1998 HIGHER SCHOOL CERTIFICATE SOLUTIONS

4 UNIT (ADDITIONAL) MATHEMATICS

QUESTION 1

(a) $$\int_0^3 \frac{6}{9+x^2}\,dx = \frac{6}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$$
$$= 2\left[\tan^{-1}1 - \tan^{-1}0\right]$$
$$= 2\left[\frac{\pi}{4} - 0\right]$$
$$= \frac{\pi}{2}.$$

(b) $$\int x^2 \ln x\,dx = \int\left(\frac{d}{dx}\frac{x^3}{3}\right)\ln x\,dx$$
$$= \frac{x^3}{3}\cdot\ln x - \int\frac{x^3}{3}\cdot\frac{1}{x}\,dx$$
$$= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3} + C$$
$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

(c) $$\int\frac{\sin^3 x}{\cos^2 x}\,dx = \int\frac{(1-\cos^2 x)\sin x}{\cos^2 x}\,dx$$
$$= \int\frac{\sin x}{\cos^2 x}\,dx - \int\sin x\,dx$$
$$= \frac{1}{\cos x} + \cos x + C.$$

OR Put $u = \cos x$.
$$\frac{du}{dx} = -\sin x.$$
$$\int\frac{\sin^3 x}{\cos^2 x}\,dx = \int\frac{\sin^2 x\cdot\sin x}{\cos^2 x}\,dx$$
$$= \int\frac{(1-u^2)\cdot(-1)\,du}{u^2}$$
$$= \int\frac{u^2-1}{u^2}\,du$$
$$= \int 1 - \frac{1}{u^2}\,du$$
$$= u + \frac{1}{u} + C$$
$$= \cos x + \sec x + C.$$

(d) $$u^2 = 4 - x^2.$$
$$2u\,du = -2x\,dx.$$
$$\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_{u=2}^{u=0}\left(4-u^2\right)\sqrt{u^2}\cdot(-u\,du)$$
$$= \int_0^2 u^2\left(4-u^2\right)du$$
$$= \int_0^2 4u^2 - u^4\,du$$
$$= \left[\frac{4u^3}{3} - \frac{u^5}{5}\right]_0^2$$
$$= \frac{32}{3} - \frac{32}{5} - 0$$
$$= \frac{64}{15}.$$

(e) (i) $x^2+6 = 1\left(x^2+x-6\right) - x + 12.$
$\therefore$ Remainder is $-x+12$.
OR
$$\begin{array}{r} 1 \\ x^2+x-6\overline{)\,x^2+0x+6} \\ \underline{x^2+x-6} \\ -x+12 \end{array}$$
$\therefore$ Remainder is $-x+12$.

(ii) $$\int\frac{x^2+6}{x^2+x-6}\,dx$$
$$= \int\left(\frac{x^2+x-6}{x^2+x-6} + \frac{-x+12}{x^2+x-6}\right)dx$$
$$= \int\left(1 + \frac{-x+12}{x^2+x-6}\right)dx.$$
Let $$\frac{-x+12}{x^2+x-6} \equiv \frac{A}{x+3} + \frac{B}{x-2}.$$
Then $$-x+12 \equiv A(x-2) + B(x+3).$$
Put $x=2$, $10 = 5B$
$\therefore$ $B = 2$.
Put $x=-3$, $15 = -5A$
$\therefore$ $A = -3$.

$$\therefore \int \frac{x^2+6}{x^2+x-6}\,dx$$
$$= \int\left(1-\frac{3}{x+3}+\frac{2}{x-2}\right)dx$$
$$= x-3\ln|x+3|+2\ln|x-2|+C.$$

QUESTION 2

(a) $1998 = 4\times 499+2$

$$\therefore \quad i^{1998} = \left(i^4\right)^{499}\times i^2$$
$$= -1.$$

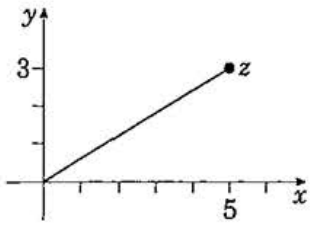
(b) $z = \dfrac{18+4i}{3-i}.$

(i) $(18+4i)\overline{(3-i)} = (18+4i)(3+i)$
$= 54+12i+18i-4$
$= 50+30i.$

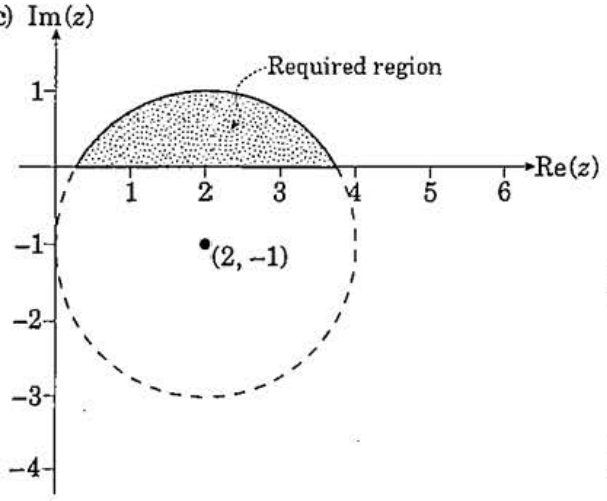
(ii) $z = \dfrac{(18+4i)}{(3-i)}\times\dfrac{(3+i)}{(3+i)}$
$= \dfrac{50+30i}{10}$
$= 5+3i.$

(iii) $|z| = \sqrt{5^2+3^2}$
$= 34.$

$\arg z = \tan^{-1}\left(\dfrac{3}{5}\right)$
$\doteqdot 0{\cdot}540$ radians.



(c) $\mathrm{Im}(z)$



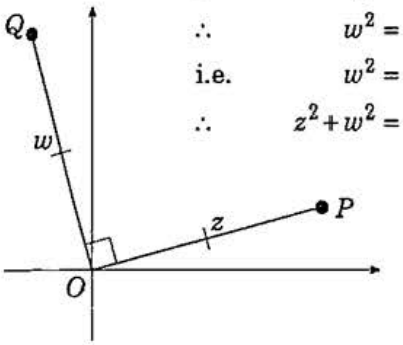
(d) $\angle POQ = 90°$ and $OP = OQ$

$\therefore \quad w = iz$

$\therefore \quad w^2 = i^2z^2$

i.e. $\quad w^2 = -z^2$

$\therefore \quad z^2+w^2 = 0.$



(e) (i) $z^3+1 = 0$

i.e. $(z+1)\left(z^2-z+1\right) = 0.$

$\therefore \quad z = -1$ or $z = \dfrac{1\pm\sqrt{1-4}}{2}$

$= \dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i.$

$\therefore$ The cube roots are

$-1,\ \dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i,\ \dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i.$

(ii) The two non-real cube roots of -1 are the numbers $z = \dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i,$ that is, these roots satisfy the equation

$z^2-z+1 = 0.$

Since λ is a non-real cube root of -1, λ satisfies the equation.

$\therefore \quad \lambda^2-\lambda+1 = 0$

i.e. $\quad \lambda^2 = \lambda-1.$

OR If $\lambda = \dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$

$\lambda^2 = \dfrac{1}{4}-\dfrac{3}{4}+2\times\dfrac{1}{2}\times\dfrac{\sqrt{3}}{2}i$
$= -\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$
$= \dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i-1$
$= \lambda-1.$

If $\lambda = \dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

$\lambda^2 = \dfrac{1}{4}-\dfrac{3}{4}-2\times\dfrac{1}{2}\times\dfrac{\sqrt{3}}{2}i$
$= -\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$
$= \dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i-1$
$= \lambda-1.$

(iii) $(1-\lambda)^6 = \left(-\lambda^2\right)^6$

$= \lambda^{12}$

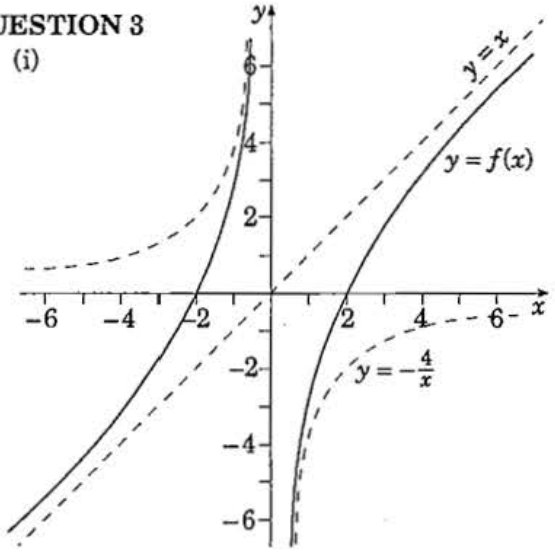
$= \left(\lambda^3\right)^4$

$= (-1)^4$

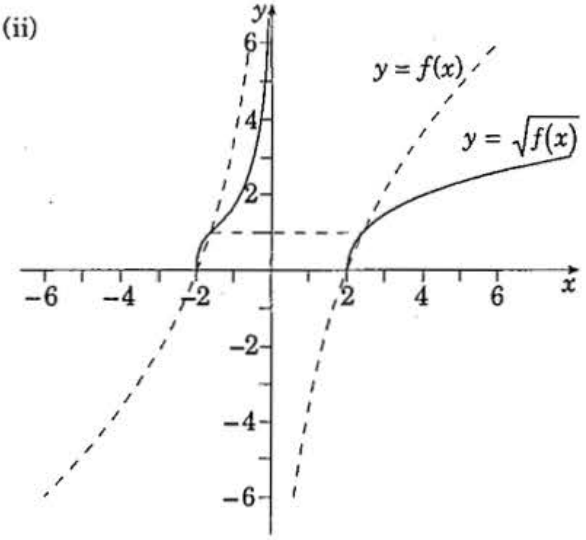
$= 1.$

QUESTION 3

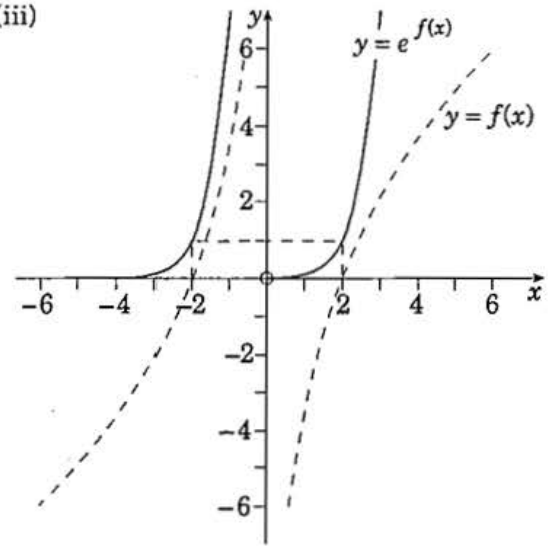
(a) (i)



(ii)



(iii)



(b) (i) $I_n = \int_1^e (\ln x)^n \, dx$

$= \left[x \cdot (\ln x)^n\right]_1^e - \int_1^e x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x} \, dx$

$= \left[e(\ln e)^n - 1(\ln 1)^n\right] - n\int_1^e (\ln x)^{n-1} \, dx$

$= e - nI_{n-1}$

provided $n \geq 1$.

(ii) $I_4 = e - 4I_3$

$= e - 4(e - 3I_2)$

$= e - 4e + 12I_2$

$= -3e + 12(e - 2I_1)$

$= 9e - 24I_1$

$= 9e - 24(e - 1I_0)$

$= -15e + 24I_0.$

$I_0 = \int_1^e dx$

$= [x]_1^e$

$= e - 1.$

$\therefore I_4 = -15e + 24(e-1)$

$= 9e - 24.$

(c) (i) $P = 1000 + Ae^{-kt}$

$\frac{dP}{dt} = 0 - k \cdot Ae^{-kt}$

$= -k(P - 1000).$

(ii) At $t = 0$, $P = 2500$

$\therefore \quad 2500 = 1000 + Ae^0$

$\therefore \quad A = 1500.$

At $t = 10$, $P = 1900$

$\therefore \quad 1900 = 1000 + 1500e^{-10k}$

$\therefore \quad 900 = 1500e^{-10k}$

$\therefore \quad e^{-10k} = 0{\cdot}6$

$-10k = \ln 0{\cdot}6$

$\therefore \quad k = -\frac{1}{10} \ln 0{\cdot}6.$

At $P = 1500$, $t = ?$

$1500 = 1000 + 1500e^{-kt}$

$e^{-kt} = \frac{500}{1500}$

$\therefore \quad e^{kt} = 3$

$\therefore \quad t = \frac{1}{k} \ln 3$

$= \frac{\ln 3}{\left(\frac{-1}{10} \ln 0{\cdot}6\right)}$

$= 21{\cdot}506\,601\,03.$

$\therefore$ The population will be 1500 after approximately 21·5 years.

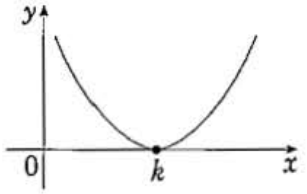
(iii) $P = 1000 + 1500e^{-kt}$.
As $t \to \infty$, $1500e^{-kt} \to 0$
$\therefore \quad P \to 1000.$
$\therefore$ The population will approach 1000.

QUESTION 4

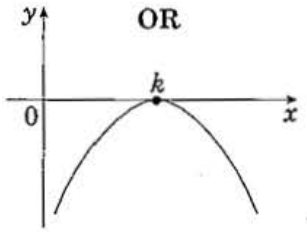
(a) (i) Since k is a double root of the polynomial equation $f(x) = 0$, then $f(x)$ can be expressed as $f(x) = (x-k)^2 \times Q(x)$, where $Q(x)$ is a polynomial expression.

$$f'(x) = 2(x-k)\times Q(x) + (x-k)^2 \times Q'(x)$$
$$= (x-k)\left[2Q(x) + (x-k)\times Q'(x)\right]$$
$$\therefore \quad f'(k) = (k-k)\times\left[2Q(k) + (k-k)Q'(k)\right]$$
$$= 0.$$

(ii) There is a stationary point at the root. If the root is at $x = k$, the graph touches the x axis at the point $(k, 0)$.



OR



(iii) $P(x) = ax^7 + bx^6 + 1$
$P'(x) = 7ax^6 + 6bx^5.$
Since $x = 1$ is a double root, $P(x) = 0$ and $P'(x) = 0$.
$\therefore \quad a + b + 1 = 0$
and $\quad 7a + 6b = 0.$
Solving simultaneously, $a = 6$ and $b = -7$.

(iv) $E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$
$E(0) = 1.$
$\therefore$ $x = 0$ is not a root of $E(x) = 0$.

$$E'(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

If k is a double root of $E(x) = 0$, then $E(k) = 0$ and $E'(k) = 0$.
$\therefore \quad E(k) - E'(k) = 0.$

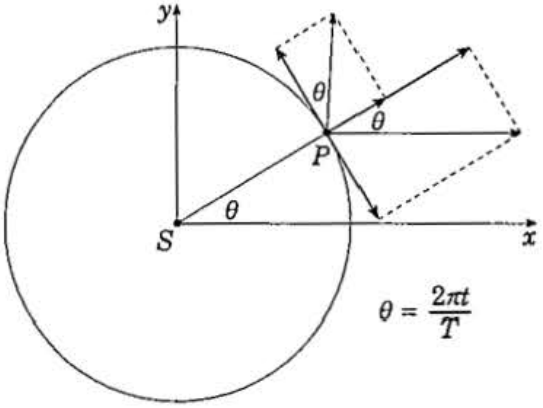
That is, $\left(1 + k + \frac{k^2}{2} + \frac{k^3}{6} + \frac{k^4}{24}\right) - \left(1 + k + \frac{k^2}{2} + \frac{k^3}{6}\right) = 0.$

That is, $\frac{k^4}{24} = 0$
$\therefore \quad k = 0.$
But $x = 0$ is not a root,
$\therefore$ there are no double roots.

(b) (i)
$$x = R\cos\frac{2\pi t}{T}$$
$$\dot{x} = -\frac{2\pi}{T}R\cos\frac{2\pi t}{T}$$
$$\ddot{x} = -\left(\frac{2\pi}{T}\right)^2 R\cos\frac{2\pi t}{T}$$
$$y = R\sin\frac{2\pi t}{T}$$
$$\dot{y} = \frac{2\pi}{T}R\cos\frac{2\pi t}{T}$$
$$\ddot{y} = -\left(\frac{2\pi}{T}\right)^2 R\sin\frac{2\pi t}{T}.$$
$$F = ma$$
$$= m\sqrt{\ddot{x}^2 + \ddot{y}^2}$$
$$= m\left(\frac{2\pi}{T}\right)^2 R\sqrt{\cos^2\frac{2\pi t}{T} + \sin^2\frac{2\pi t}{T}}$$
$$= m\left(\frac{2\pi}{T}\right)^2 R, \text{ which is a constant.}$$

Note that $\ddot{x} = -\left(\frac{2\pi}{T}\right)^2 x$ and $\ddot{y} = -\left(\frac{2\pi}{T}\right)^2 y$.
This means that the force is in the opposite direction to SP, that is, it is directed towards the origin (S).

OR We could resolve the accelerations in the radial and tangential directions.



$$\theta = \frac{2\pi t}{T}$$

$$a_T = -\ddot{x}\sin\theta + \ddot{y}\cos\theta$$
$$= -\left(\frac{2\pi}{T}\right)^2 R\sin\frac{2\pi t}{T}\cos\frac{2\pi t}{T}$$
$$-(-1)\left(\frac{2\pi}{T}\right)^2 R\cos\frac{2\pi t}{T}\sin\frac{2\pi t}{T}$$
$$= 0.$$

$$a_R = \ddot{x}\cos\theta + \ddot{y}\sin\theta$$
$$= -\left(\frac{2\pi}{T}\right)^2 R\cos\frac{2\pi t}{T}\cos\frac{2\pi t}{T} + (-1)\left(\frac{2\pi}{T}\right)^2 R\sin\frac{2\pi t}{T}\sin\frac{2\pi t}{T}$$
$$= -\left(\frac{2\pi}{T}\right)^2 R\left(\cos^2\frac{2\pi t}{T} + \sin^2\frac{2\pi t}{T}\right)$$
$$= -\left(\frac{2\pi}{T}\right)^2 R.$$

Since $a_T = 0$, $a = |a_R|$ and so the force is
$F = ma$
$= m\left(\frac{2\pi}{T}\right)^2 R$, which is a constant.

Note that the minus sign in the formula for a_R indicates that the force is directed towards the centre of the circle.

(ii) $F = \frac{GMm}{R^2}$ and $F = m\frac{4\pi^2 R}{T^2}$

$\therefore \quad \frac{GMm}{R^2} = \frac{m4\pi^2 R}{T^2}$

$\therefore \quad T^2 = \frac{4\pi^2 R^3}{GM}$

$\therefore \quad T = \sqrt{\frac{4\pi^2 R^3}{GM}}$
$= 2\pi R\sqrt{\frac{R}{GM}}.$

(c) **Sue:** P(same colour)
$= \frac{3}{3+W}\times\frac{2}{2+W} + \frac{W}{3+W}\times\frac{W-1}{2+W}$
$= \frac{W^2 - W + 6}{(3+W)(2+W)}.$

If this is $\frac{1}{2}$, then

$2(W^2 - W + 6) = (3+W)(2+W)$
$2W^2 - 2W + 12 = W^2 + 5W + 6$
$W^2 - 7W + 6 = 0$
$(W-6)(W-1) = 0$
$\therefore \quad W = 1$ or $W = 6$.

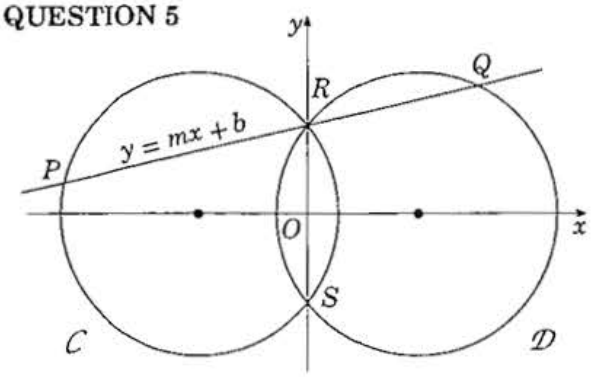
Bill: P(same colour)
$= \left(\frac{3}{3+W}\right)^2 + \left(\frac{W}{3+W}\right)^2$
$= \frac{9+W^2}{(3+W)^2}.$

If this is $\frac{5}{8}$, then

$8(9+W^2) = 5(9+6W+W^2)$
$8W^2 + 72 = 5W^2 + 30W + 45$
$3W^2 - 30W + 27 = 0$
$W^2 - 10W + 9 = 0$
$(W-9)(W-1) = 0$
$\therefore \quad W = 9$ or $W = 1$.

Since both conditions must be true, $W = 1$ is the only solution.

QUESTION 5



(a) (i) P is the point of intersection of $y = mx + b$ and $(x+a)^2 + y^2 = a^2 + b^2$.

$\therefore \quad (x+a)^2 + (mx+b)^2 = a^2 + b^2$
$x^2 + 2ax + a^2 + m^2x^2 + 2mbx + b^2 = a^2 + b^2$
$x^2(1+m^2) + x(2a+2mb) = 0$
$x\left[x(1+m^2) + 2(a+mb)\right] = 0$
$\therefore \quad x = 0$ or $x = -\frac{2(a+mb)}{(1+m^2)}.$

At R, $x = 0$,
$\therefore$ at P, $x = -\frac{2(a+mb)}{(1+m^2)}.$

(ii) Q is the point of intersection of $y = mx + b$ and $(x-a)^2 + y^2 = a^2 + b^2$.

$\therefore \quad (x-a)^2 + (mx+b)^2 = a^2 + b^2$
$x^2 - 2ax + a^2 + m^2x^2 + 2mbx + b^2 = a^2 + b^2$
$x^2(1+m^2) - 2x(a-mb) = 0$
$\therefore \quad x = 0$ at R or $x = \frac{2(a-mb)}{(1+m^2)}$ at Q.

(iii) The midpoint of PQ has x value
$\frac{1}{2}\left[\frac{-2(a+mb)}{1+m^2} + \frac{2(a-mb)}{1+m^2}\right]$
$= \frac{2}{2}\left[\frac{-a-mb+a-mb}{1+m^2}\right]$
$= \frac{-2mb}{1+m^2}.$

Since P, Q lie on $y = mx + b$, the midpoint of PQ lies on $y = mx + b$.

$\therefore$ The y value is $\frac{-2m^2b}{1+m^2} + b$,

that is, $\frac{-2m^2b + b + bm^2}{1+m^2}$,

that is, $\dfrac{b\left(1-m^2\right)}{1+m^2}$.

$\therefore\ x=\dfrac{-2mb}{1+m^2}$ and $y=\dfrac{b\left(1-m^2\right)}{1+m^2}$.

To eliminate m, square both x and y and add.

$$x^2+y^2=\left(\frac{-2mb}{1+m^2}\right)^2+\left[\frac{b\left(1-m^2\right)}{1+m^2}\right]^2$$

$$=\frac{4m^2b^2+b^2-2m^2b+b^2m^4}{\left(1+m^2\right)^2}$$

$$=\frac{b^2\left(1+2m^2+m^4\right)}{\left(1+m^2\right)^2}$$

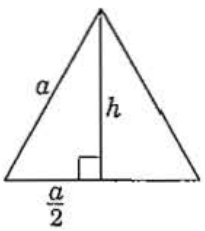
$$=\frac{b^2\left(1+m^2\right)^2}{\left(1+m^2\right)^2}.$$

$\therefore\ x^2+y^2=b^2$ is the required locus.

This is a circle, centre the origin, with radius equal to b. If the line PRQ has the form $y=mx+b$ as given in the question, then the point $S(0,-b)$ is excluded from the locus. This is because when $P=S$, $Q=S$, and so the line PRQ is vertical, with equation $x=0$. This equation does not match the given form.

If the line PRQ is any line through R, then this exclusion would not apply and the locus would be the complete circle.

(b) (i)



$$h^2=a^2-\left(\frac{a}{2}\right)^2$$

$$=\frac{3a^2}{4}$$

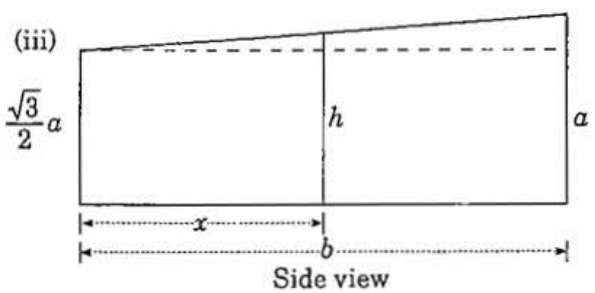
$\therefore\ h=\dfrac{\sqrt{3}}{2}a.$

(ii) x and b are the altitudes of similar triangles CYZ and CRS.

$\therefore\ \dfrac{YZ}{x}=\dfrac{SR}{b}$ and $SR=a$.

$\therefore\ YZ=\dfrac{ax}{b}$.

(iii)



Side view

Using similar triangles,

$$\frac{h-\frac{\sqrt{3}}{2}a}{a-\frac{\sqrt{3}}{2}a}=\frac{x}{b}$$

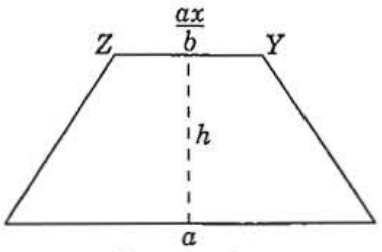
$$\therefore\quad h-\frac{\sqrt{3}}{2}a=\frac{ax}{b}-\frac{\sqrt{3}ax}{2b}$$

$$\therefore\quad h=\frac{\sqrt{3}}{2}a+\frac{ax}{b}-\frac{\sqrt{3}ax}{2b}$$

$$=\frac{a}{2}\left(\sqrt{3}+\frac{2x}{b}-\frac{\sqrt{3}x}{b}\right)$$

$$=\frac{a}{2}\left[\sqrt{3}+\left(2-\sqrt{3}\right)\frac{x}{b}\right].$$

(iv)



Cross-section

$$\text{Area}=\frac{1}{2}\left(\frac{ax}{b}+a\right)\times h$$

$$=\frac{1}{2}\left(\frac{ax}{b}+a\right)\times\frac{a}{2}\left[\sqrt{3}+\left(2-\sqrt{3}\right)\frac{x}{b}\right]$$

$$=\frac{a^2}{4}\left(\frac{x}{b}+1\right)\left[\sqrt{3}+\left(2-\sqrt{3}\right)\frac{x}{b}\right]$$

$$=\frac{a^2}{4b^2}(x+b)\left[\sqrt{3}b+\left(2-\sqrt{3}\right)x\right].$$

(v) Volume

$$=\int_0^b\frac{a^2}{4b^2}(x+b)\left[\sqrt{3}b+\left(2-\sqrt{3}\right)x\right]dx$$

$$=\frac{a^2}{4b^2}\int_0^b\left[x^2\left(2-\sqrt{3}\right)+x\left(\sqrt{3}b+2b-\sqrt{3}b\right)+\sqrt{3}b^2\right]dx$$

$$=\frac{a^2}{4b^2}\int_0^b\left[x^2\left(2-\sqrt{3}\right)+2bx+\sqrt{3}b^2\right]dx$$

$$=\frac{a^2}{4b^2}\left[\frac{x^3}{3}\left(2-\sqrt{3}\right)+bx^2+\sqrt{3}b^2x\right]_0^b$$

$$=\frac{a^2}{4b^2}\left[\frac{b^3}{3}\left(2-\sqrt{3}\right)+b^3+\sqrt{3}b^3\right]-0$$

$$=\frac{a^2b^3}{4\times3\times b^2}\left(2-\sqrt{3}+3+3\sqrt{3}\right)$$

$$=\frac{a^2b}{12}\left(5+2\sqrt{3}\right).$$

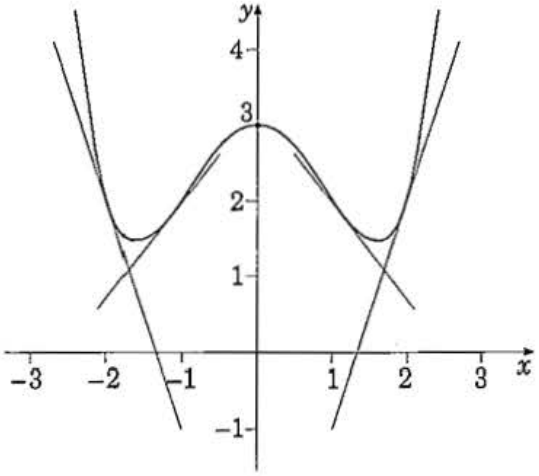
QUESTION 6

(a) (i) If $Q(x)$ is even, then terms are of the form $a_n x^{2n}$, where n is an integer.

$\therefore$ $Q'(x)$ has terms of the form $2na_n x^{2n-1}$, that is, all odd powers of x, that is, $Q'(x)$ is odd.

$\therefore$ The statement is true.

OR The symmetric property of even functions implies that tangents at equal distances from the y axis are also symmetric, that is, they have gradients which are equal in magnitude but opposite in sign. Hence if $Q(x)$ is even, then $Q'(x)$ is odd.



OR
$$Q'(x) = \lim_{h\to 0}\frac{Q(x+h)-Q(x)}{h}$$

$\therefore$
$$Q'(-x) = \lim_{h\to 0}\frac{Q(-x+h)-Q(-x)}{h}$$
$$= \lim_{h\to 0}\frac{Q(x-h)-Q(x)}{h}$$
since $Q(x)$ is even,
$$= -\lim_{h\to 0}\frac{Q(x)-Q(x-h)}{h}$$
$$= -Q'(x).$$

$\therefore$ If $Q(x)$ is even, then $Q'(x)$ is odd.

(ii) If $Q'(x)$ is even, then $Q'(x)$ contains terms of the form $a_n x^{2n}$, where n is an integer.

Then $Q(x)$ contains terms $\frac{a_n}{2n+1}x^{2n+1}$, which are odd powers of x, but $Q(x)$ also possibly contains a constant.

Hence $Q(x)$ is not necessarily odd.

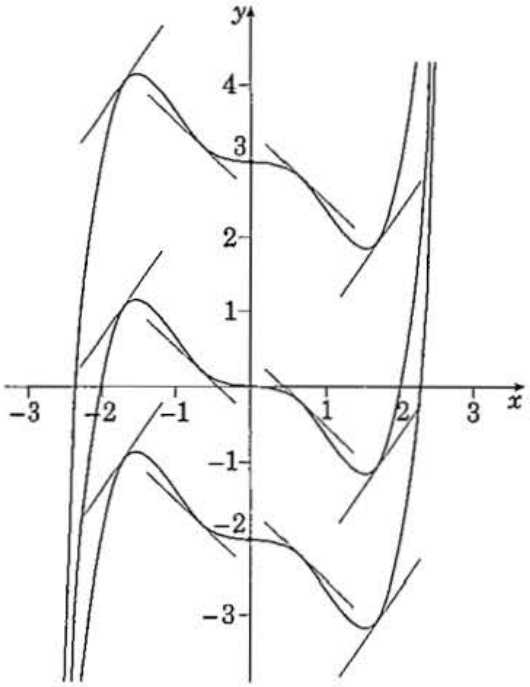
$\therefore$ The statement is false.

OR A similar argument to that given in part (i) applies if $Q(x)$ is odd.

That is, tangents at equal distances from the y axis will be symmetric, in this case with gradients equal in magnitude and sign.

However this argument is just as valid if $Q(x)$ is an odd polynomial plus a constant.

Hence if $Q'(x)$ is even, $Q(x)$ may or may not be odd.



OR A single counter-example will suffice.

$Q(x) = x^3 + x + 3$ is not odd,

but $Q'(x) = 3x^2 + 1$ is even.

$\therefore$ If $Q'(x)$ is even, then $Q(x)$ is not necessarily odd.

(b) $P_n = e^{-2\cdot 6}\frac{(2\cdot 6)^n}{n!}$, $n = 0, 1, 2, 3, \ldots$

(i) $P_0 = e^{-2\cdot 6}\frac{(2\cdot 6)^0}{0!}$

$= e^{-2\cdot 6}$

$= 0\cdot 074\,27\ldots$

$\approx 0\cdot 074$ correct to three decimal places.

(ii) This is a binomial distribution based on $(p+q)^{10}$, where $p = e^{-2\cdot 6} = 0\cdot 074$ and $q = 1 - e^{-2\cdot 6} = 0\cdot 926$.

P(at least 2 years)

$= 1 - $ P(1 year or 0 years)

$= 1 - {}^{10}C_1(0\cdot 926)^9 \times 0\cdot 074 - {}^{10}C_0(0\cdot 926)^{10}$

$= 0\cdot 165\,98\ldots$

$\approx 0\cdot 166$, correct to 3 dec. places.

OR Using the exact values:

P(at least 2 years)

$= 1 - {}^{10}C_1\left(1-e^{-2\cdot6}\right)^9 \times e^{-2\cdot6} - {}^{10}C_0\left(1-e^{-2\cdot6}\right)^{10}$

$= 0\cdot166\,970\ldots$

$\approx 0\cdot167$ correct to 3 dec. places.

(iii) $\dfrac{P_{n+1}}{P_n} = \dfrac{\dfrac{e^{-2\cdot6}(2\cdot6)^{n+1}}{(n+1)!}}{\dfrac{e^{-2\cdot6}(2\cdot6)^{n}}{n!}} = \dfrac{2\cdot6}{n+1}.$

But $\dfrac{P_{n+1}}{P_n} \geq 1.$

$\therefore \quad \dfrac{2\cdot6}{n+1} \geq 1$

$\therefore \quad n+1 \leq 2\cdot6$

$\therefore \quad n \leq 1\cdot6.$

Hence for $\dfrac{P_{n+1}}{P_n}$ to be greater than 1, n must equal 1.

That is, if $n \geq 2$,

$\dfrac{P_{n+1}}{P_n} < 1.$

$\therefore \quad P_n > P_{n+1}$

$\therefore \quad P_2 > P_3 > P_4$, etc.

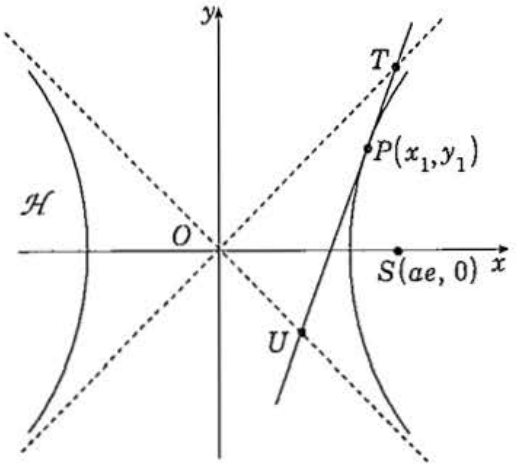
If $n = 1$, $\dfrac{P_2}{P_1} > 1$

$\therefore \quad P_2 > P_1.$

Hence the greatest probability occurs when $n = 2$.

That is, the most likely number of accidents is 2.

(c)



(i) $x^2 - y^2 = a^2$

$\therefore \quad 2x - 2y\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{x}{y}.$

At (x_1, y_1) $\dfrac{dy}{dx} = \dfrac{x_1}{y_1}.$

Hence tangent is $y - y_1 = \dfrac{x_1}{y_1}(x - x_1),$

that is, $yy_1 - y_1^2 = xx_1 - x_1^2,$

that is, $xx_1 - yy_1 = x_1^2 - y_1^2.$

But since (x_1, y_1) lies on $x^2 - y^2 = a^2$,

$x_1^2 - y_1^2 = a^2.$

$\therefore$ Equation of tangent is $xx_1 - yy_1 = a^2.$

(ii) U lies on tangent $xx_1 - yy_1 = a^2$ and on asymptote $y = -x$.

$\therefore \quad x_1x_U + y_1x_U = a^2$

$\therefore \quad x_U = \dfrac{a^2}{x_1 + y_1}.$

$\therefore$ U is the point $\left(\dfrac{a^2}{x_1+y_1}, \dfrac{-a^2}{x_1+y_1}\right).$

$\therefore$ Gradient of $SU = \dfrac{0 + \dfrac{a^2}{x_1+y_1}}{ae - \dfrac{a^2}{x_1+y_1}}$

$= \dfrac{a^2}{(x_1+y_1)ae - a^2}$

$= \dfrac{a}{e(x_1+y_1) - a}.$

(iii) T lies on $y = x$ and on $xx_1 - yy_1 = a^2$.

$\therefore \quad x_T = \dfrac{a^2}{x_1 - y_1}$

$\therefore$ T is $\left(\dfrac{a^2}{x_1-y_1}, \dfrac{a^2}{x_1-y_1}\right)$

$\therefore$ Gradient of $ST = \dfrac{\dfrac{a^2}{x_1-y_1} - 0}{\dfrac{a^2}{x_1-y_1} - ae}$

$= \dfrac{a}{a - e(x_1 - y_1)}.$

Angle between US and ST is given by

$\tan\theta = \dfrac{m_2 - m_1}{1 + m_2m_1},$

where m_1 is the slope of ST and m_2 is the slope of SU.

(N.B. Note the anticlockwise order.)

$\therefore \tan\theta = \dfrac{\dfrac{a}{e(x_1+y_1)-a} - \dfrac{a}{a-e(x_1-y_1)}}{1 + \dfrac{a}{e(x_1+y_1)-a} \times \dfrac{a}{a-e(x_1-y_1)}}$

$= \dfrac{a\left[a - e(x_1 - y_1) - e(x_1 + y_1) + a\right]}{\left[e(x_1+y_1) - a\right]\left[a - e(x_1 - y_1)\right] + a^2}$

$$= \frac{a(2a-2ex_1)}{ae(x_1+y_1)+ae(x_1-y_1)-e^2(x_1^2-y_1^2)-a^2+a^2}$$

$$= \frac{2a(a-ex_1)}{2aex_1-e^2a^2}, \text{ since } x_1^2-y_1^2=a^2$$

$$= \frac{2a(a-ex_1)}{2aex_1-2a^2}, \text{ since } e^2 = 1+\frac{1^2}{1^2} = 1+1 = 2,$$

$$= \frac{2a(a-ex_1)}{2a(ex_1-a)}$$

$= -1$ as required.

QUESTION 7

(a) (i) $P(z) = z^8 - \frac{5}{2}z^4 + 1.$

w is a root of $P(z) = 0$

$\therefore \quad w^8 - \frac{5}{2}w^4 + 1 = 0$

$$P(iw) = (iw)^8 - \frac{5}{2}(iw)^4 + 1$$
$$= i^8w^8 - \frac{5}{2}i^4w^4 + 1$$
$$= w^8 - \frac{5}{2}w^4 + 1$$
$$= 0.$$

$$P\left(\frac{1}{w}\right) = \left(\frac{1}{w}\right)^8 - \frac{5}{2}\left(\frac{1}{w}\right)^4 + 1$$
$$= \frac{1}{w^8}\left(1 - \frac{5}{2}w^4 + w^8\right)$$
$$= \frac{1}{w^8} \times 0$$
$$= 0.$$

$\therefore$ iw and $\frac{1}{w}$ are also roots of $P(z) = 0$.

(ii) $z^8 - \frac{5}{2}z^4 + 1 = 0.$

$\therefore \quad 2z^8 - 5z^4 + 2 = 0,$

that is, $(2z^4 - 1)(z^4 - 2) = 0$

$\therefore$ $z^4 = \frac{1}{2}$ or 2.

$\therefore$ One root of $P(z) = 0$ is $z = \sqrt[4]{2}$.

(iii) Roots are $\pm\sqrt[4]{2}, \ \pm\sqrt[4]{2}\,i,$
$\pm\frac{1}{\sqrt[4]{2}}, \ \pm\frac{1}{\sqrt[4]{2}}\,i.$

(b) (i) $y = \sin^{-1}(u) - \sqrt{1-u^2}$
(defined for $-1 \le u \le 1$).

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} - \frac{1}{2}(1-u^2)^{-\frac{1}{2}} \times -2u$$
(defined for $-1 < u < 1$).
$$= \frac{1}{\sqrt{1-u^2}} + \frac{u}{\sqrt{1-u^2}}$$
$$= \frac{1+u}{\sqrt{1-u^2}}$$
$$= \sqrt{\frac{1+u}{1-u}}.$$

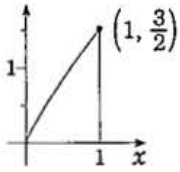
(This result is also valid when $u = -1$.)

(ii) $\int_0^\alpha \sqrt{\frac{1+u}{1-u}}\,du$

$$= \left[\sin^{-1}u - \sqrt{1-u^2}\right]_0^\alpha$$
$$= \sin^{-1}\alpha - \sqrt{1-\alpha^2} - \left(\sin^{-1}0 - \sqrt{1}\right)$$
$$= \sin^{-1}\alpha - \sqrt{1-\alpha^2} + 1.$$

This result is valid for $-1 \le \alpha < 1$ and therefore for $0 < \alpha < 1$.

(c) (i) $y = \frac{3}{2}x^{\frac{2}{3}}, \quad 0 \le x \le 1$



$\dot{x} = x^{\frac{1}{3}}\dot{y}$

$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy = E.$

$t = 0, \ \dot{x} = 0, \ \dot{y} = 0, \ x = 1, \ y = \frac{3}{2}$

$\therefore \quad \frac{1}{2}m \times 0 + \frac{1}{2}m \times 0 + mg \cdot \frac{3}{2} = E$

$$E = \frac{3mg}{2}.$$

$\therefore \quad \frac{1}{2}m \cdot x^{\frac{2}{3}}\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + mgy = \frac{3mg}{2}$

$$x^{\frac{2}{3}}\dot{y}^2 + \dot{y}^2 + 2gy = 3g$$

$$\dot{y}^2\left(x^{\frac{2}{3}} + 1\right) = 3g - 2gy$$

$\therefore \quad \dot{y}^2 = \frac{g(3-2y)}{\frac{2y}{3}+1} \quad \left(\text{since } \frac{2y}{3} = x^{\frac{2}{3}}\right)$

$$= \frac{3g(3-2y)}{3+2y}.$$

(ii) At t_1, $(x, y) = (0, 0)$,

so $\dot{x}(t_1) = 0^{\frac{1}{3}} \times \dot{y}(t_1)$
$= 0.$

and $\left[\dot{y}(t_1)\right]^2 = 3g.$

$\therefore \quad \dot{y}(t_1) = -\sqrt{3g}$ since $\dot{y}(t_1) < 0.$

(iii) $\frac{dt}{dy} = \frac{-1}{\sqrt{3g}}\sqrt{\frac{3+2y}{3-2y}}$

$\therefore \quad t = -\frac{1}{\sqrt{3g}}\int_{\frac{3}{8}}^0 \sqrt{\frac{3+2y}{3-2y}}\,dy$

$$= \frac{1}{\sqrt{3g}}\int_0^{\frac{3}{8}} \sqrt{\frac{3+2y}{3-2y}}\,dy$$

$$= \frac{3}{2\sqrt{3g}} \int_0^{\frac{1}{4}} \sqrt{\frac{3+3u}{3-3u}}\, du \left(\text{using } u = \frac{2}{3}y\right)$$

$$= \frac{3}{2\sqrt{3g}} \int_0^{\frac{1}{4}} \sqrt{\frac{1+u}{1-u}}\, du$$

$$= \frac{3}{2\sqrt{3g}} \left(\sin^{-1}\frac{1}{4} + 1 - \sqrt{1-\frac{1}{16}}\right)$$

$$= \frac{3}{2\sqrt{3g}} \left(\sin^{-1}\frac{1}{4} + 1 - \frac{\sqrt{15}}{4}\right).$$

QUESTION 8

(a) $p>1,\ q>1,\ p = \dfrac{q}{q-1}$.

(i) $f(t) = \dfrac{s^p}{p} + \dfrac{t^q}{q} - st$

$$f'(t) = 0 + \frac{qt^{q-1}}{q} - s.$$

That is, $f'(t) = t^{q-1} - s.$

$$f''(t) = (q-1)t^{q-2} > 0,$$

(since $q>1$ and $t>0$).

Therefore the curve is concave up and so any stationary point must be an absolute minimum.

If $f'(t) = 0,\quad t^{q-1} = s,$

$\therefore\ t = s^{\frac{1}{q-1}}$ minimises the expression.

(ii) Since $f\left(s^{\frac{1}{q-1}}\right)$ is the minimum value of the expression, $f(t) \geq f\left(s^{\frac{1}{q-1}}\right)$ for all $t>0$.

$$f\left(s^{\frac{1}{q-1}}\right) = \frac{s^p}{p} + \frac{s^{\frac{q}{q-1}}}{q} - s\cdot s^{\frac{1}{q-1}}$$

$$= \frac{s^p}{p} + \frac{s^p}{q} - s^{1+\frac{1}{q-1}}$$

$$= \frac{s^p}{p} + \frac{s^p}{q} - s^{\frac{q}{q-1}}$$

$$= \frac{s^p}{p} + \frac{s^p}{q} - s^p, \text{ since } p = \frac{q}{q-1},$$

$$= s^p\left(\frac{1}{p} + \frac{1}{q} - 1\right)$$

$$= s^p\left(\frac{1}{p} + \frac{1-q}{q}\right)$$

$$= s^p\left(\frac{q-1}{q} + \frac{1-q}{q}\right), \text{ since } p = \frac{q}{q-1},$$

$$= s^p \times 0$$

$$= 0.$$

$\therefore$ Minimum value of expression is 0.

$\therefore\ f(t) \geq 0.$

$$\therefore\quad \frac{s^p}{p} + \frac{t^q}{q} - st \geq 0$$

$$\therefore\quad \frac{s^p}{p} + \frac{t^q}{q} \geq st.$$

(iii) To prove $\left(x_1x_2 \ldots x_n\right)^{\frac{1}{n}} \leq \dfrac{x_1+x_2+\cdots+x_n}{n}$ by induction.

Step 1 (Grounding)

When $n=1$, LHS $= x_1^{\frac{1}{1}} = x_1$, RHS $= \dfrac{x_1}{1} = x_1$, so the statement is true when $n=1$, (in this case equality).

Step 2

Assume the statement is true when $n=k$, that is, assume $\left(x_1x_2 \ldots x_k\right)^{\frac{1}{k}} \leq \dfrac{x_1+x_2+\cdots+x_k}{k}$ for $x_1, x_2, \ldots, x_k > 0$.

Required to prove

$$\left(x_1x_2 \ldots x_kx_{k+1}\right)^{\frac{1}{k+1}} \leq \frac{x_1+x_2+\cdots+x_{k+1}}{k+1}.$$

Now LHS $= \left(x_1x_2 \ldots x_kx_{k+1}\right)^{\frac{1}{k+1}}$

$$= \left(x_1x_2 \ldots x_k\right)^{\frac{1}{k+1}} x_{k+1}^{\frac{1}{k+1}}$$

$$= \left[\left(x_1x_2 \ldots x_k\right)^{\frac{1}{k}}\right]^{\frac{k}{k+1}} x_{k+1}^{\frac{1}{k+1}}$$

$$\leq \left(\frac{x_1+x_2+\cdots+x_k}{k}\right)^{\frac{k}{k+1}} x_{k+1}^{\frac{1}{k+1}}$$

by assumption,

$$= st, \text{ if } s = \left(\frac{x_1+x_2+\cdots+x_k}{k}\right)^{\frac{k}{k+1}},\ t = x_{k+1}^{\frac{1}{k+1}}$$

$$\leq \frac{s^{\frac{k+1}{k}}}{\frac{k+1}{k}} + \frac{t^{k+1}}{k+1}.$$

N.B. This is the inequality in (ii), with $p = \dfrac{k+1}{k},\ q = k+1.$

N.B. $\dfrac{q}{q-1} = \dfrac{k+1}{k} = p,$ as required by this inequality.

$$= \frac{k}{k+1}\left(\frac{x_1+x_2+\cdots+x_k}{k}\right) + \frac{1}{k+1}\left(x_{k+1}\right)$$

$$= \frac{x_1+x_2+\cdots+x_k}{k+1} + \frac{x_{k+1}}{k+1}$$

$$= \frac{x_1+x_2+\cdots+x_k+x_{k+1}}{k+1} \text{ as required.}$$

Step 3

We have shown in Step 2 that if the result is true for $n=k$, it is also true for $n=k+1$.

In Step 1 we showed it was true for $n=1$.

Therefore it is true for $n = 1+1 = 2$, and hence for $n = 2+1 = 3$, and so on for all n.

Comments on this question

The notion that for positive numbers the arithmetic mean is greater than the geometric mean might be approached from the point of view of weighted means.

If two positive numbers x and y are averaged with weightings a and b, we have

$$\text{AM} = \frac{ax+by}{a+b}, \quad \text{GM} = \left(x^a y^b\right)^{\frac{1}{a+b}},$$

$$\text{thus AM} - \text{GM} = \frac{a}{a+b}x + \frac{b}{a+b}y - x^{\frac{a}{a+b}}y^{\frac{b}{a+b}}.$$

By substituting $s = x^{\frac{a}{a+b}}, \quad t = y^{\frac{b}{a+b}},$

$$p = \frac{a+b}{a}, \quad q = \frac{a+b}{b}$$

$$\left(\text{N.B. } \frac{q}{q-1} = \frac{a+b}{b} \div \frac{a}{b} = \frac{a+b}{a} = p\right),$$

we have $\text{AM} - \text{GM} = \dfrac{s^p}{p} + \dfrac{t^q}{q} - st.$

Thus parts (i) and (ii) of this problem prove that the weighted arithmetic mean of two positive numbers is greater than their weighted geometric mean for any weightings.

The inductive proof in (iii) essentially uses the weightings $a = n$ applied to the mean of the first n numbers and $b = 1$ applied to the extra number.

(iv) $$\frac{\frac{y_1}{y_2} + \frac{y_2}{y_3} + \frac{y_3}{y_4} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1}}{n}$$

$$\geq \left(\frac{y_1}{y_2} \times \frac{y_2}{y_3} \times \frac{y_3}{y_4} \times \cdots \times \frac{y_{n-1}}{y_n} \times \frac{y_n}{y_1}\right)^{\frac{1}{n}}, \text{ from (iii)},$$

$$= 1^{\frac{1}{n}}$$

$$= 1.$$

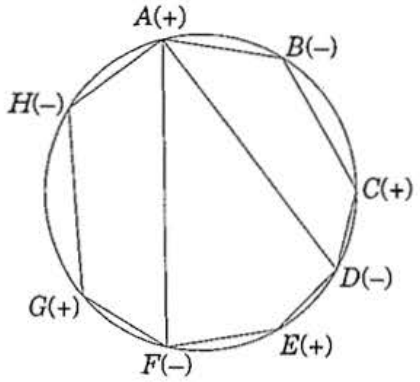
$$\therefore \frac{y_1}{y_2} + \frac{y_2}{y_3} + \frac{y_3}{y_4} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n.$$

(b) (i) $\angle DAB$ and $\angle BCD$ are opposite angles in the cyclic quadrilateral $ABCD$, and are supplementary.
That is, $\angle DAB + \angle BCD = 180°$.
Similarly, $\angle ABC$ and $\angle CDA$ are opposite angles in the same cyclic quadrilateral, and so also sum to 180°.
$\therefore \angle DAB + \angle BCD = \angle ABC + \angle CDA \ (= 180°)$.

(ii) As in (i), both pairs of angles are opposite angles in the same cyclic quadrilateral (in this case $ADEF$), and so
$\angle FAD + \angle DEF = \angle EFA + \angle ADE \ (= 180°)$.

(iii) 'Adding' equations in (i) and (ii) gives
$\angle DAB + \angle BCD + \angle FAD + \angle DEF$
$= \angle ABC + \angle CDA + \angle EFA + \angle ADE,$
but $\angle FAB = \angle FAD + \angle DAB,$
and $\angle CDE = \angle CDA + \angle ADE.$
$\therefore \angle FAB + \angle BCD + \angle DEF$
$= \angle ABC + \angle CDE + \angle EFA.$
$\therefore \angle ABC - \angle BCD + \angle CDE - \angle DEF$
$+ \angle EFA - \angle FAB = 0.$

(iv) and (v) Any even-sided cyclic polygon (including the octagon) can:
(A) have its vertices labelled sequentially (+) or (−);
(B) be divided into quadrilaterals.
The division into quadrilaterals can be done in many ways, but a simple approach is to have a common vertex (A in the diagram) for each quadrilateral, as shown for the octagon.



The $A(+)$ point connects only to (−) points, so the sequential labelling of (+) and (−) points applies to each quadrilateral, as well as to the whole polygon. Each quadrilateral is cyclic, so within each quadrilateral the sum of the two (+) angles equals the sum of the two (−) angles (which are both 180°).
Applying this to the whole polygon, the sum of all (+) angles equals the sum of all (−) angles (which equals 180° times the number of quadrilaterals).
After combining angles at the vertices, this means that the sum of the angles at the (+) vertices equals the sum of the angles at the (−) vertices, and therefore that alternately adding and subtracting angles as you move around an even-sided polygon results in zero.
For the octagon as shown, the argument proceeds as follows:
$\angle BAD + \angle BCD = \angle ABC + \angle CDA$
$\angle DAF + \angle DEF = \angle ADE + \angle EFA$
$\angle FAH + \angle HGF = \angle AFG + \angle FGH.$

$\therefore (\angle BAD + \angle DAF + \angle FAH) + \angle BCD + \angle DEF + \angle FGH$
$= \angle ABC + (\angle CDA + \angle ADE) + (\angle EFA + \angle AFG) + \angle FGH$
(The brackets indicate angles to be combined at a vertex.)

That is,
$\angle BAH + \angle BCD + \angle DEF + \angle FGH = \angle ABC + \angle CDE + \angle EFG + \angle GHA.$

That is,
$\angle ABC - \angle BCD + \angle CDE - \angle DEF + \angle EFG - \angle FGH + \angle GHA - \angle HAB = 0.$

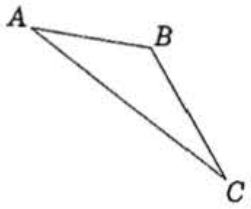
Note that the angles are presented sequentially with alternating signs, the pattern that generalises for even-sided cyclic polygons.

Does the pattern apply to odd-sided cyclic polygons?

With odd-sided polygons we cannot label the vertices (+) and (−) sequentially without finishing with a double (+) [or double (−) if you start with a minus].

So the pattern does not apply.

However, we might consider going around the polygon *twice*. Every vertex would have both a (+) and a (−), and so would cancel itself out. In this sense the result would be true (in a trivial way) and the pattern would include odd polygons as well. For example, for a triangle:



$\angle ABC - \angle BCA + \angle CAB - \angle ABC + \angle BCA - \angle CAB = 0.$

END OF 4 UNIT (ADDITIONAL) SOLUTIONS