$\therefore \quad M\left(\overrightarrow{OM}\right) = \frac{\overrightarrow{OP}+\overrightarrow{OQ}}{2}$

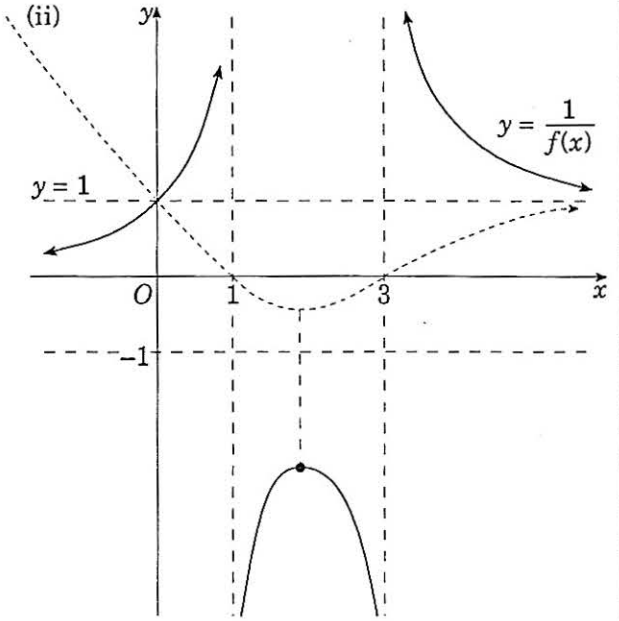
$= \frac{z_1(1+i)+z_2(1-i)}{2}.$

QUESTION 3

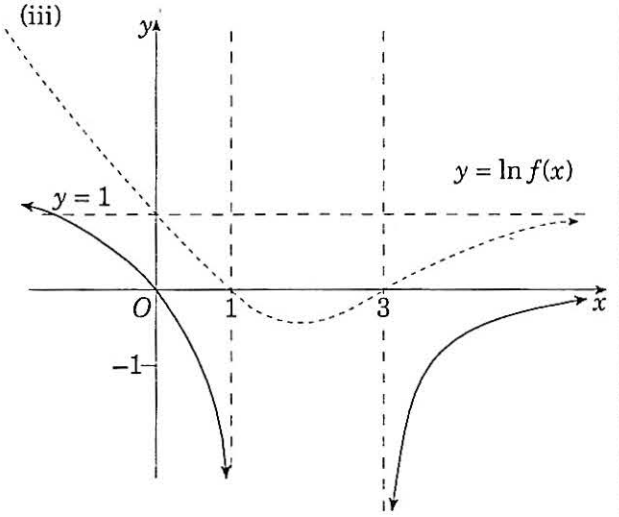
(a) (i)



(ii)



(iii)



(b) $\dfrac{x^2}{5^2}+\dfrac{y^2}{3^2}=1, \quad P(x_0, y_0)$

(i) $a=5, \quad b=3$

$$b^2 = a^2\left(1-e^2\right)$$
$$9 = 25\left(1-e^2\right)$$
$$e^2 = 1-\frac{9}{25}$$
$$e^2 = \frac{16}{25}$$
$$e = \frac{4}{5}, \qquad (e>0).$$

(ii) Foci: $(ae, 0) = (4, 0)$

$(-ae, 0) = (-4, 0)$

Directrices: $x = \pm\dfrac{a}{e}$

$$x = \pm\frac{5}{\frac{4}{5}}$$

$\therefore\ x = \dfrac{25}{4}, \ x = -\dfrac{25}{4}.$

(iii) $\dfrac{x^2}{5^2}+\dfrac{y^2}{3^2}=1$

Differentiate w.r.t. x:

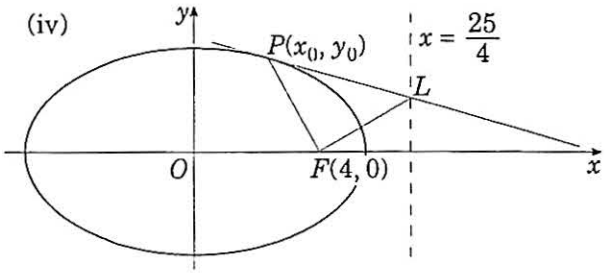
$$\frac{2x}{5^2}+\frac{2y}{3^2}\cdot\frac{dy}{dx}=0$$
$$\frac{dy}{dx} = -\frac{2x}{5^2}\times\frac{3^2}{2y}$$
$$= -\frac{3^2}{5^2}\cdot\frac{x}{y}.$$

Gradient of tangent at $P = -\dfrac{3^2}{5^2}\cdot\dfrac{x_0}{y_0}$.

Equation of tangent at P is

$$y-y_0 = -\frac{3^2}{5^2}\cdot\frac{x_0}{y_0}(x-x_0)$$
$$\frac{y_0}{3^2}\cdot y-\frac{y_0^2}{3^2} = -\frac{x_0x}{5^2}+\frac{x_0^2}{5^2}$$
$$\frac{x_0x}{5^2}+\frac{y_0y}{3^2} = \frac{x_0^2}{5^2}+\frac{y_0^2}{3^2}$$
$$= 1, \text{ since } (x_0, y_0) \text{ lies on } \mathcal{E}.$$

(iv)



L is the point where $\dfrac{x_0x}{5^2}+\dfrac{y_0y}{3^2}=1$ meets $x=\dfrac{25}{4}$.

$$\therefore\ \frac{x_0}{25}\times\frac{25}{4}+\frac{y_0y}{9} = 1$$
$$\frac{y_0y}{9} = 1-\frac{x_0}{4}$$
$$y = \frac{9}{y_0}\left(\frac{4-x_0}{4}\right)$$

Coordinates of L are $\left(\dfrac{25}{4}, \dfrac{9(4-x_0)}{4y_0}\right)$.

Let grad. $PF = m_1$ and grad. $FL = m_2$.

$$\therefore\ m_1 = \frac{y_0-0}{x_0-4}, \ m_2 = \frac{\frac{9(4-x_0)}{4y_0}-0}{\frac{25}{4}-4}$$
$$= \frac{\frac{9}{4}\left(\frac{4-x_0}{y_0}\right)}{\frac{9}{4}}$$
$$= \frac{4-x_0}{y_0}.$$
$$\therefore\ m_1m_2 = \frac{y_0}{x_0-4}\times\frac{4-x_0}{y_0} = -1.$$

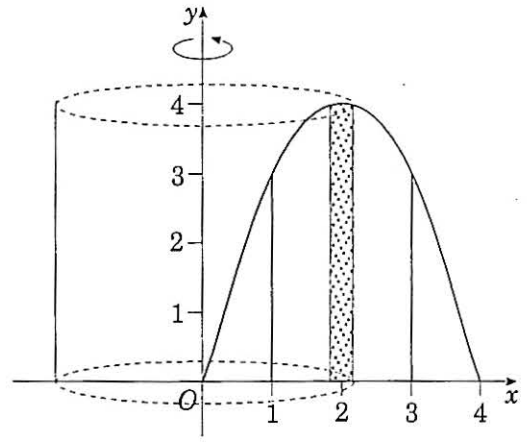
$\therefore\ PF \perp LF$

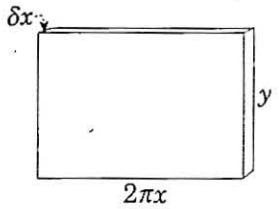
That is, $\angle PFL$ is a right angle.

By symmetry, this also works for the directrix $x = -\frac{25}{4}$.

QUESTION 4

(a)





Let δV be the volume of the cylindrical shell formed by rotating the curve from x to $x+\delta x$ about the y axis.

$$\begin{aligned}
\delta V &= 2\pi x.y.\delta x \\
V &= \lim_{\delta V \to 0} \sum \delta V \\
&= \lim_{\delta x \to 0} \sum 2\pi xy.\delta x \\
&= 2\pi \int_1^3 xy\ dx \\
&= 2\pi \int_1^3 x\left(4x - x^2\right) dx \\
&= 2\pi \int_1^3 4x^2 - x^3\ dx \\
&= 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_1^3 \\
&= 2\pi \left(36 - 20\frac{1}{4} - 1\frac{1}{3} + \frac{1}{4}\right) \\
&= \frac{88\pi}{3}.
\end{aligned}$$

(b) (i) If $P(x)$ has a double root at $x = \alpha$,

then $P(x) = (x-\alpha)^2 Q(x)$, where $Q(\alpha) \neq 0$.

$$\begin{aligned}
\therefore\ P'(x) &= 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x) \\
&= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right] \\
\therefore\ P'(\alpha) &= (\alpha-\alpha)\left[2Q(\alpha) + (\alpha-\alpha)Q'(\alpha)\right] \\
&= 0.
\end{aligned}$$

$\therefore\ x = \alpha$ is also a root of $P'(x)$.

(ii) $A(x) = x^4 + ax^2 + bx + 36$

$A'(x) = 4x^3 + 2ax + b$

$A(2) = A'(2) = 0$, since $x = 2$ is a double root of $A(x)$.

$$\begin{aligned}
\therefore\ 16 + 4a + 2b + 36 &= 0 \\
4a + 2b &= -52 \\
2a + b &= -26. \qquad \text{—①} \\
32 + 4a + b &= 0 \\
4a + b &= -32. \qquad \text{—②}
\end{aligned}$$

②−①: $2a = -6$

$a = -3$

$b = -20.$

(iii) $A(x) = x^4 - 3x^2 - 20x + 36$

$= (x-2)^2 Q(x)$

$= (x-2)^2\left(kx^2 + \ell x + m\right).$

By inspection, $m = 9,\ k = 1$

$\Rightarrow\ x^4 - 3x^2 - 20x + 36$

$= (x-2)^2\left(x^2 + \ell x + 9\right).$

When $x = 1$,

$1 - 3 - 20 + 36 = 1 + \ell + 9$

$14 = 10 + \ell$

$4 = \ell.$

$\Rightarrow A(x) = (x-2)^2\left(x^2 + 4x + 9\right)$ over $\mathbb{R}$.

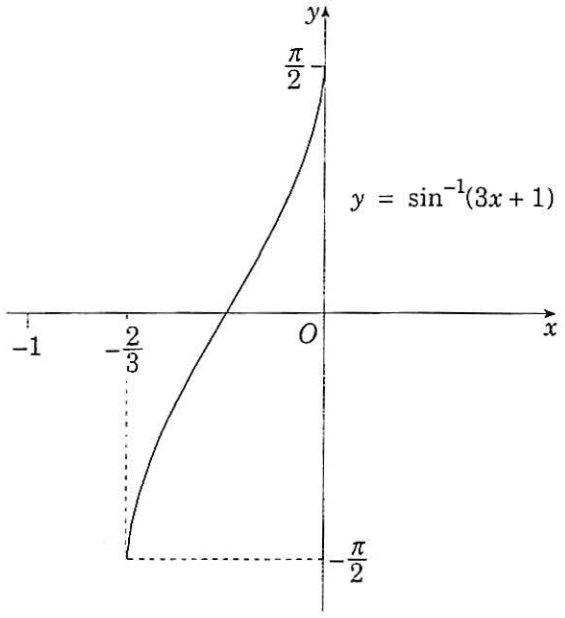
N.B. $Q(x)$ can also be found by polynomial division.

(c) (i) $\sin^{-1}x$ has domain $-1 \le x \le 1$,

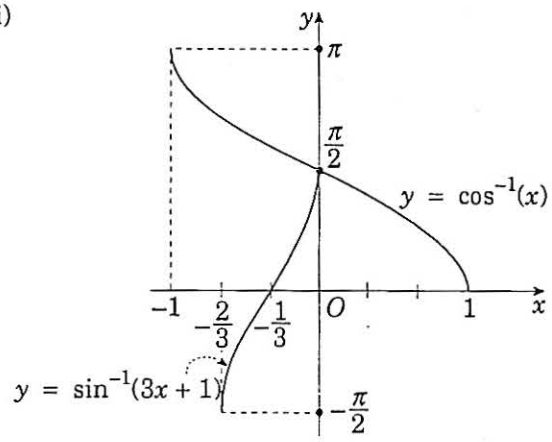
$\therefore\ \sin^{-1}(3x+1)$ has domain $-1 \le 3x+1 \le 1$.

That is, $-2 \le 3x \le 0$, that is, $-\frac{2}{3} \le x \le 0$.

(ii) N.B. Range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.



(iii)



From the graph, we see that
$\sin^{-1}(3x+1) = \cos^{-1}x$ when $x = 0$.

OR $\sin^{-1}(3x+1) = \cos^{-1}x$

$3x+1 = \sin\left(\cos^{-1}x\right)$

$= \sqrt{1-x^2}$

$\therefore \quad (3x+1)^2 = 1-x^2.$

{ Alternatively, let
$\sin^{-1}(3x+1) = \cos^{-1}x = y$
$\therefore \quad 3x+1 = \sin y$
$x = \cos y$
$(3x+1)^2 + x^2 = 1.$ }

$\therefore \quad 10x^2 + 6x = 0$

$x(5x+3) = 0.$

$\therefore \ x = 0$ or $x = -\frac{3}{5}.$

Note: We must check these solutions since we squared in the working and may have introduced unwanted solutions.

At $x = 0$, $\sin^{-1}(3\times 0+1) = \frac{\pi}{2}$

$\cos^{-1}(0) = \frac{\pi}{2}.$

This is a solution.

At $x = -\frac{3}{5}$,

$\sin^{-1}\left(3\times -\frac{3}{5}+1\right) = \sin^{-1}\left(-\frac{4}{5}\right) < 0$

$\cos^{-1}\left(-\frac{3}{5}\right) > 0.$

This is *not* a solution.
The only solution is $x = 0$.

QUESTION 5

(a) (i) Let $y = x^2$

$\therefore \quad x = \sqrt{y}.$

Using substitution:

$\left(\sqrt{y}\right)^3 + 5\left(\sqrt{y}\right)^2 + 11 = 0$

$y\sqrt{y} + 5y + 11 = 0$

$\therefore \quad \left(y\sqrt{y}\right)^2 = (-5y-11)^2$

$y^3 = 25y^2 + 110y + 121.$

$\therefore \ y^3 - 25y^2 - 110y - 121 = 0.$

OR $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 25 - 0$

$= 25.$

$(\alpha\beta\gamma)^2 = (-11)^2$

$= 121.$

$\alpha^2\beta^2+\alpha^2\gamma^2+\gamma^2\beta^2$

$= (\alpha\beta+\alpha\gamma+\gamma\beta)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= 0 - 2\times -11\times -5$

$= -110.$

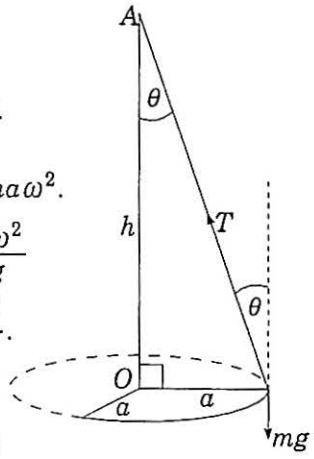
$\therefore \ y^3 - 25y^2 - 110y - 121 = 0.$

(ii) $\alpha^2+\beta^2+\gamma^2 = 25.$

This calculation is done in the second method above, or can be read from the coefficient of y^2 in the result from the first method.

(b) (i) Force is $ma\omega^2$.

(ii) Vertically,
$T\cos\theta - mg = 0.$
Horizontally,
$T\sin\theta = ma\omega^2.$



$\therefore \ \frac{T\sin\theta}{T\cos\theta} = \frac{ma\omega^2}{mg}$

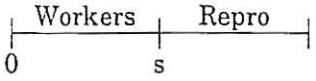
$\therefore \quad \tan\theta = \frac{a\omega^2}{g}.$

But $\tan\theta = \frac{a}{h}$

$\therefore \quad \frac{a}{h} = \frac{a\omega^2}{g}.$

So $\omega^2 = \frac{g}{h}.$

(c)



(i) $\frac{dw}{dt} = k_1 w$ when $0 \le t \le s.$

$\frac{dt}{dw} = \frac{1}{k_1 w}$

$dt = \frac{1}{k_1}\times\frac{dw}{w}$

$t = \frac{1}{k_1}\int\frac{dw}{w}$

$t = \frac{1}{k_1}\ln w + C$ —①

But when $t = 0$, $w = 1$.

$\therefore \quad 0 = \frac{1}{k_1}\ln 1 + C$

$= 0 + C \Rightarrow C = 0.$

① becomes $t = \frac{\ln w}{k_1}$

$k_1 t = \ln w$

$e^{k_1 t} = w(t), \quad 0 \le t \le s.$

$\therefore \quad w(s) = e^{k_1 s}.$

N.B. This full derivation is not required here. From 3 Unit work we could simply say:

$\frac{dw}{dt} = k_1 w, \quad 0 \le t \le s.$

$\therefore \quad w = w(0)e^{k_1 t}$

$= e^{k_1 t}.$

$\therefore \quad w(s) = e^{k_1 s}.$

(ii) $\frac{dr}{dt} = k_2 w(s), \quad t \ge s$

$= k_2 e^{k_1 s}$

$\therefore \quad r = k_2 e^{k_1 s} t + C.$

Substitute $r(s) = 0$.

$\therefore \quad 0 = k_2 e^{k_1 s} s + C$

$C = -k_2 e^{k_1 s} s$

$\therefore \quad r = k_2 e^{k_1 s} t - k_2 e^{k_1 s} s$

$= k_2 e^{k_1 s}(t - s), \quad t \ge s.$

(iii) $k_1 = 0{\cdot}04$

$r(100) = k_2 e^{0{\cdot}04 s}(100 - s)$

Let $R = r(100)$

$\frac{dR}{ds} = k_2 \cdot 0{\cdot}04 e^{0{\cdot}04 s}(100 - s) + k_2 e^{0{\cdot}04 s}(-1)$

$= k_2 e^{0{\cdot}04 s}(4 - 0{\cdot}04 s - 1)$

$= k_2(3 - 0{\cdot}04 s)e^{0{\cdot}04 s}$ —*

Stationary points when $\frac{dR}{ds} = 0$,

that is, $3 - 0{\cdot}04 s = 0$

$s = \frac{3}{0{\cdot}04}$

$= 75.$

s	< 75	75	> 75
$\frac{dR}{ds}$	+	0	−
R	/	—	\

N.B. The sign of $\frac{dR}{ds}$ can be determined from the factored form * above.

$\therefore$ $s = 75$ maximises $r(100)$.

QUESTION 6

(a) (i) *Step 1:*

For $n = 2$, LHS $= (1+x)^2$

$= 1 + 2x + x^2.$

RHS $= 1 + 2x.$

Since $x^2 > 0$, $1 + 2x + x^2 > 1 + 2x.$

$\therefore (1+x)^n > 1 + nx$ when $n = 2.$

Step 2:

Assume true for $n = k$, $k \ge 2$, k is a positive integer.

That is, assume $(1+x)^k > 1 + kx.$

Need to prove true for $n = k + 1.$

That is, need to prove

$(1+x)^{k+1} > 1 + (k+1)x.$

Now $(1+x)^{k+1}$

$= (1+x)(1+x)^k$

$> (1+x)(1+kx)$, using assumption, and since $x > -1 \Rightarrow 1 + x > 0.$

$= 1 + kx + x + kx^2$

$> 1 + (k+1)x$, since $kx^2 > 0.$

Step 3:

We have shown that if the inequality is true when $n = k$, it is also true when $n = k + 1$. Since it is true when $n = 2$, it is also true when $n = 2 + 1 = 3$, and hence when $n = 3 + 1 = 4$, and so on. By the principle of mathematical induction, the inequality is true for all integers $n \ge 2$.

(ii) Let $x = -\frac{1}{2n}$. If $n \ge 2$, then this satisfies the conditions of part (i), including $x > -1$.

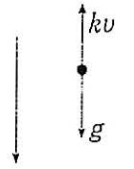
$\therefore \quad (1+x)^n > 1 + nx.$

$\therefore \quad \left(1 - \frac{1}{2n}\right)^n > 1 - \frac{n}{2n}$

$= \frac{1}{2}.$

$\therefore \quad \left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, \ldots$

(b) (i)



Note: $m = 1$, so forces acting are equal in magnitude to the acceleration.

Newton's Second Law: $\ddot{y} = g - kv.$

That is, $\frac{dv}{dt} = g - kv.$

When $t = 0$, the ball is at maximum height so the velocity is zero. That is, $v(0) = 0$.

(ii) $\dfrac{dt}{dv} = \dfrac{1}{g-kv}$

$$\int \frac{dt}{dv}\,dv = \int \frac{dv}{g-kv}.$$

$$t = -\frac{1}{k}\ln(g-kv) + c. \qquad \text{—①}$$

Note: $g - kv > 0$.

Initial conditions: $t = 0,\ v = 0$.

$$0 = -\frac{1}{k}\ln g + c$$

$$c = \frac{1}{k}\ln g.$$

From ①, $t = -\dfrac{1}{k}\ln(g-kv) + \dfrac{1}{k}\ln g$

$$t = -\frac{1}{k}\ln\left(\frac{g-kv}{g}\right)$$

$$-kt = \ln\left(\frac{g-kv}{g}\right)$$

$$e^{-kt} = \frac{g-kv}{g}$$

$$= 1 - \frac{kv}{g}$$

$$v = \frac{g}{k}\left(1 - e^{-kt}\right).$$

N.B. This applies for $t > 0$, that is, for the downward motion.

(iii) $\dfrac{dv}{dt} = v\dfrac{dv}{dy}$

$$v\frac{dv}{dy} = g - kv$$

$$\frac{dv}{dy} = \frac{g-kv}{v}$$

$$\frac{dy}{dv} = \frac{v}{g-kv}$$

$$= -\frac{1}{k}\cdot\frac{-kv}{g-kv}$$

$$= -\frac{1}{k}\cdot\frac{g-kv-g}{g-kv}$$

$$-k\frac{dy}{dv} = 1 - \frac{g}{g-kv}$$

$$-ky = v + \frac{g}{k}\ln(g-kv) + c.$$

When $y = 0,\ v = 0$,

$$0 = 0 + \frac{g}{k}\ln(g) + c$$

$$c = -\frac{g}{k}\ln g$$

$$\therefore \quad -ky = v + \frac{g}{k}\ln(g-kv) - \frac{g}{k}\ln g$$

$$= v + \frac{g}{k}\ln\left(\frac{g-kv}{g}\right).$$

That is, $\dfrac{g}{k}\ln\left(\dfrac{g-kv}{g}\right) + v = -ky.$

(iv) From (iii), $v + ky = -\dfrac{g}{k}\ln\left(1 - \dfrac{k}{g}v\right)$

Using (ii),

$$\frac{v+ky}{g} = -\frac{1}{k}\ln\left[1 - \frac{k}{g}\cdot\frac{g}{k}\left(1 - e^{-kt}\right)\right]$$

$$= -\frac{1}{k}\ln\left(e^{-kt}\right)$$

$$= -\frac{1}{k}\cdot(-kt)$$

$$= t.$$

(v) Max. height: $h = \dfrac{1}{k}\left[U - \dfrac{g}{k}\log_e\left(\dfrac{g+kU}{g}\right)\right]$.

Time to reach h: $t_h = \dfrac{1}{k}\log_e\left(\dfrac{g+kU}{g}\right)$.

The ball reaches ground level again when $y = h$.

Let T_h be the time taken.

V is the final speed.

$$T_h = \frac{V+kh}{g}, \text{ from (iv)}$$

$$= \frac{V}{g} + \frac{k}{g}\cdot h$$

$$= \frac{V}{g} + \frac{k}{g}\cdot\frac{1}{k}\left[U - \frac{g}{k}\log_e\left(\frac{g+kU}{g}\right)\right]$$

$$= \frac{V}{g} + \frac{U}{g} - \frac{1}{k}\log_e\left(\frac{g+kU}{g}\right).$$

T is the total time in the air,

so $T = t_h + T_h$

$$= \frac{1}{k}\log_e\left(\frac{g+kU}{g}\right) + \frac{V}{g} + \frac{U}{g}$$

$$- \frac{1}{k}\log_e\left(\frac{g+kU}{g}\right)$$

$$= \frac{V}{g} + \frac{U}{g}$$

$$= \frac{V+U}{g}.$$

(vi) If air resistance is ignored, then $V_0 = U$.

Also, $U > V$, since air resistance must (a) reduce height, and (b) reduce speed on descent, both of which reduce V.

Therefore $V_0 > V$, and so

$$T_0 = \frac{U+V_0}{g}$$

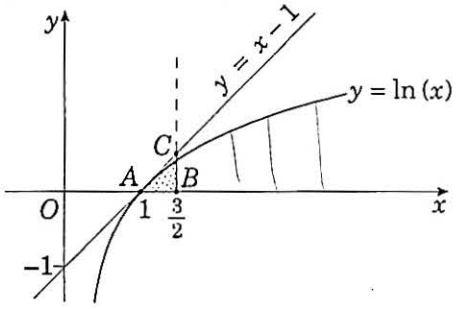
$$> \frac{U+V}{g}$$

$$= T.$$

$$\therefore \quad T_0 > T.$$

QUESTION 7

(a) (i)



(ii) $y = \ln x$

$\dfrac{dy}{dx} = \dfrac{1}{x}$.

At $x = 1$, $\dfrac{dy}{dx} = 1$.

Equation of the tangent is $y = x - 1$.

Shaded area $= \displaystyle\int_1^{\frac{3}{2}} \ln x \, dx$

$<$ area of ΔABC

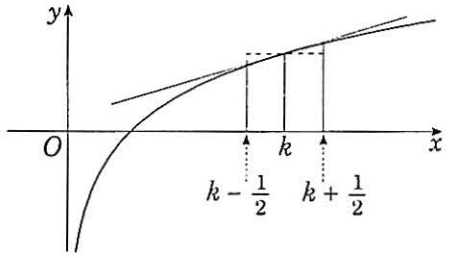
$= \dfrac{1}{2} \times AB \times BC$

$= \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2}$

$= \dfrac{1}{8}$.

$\therefore \displaystyle\int_1^{\frac{3}{2}} \ln x \, dx < \frac{1}{8}$.

(iii)



$\ln x$ is concave down, so the tangent at $x = k$ is above the curve $y = \ln x$ for all $k > 0$.

$\displaystyle\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \ln x \, dx <$ area under the tangent from $x = k - \frac{1}{2}$ to $x = k + \frac{1}{2}$,

provided $k > 1\frac{1}{2}$, that is, for $k = 2, 3, 4, \ldots$

Area under the tangent
= area of trapezium
= area of rectangle (see diagram)
$= \ln k \times 1$
$= \ln k$.

Hence $\displaystyle\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \ln x \, dx < \ln k$ for $k = 2, 3, 4, \ldots$

(iv) $\displaystyle\int_1^n \ln x \, dx$

$= \displaystyle\int_1^{\frac{3}{2}} \ln x \, dx + \int_{\frac{3}{2}}^{\frac{5}{2}} \ln x \, dx + \int_{\frac{5}{2}}^{\frac{7}{2}} \ln x \, dx$

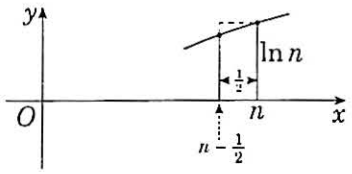
$+ \cdots + \displaystyle\int_{n-\frac{3}{2}}^{n-\frac{1}{2}} \ln x \, dx + \int_{n-\frac{1}{2}}^{n} \ln x \, dx$

$< \dfrac{1}{8} + \ln 2 + \ln 3 + \cdots + \ln(n-1) + \displaystyle\int_{n-\frac{1}{2}}^{n} \ln x \, dx$,

from (ii) and (iii).

Now $\displaystyle\int_{n-\frac{1}{2}}^{n} \ln x \, dx < \int_{n-\frac{1}{2}}^{n} \ln n \, dx$

$= \dfrac{1}{2} \ln n$.



So $\displaystyle\int_1^n \ln x \, dx < \frac{1}{8} + \ln 2 + \ln 3 + \cdots$

$+ \ln(n-1) + \dfrac{1}{2} \ln n$

for $n = 2, 3, 4, \ldots$

(v) $\displaystyle\int_1^n \ln x \, dx = n \ln n - n + 1$

Using (iv),

$n \ln n - n + 1 < \dfrac{1}{8} + \ln 2 + \ln 3 + \cdots$

$+ \ln(n-1) + \dfrac{1}{2} \ln n$

$\therefore \left(n + \dfrac{1}{2}\right) \ln n - n + 1 - \dfrac{1}{8}$

$< \ln 2 + \ln 3 + \cdots + \ln(n-1) + \ln n$

$= \ln\left[2 \times 3 \times \cdots \times (n-1) \times n\right]$

$= \ln(n!)$.

$\therefore$ Since e^x is a monotonic increasing function, $e^{\ln(n!)} > e^{\left(n+\frac{1}{2}\right)\ln n - n + \frac{7}{8}}$.

That is, $n! > e^{\ln\left(n^{n+\frac{1}{2}}\right)} \times e^{-n} \times e^{\frac{7}{8}}$

$= n^{n+\frac{1}{2}} \times e^{-n} \times e^{\frac{7}{8}}$

$= e^{\frac{7}{8}} n^n \sqrt{n} e^{-n}$ for $n = 2, 3, 4, \ldots$

(b) (i) P(not getting four and losing)

$= P(1 \to 0) + P(1 \to 2 \to 0)$

$= q + pq$

$[= q(1 + p)$

$= 1 - p^2]$.

(ii) P(getting four once and losing)
$= P(1\to 2\to 4\to 3\to 1\to 0)$
$\quad + P(1\to 2\to 4\to 3\to 1\to 2\to 0)$
$= ppqqq + ppqqpq$
$= p^2q^3 + p^3q^3.$

OR P(getting four once and losing)
$= P(1\to 2\to 4\to 3\to 1)$
$\quad \times P$(not getting four and losing)
$= ppqq\times(q+pq)$, from (i)
$= p^2q^2(q+pq).$

(iii) P(getting four twice and losing)
$= P(1\to 2\to 4\to 3\to 1)$
$\quad \times P$(getting four once and losing)
$= p^2q^2 \times p^2q^2(q+pq)$, from (ii)
$= \left(p^2q^2\right)^2(q+pq).$

$$\therefore\ P(\text{losing}) = (q+pq)\left[1+p^2q^2+\left(p^2q^2\right)^2 + \left(p^2q^2\right)^3+\cdots\right]$$
$$= (q+pq)\times\frac{a}{1-r},\quad \text{where } a=1,\ r=p^2q^2$$
$$= \frac{q+pq}{1-p^2q^2}.$$

For $p=\frac{1}{2}$, $q=\frac{1}{2}$ and $P(\text{losing}) = \dfrac{\frac{1}{2}+\frac{1}{4}}{1-\frac{1}{16}}$
$= \frac{4}{5}.$

$\therefore\ P(\text{winning}) = \frac{1}{5}.$

QUESTION 8

(a) (i) $\rho = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\rho^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7$
$= \cos 2\pi + i\sin 2\pi$
$= 1.$

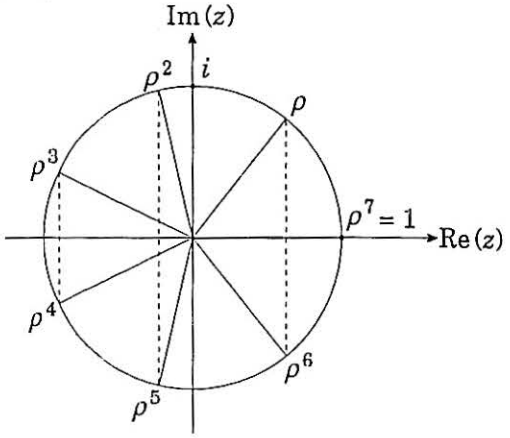
$\therefore$ ρ is a root of $x^7-1=0$, that is of
$(x-1)\left(x^6+x^5+x^4+x^3+x^2+x+1\right)=0.$

Since $\rho\neq 1$,
then ρ satisfies $x^6+x^5+\cdots+x+1=0.$
That is, $\rho^6+\rho^5+\rho^4+\cdots+\rho+1=0.$

(ii) Since coefficients of $x^2+ax+b=0$ are real and $\alpha=\rho+\rho^2+\rho^4$ is a complex root, then $\overline{\alpha}$ is also a root.

$\therefore\ \beta=\overline{\alpha}=\overline{\rho+\rho^2+\rho^4}$
$= \overline{\rho}+\overline{\rho^2}+\overline{\rho^4}$
$= \rho^6+\rho^5+\rho^3,$

since $\overline{\rho}=\rho^6$, $\overline{\rho^2}=\rho^5$ and $\overline{\rho^4}=\rho^3$,
as the following diagram illustrates.



(iii) $a=-(\alpha+\beta)$
$= -\left(\rho+\rho^2+\rho^3+\rho^4+\rho^5+\rho^6\right)$, from (ii)
$= 1$, from (i).

$b=\alpha\beta$
$= \left(\rho+\rho^2+\rho^4\right)\left(\rho^3+\rho^5+\rho^6\right)$
$= \rho^4\left(1+\rho+\rho^3\right)\left(1+\rho^2+\rho^3\right)$
$= \rho^4\left(1+\rho^2+\rho^3+\rho+\rho^3+\rho^4+\rho^3+\rho^5+\rho^6\right)$
$= \rho^4\left(1+\rho+\rho^2+\rho^3+\rho^4+\rho^5+\rho^6+2\rho^3\right)$
$= \rho^4\left(0+2\rho^3\right)$
$= 2\rho^7$
$= 2.$

(iv) From (iii), the quadratic equation is
$x^2+x+2=0.$

$$\therefore\quad x = \frac{-1\pm\sqrt{1-8}}{2}$$
$$= \frac{-1\pm i\sqrt{7}}{2}$$

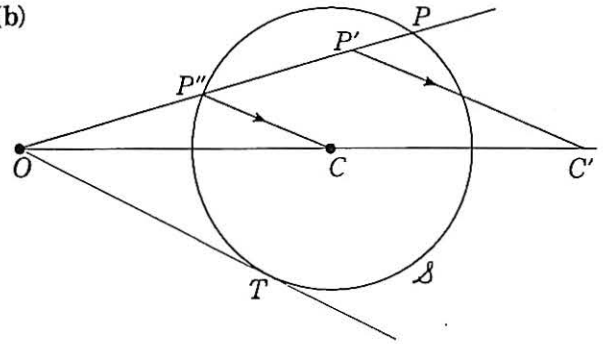
$\therefore\ \text{Im}(\alpha) = \pm\frac{\sqrt{7}}{2}.$

However, $\alpha=\rho+\rho^2+\rho^4.$
From the diagram in (ii),
$\text{Im}\left(\rho+\rho^2+\rho^4\right) > 0.$

$\therefore\ \text{Im}(\alpha) = \frac{\sqrt{7}}{2}.$

Now,
$\text{Im}(\alpha) = \text{Im}\left(\rho+\rho^2+\rho^4\right)$
$= \sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$
$= \sin\frac{2\pi}{7}+\sin\left(\pi-\frac{3\pi}{7}\right)+\sin\left(\pi+\frac{\pi}{7}\right)$
$= \sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}$
$= \frac{\sqrt{7}}{2}$, as required.

(b)



(i) Draw a tangent to $\mathcal{S}$ from O, touching at T.

$\therefore OT^2 = OP \times OP''$ (square of tangent = product of intercepts)

Since O, T are fixed, $OP \times OP''$ is constant.

(ii) $\dfrac{OP''}{OP'} = \dfrac{OP \cdot OP''}{OP \cdot OP'}$

$= \dfrac{OT^2}{k^2}$, which is a constant since OT and k are constants.

(iii) $\Delta OP''C \parallel\!\parallel \Delta OP'C'$ (equiangular)

$\therefore \dfrac{OC}{OC'} = \dfrac{OP''}{OP'}$ (corresp. sides in sim. Δs in same ratio)

$= \dfrac{OT^2}{k^2}$, a constant (from (ii)).

Since O and C are fixed, OC is a constant and so $OC' = \dfrac{k^2}{OT^2} \cdot OC$ is a constant.

Hence C' is a fixed point.

(iv) Since $\Delta OP''C \parallel\!\parallel \Delta OP'C'$

$\dfrac{P''C}{P'C'} = \dfrac{OC}{OC'}$ (corresp. sides in sim. Δs in same ratio)

$\therefore P'C' = P''C \cdot \dfrac{OC'}{OC}$.

Now $P''C$ = radius of $\mathcal{S}$ and so is constant.

$\dfrac{OC'}{OC}$ is constant (shown in (iii)),

$\therefore P'C'$ is a constant.

Since C' is a fixed point, P' must lie on a circle with centre C'.

To describe the locus of P' fully, we need to establish whether P' traverses the whole circle.

Also, we could find the radius of the circle in terms of the data.

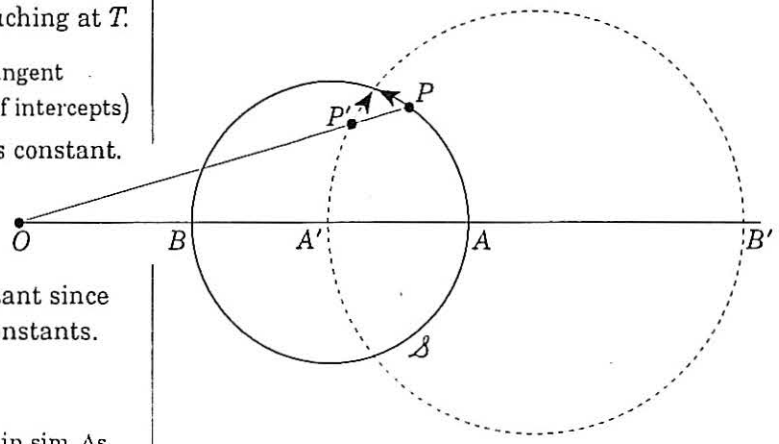
Radius:

Let the radius of $\mathcal{S}$ be R.

Now $P'C' = P''C \cdot \dfrac{OC'}{OC}$

$= R \cdot \dfrac{k^2}{OT^2}$, from (iii)

$= \dfrac{k^2 R}{OC^2 - R^2}$.



As P moves around $\mathcal{S}$ from A to B, OP decreases, which means that OP' must increase (since $OP \cdot OP'$ is constant).

Thus P' moves around its circle from A' to B'.

So the locus of P' is the whole circle.

END OF 4 UNIT (ADDITIONAL) SOLUTIONS