1997 Higher School Certificate Solutions

4 UNIT (ADDITIONAL) MATHEMATICS

QUESTION 1

(a) $\displaystyle\int_0^5 \frac{2}{\sqrt{x+4}}\,dx = 2\int_0^5 (x+4)^{-\frac{1}{2}}\,dx$

$\displaystyle = 2\left[\frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^5$

$= 4\left(\sqrt{9}-\sqrt{4}\right)$

$= 4.$

OR

Put $u = x+4$ so that $\dfrac{du}{dx} = 1.$

$x = 0, \quad u = 4$

$x = 5, \quad u = 9.$

$\displaystyle\int_0^5 \frac{2}{\sqrt{x-4}}\,dx = \int_4^9 \frac{2}{\sqrt{u}}\,du$

$= \left[4\sqrt{u}\right]_4^9$

$= 4(3-2)$

$= 4.$

(b) *Method 1*

Put $u = \cos\theta$ so that $\dfrac{du}{d\theta} = -\sin\theta.$

$\theta = 0, \quad u = 1$

$\theta = \dfrac{\pi}{4}, \quad u = \dfrac{1}{\sqrt{2}}.$

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\,d\theta = \int_1^{\frac{1}{\sqrt{2}}} \frac{-1}{u^4}\,du$

$\displaystyle = \left[\frac{1}{3u^3}\right]_1^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{3}\left(2\sqrt{2}-1\right).$

Method 2

Put $u = \sec\theta$ so that $\dfrac{du}{d\theta} = \sec\theta\tan\theta.$

$\theta = 0, \quad u = 1$

$\theta = \dfrac{\pi}{4}, \quad u = \sqrt{2}.$

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \tan\theta\sec^3\theta\,d\theta$

$\displaystyle = \int_1^{\sqrt{2}} u^2\,du$

$\displaystyle = \left[\frac{1}{3}u^3\right]_1^{\sqrt{2}}$

$= \frac{1}{3}\left(2\sqrt{2}-1\right).$

(c) $\displaystyle\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$

$\displaystyle = \int \frac{1}{(x+1)^2+\sqrt{2}^2}\,dx$

$\displaystyle = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + C.$

(d) $\dfrac{4t-6}{(t+1)\left(2t^2+3\right)} \equiv \dfrac{A}{t+1} + \dfrac{Bt+C}{2t^2+3}$

$\therefore \quad 4t-6 = A\left(2t^2+3\right) + (Bt+C)(t+1)$

$= (2A+B)t^2 + (B+C)t + (3A+C).$

Equating like coefficients:

$$\begin{cases} 2A+B = 0 & -1 \\ B+C = 4 & -2 \\ 3A+C = -6 & -3 \end{cases}$$

$\therefore \quad ① - ②: \quad 2A - C = -4 \quad -④$

$\therefore \quad ③ + ④: \quad 5A = -10$

$\therefore \quad A = -2$

$\therefore \quad B = 4$

$\therefore \quad C = 0.$

$\therefore \displaystyle\int \frac{4t-6}{(t+1)\left(2t^2+3\right)}\,dt$

$\displaystyle = \int -\frac{2}{t+1}\,dt + \int \frac{4t}{2t^2+3}\,dt$

$= -2\ln|t+1| + \ln\left(2t^2+3\right) + C$

$\displaystyle = \ln\left[\frac{2t^2+3}{(t+1)^2}\right] + C.$

(e) Let $u = x \qquad v = \tan x$

$\dfrac{du}{dx} = 1 \qquad \dfrac{dv}{dx} = \sec^2 x.$

$\therefore \displaystyle\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx = \left[x\tan x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x\,dx$

$\displaystyle = \frac{\pi}{3}\tan\frac{\pi}{3} - \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx$

$\displaystyle = \frac{\pi}{3}\times\sqrt{3} + \left[\ln(\cos x)\right]_0^{\frac{\pi}{3}}$

$\displaystyle = \frac{\pi}{\sqrt{3}} + \ln\frac{1}{2}$

$\displaystyle = \frac{\pi\sqrt{3}}{3} - \ln 2.$

QUESTION 2

(a) (i) $\sqrt{3}-i \qquad r = \sqrt{\sqrt{3}^2+(-1)^2}$

$= \sqrt{4}$

$= 2.$

$\tan\theta = -\dfrac{1}{\sqrt{3}}$

$\therefore \quad \theta = -\dfrac{\pi}{6}$

$\therefore \quad \sqrt{3}-i = 2\left\lfloor -\dfrac{\pi}{6}\right.$

$= 2\,\mathrm{cis}\left(-\dfrac{\pi}{6}\right)$

$= 2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right).$

(ii) $\left(\sqrt{3}-i\right)^6 = 2^6\,\mathrm{cis}\left(-\dfrac{\pi}{6}\times 6\right)$

$= 2^6(\cos\pi - i\sin\pi)$

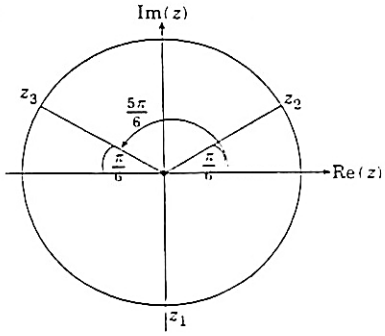
$= -2^6$

$= -64.$

(b) (i) $(-2i)^3 = -8i^3$

$= 8i.$

(ii)



The three complex numbers z will lie on the circle, centre origin, radius 2 units, $\dfrac{2\pi}{3}$ radians (120°) apart.

$z_1 = -2i \quad$ (from (i)).

$z_2 = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

$= \sqrt{3} + i.$

$z_3 = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

$= 2\left(-\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$

$= -\sqrt{3} + i.$

OR

$z^3 = 8i$

$z = \sqrt[3]{8i}$

$= \left[2^3\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)\right]^{\frac{1}{3}}$

$= 2\left[\mathrm{cis}\left(\dfrac{2n\pi}{3} + \dfrac{\pi}{6}\right)\right]$

for $n = 0, \pm 1, \pm 2, \ldots$

$n = 0, \quad z = 2\,\mathrm{cis}\dfrac{\pi}{6} = \sqrt{3} + i.$

$n = 1, \quad z = 2\,\mathrm{cis}\dfrac{5\pi}{6} = -\sqrt{3} + i.$

$n = 2, \quad z = 2\,\mathrm{cis}\dfrac{3\pi}{2} = -2i.$

OR

Use the sum of two cubes:

$z^3 = 8i$

$z^3 - 8i = 0$

$z^3 - 8i^3 = 0$

$(z+2i)\left(z^2 - 2iz - 4i^2\right) = 0$

$(z+2i)\left(z^2 - 2iz - 4\right) = 0.$

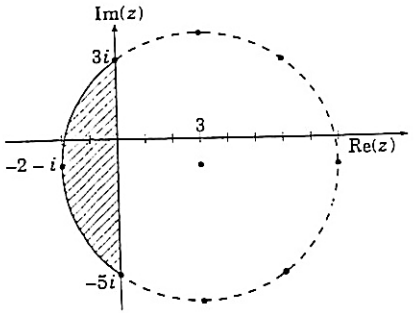
$\therefore z = -2i \quad$ or $\quad z = \dfrac{2i \pm \sqrt{4i^2+16}}{2}$

$= \dfrac{2i \pm \sqrt{12}}{2}$

$= \pm\sqrt{3} + i.$

(c) $|z-3+i| \le 5$ (Interior of circle centre $3-i$ and radius 5.)

$|z+1| \le |z-1|$ (Distance from -1 is less than or equal to distance from -1, that is, $\mathrm{Re}(z) \le 0$.)



(d) $w = \dfrac{3+4i}{5}$, $z = \dfrac{5+12i}{13}$.

(i) $wz = \dfrac{(3+4i)(5+12i)}{65}$

$= -\dfrac{33+56i}{65}$

$= -\dfrac{33}{65} + \dfrac{56}{65}i.$

$w\bar{z} = \dfrac{(3+4i)(5-12i)}{65}$

$= \dfrac{63-16i}{65}$

$= \dfrac{63}{65} - \dfrac{16}{65}i.$

(ii) $|wz| = \sqrt{\left(\dfrac{33}{65}\right)^2 + \left(\dfrac{56}{65}\right)^2}$

$\therefore\ |w|.|z| = \sqrt{\left(\dfrac{33}{65}\right)^2 + \left(\dfrac{56}{65}\right)^2}.$

But $|w| = |z| = 1$,

$\therefore\ 1 = \sqrt{\left(\dfrac{33}{65}\right)^2 + \left(\dfrac{56}{65}\right)^2}$

$\therefore\ 1 = \left(\dfrac{33}{65}\right)^2 + \left(\dfrac{56}{65}\right)^2$

$\therefore\ 65^2 = 33^2 + 56^2 \quad (a = 33,\ b = 56).$

Also, $|w\bar{z}| = \sqrt{\left(\dfrac{63}{65}\right)^2 + \left(\dfrac{16}{65}\right)^2}$

$\therefore\ |w|.|\bar{z}| = \sqrt{\left(\dfrac{63}{65}\right)^2 + \left(\dfrac{16}{65}\right)^2}.$

But $|w| = |z| = 1$, and $|\bar{z}| = |z|$,

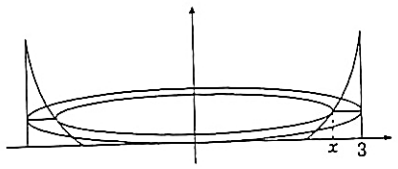
$\therefore\ 1 = \sqrt{\left(\dfrac{63}{65}\right)^2 + \left(\dfrac{16}{65}\right)^2}$

$\therefore\ 65^2 = 16^2 + 63^2 \quad (a = 16,\ b = 63).$

QUESTION 3

(a) The curve is $x^2 + (y-2)^2 = 9$

$x^2 = 9 - (y-2)^2.$



A horizontal section is an annulus with inner radius x, outer radius 3, and area $A = \pi\left(3^2 - x^2\right)$.

$\therefore\quad V = \int_0^2 \pi\left(3^2 - x^2\right) dy$

$= \pi\int_0^2 (y-2)^2\, dy$

$= \dfrac{\pi}{3}\left[(y-2)^3\right]_0^2$

$= \dfrac{\pi}{3}\left[0 - (-2)^3\right]$

$= \dfrac{8\pi}{3}.$

(b) $f(x) = 3x^5 - 10x^3 + 16x.$

(i) $f'(x) = 15x^4 - 30x^2 + 16$

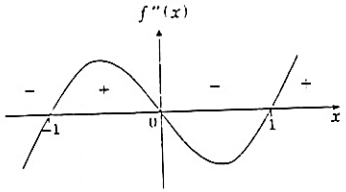
$= 15\left(x^2 - 1\right)^2 + 1$

≥ 1, since $15\left(x^2-1\right)^2 \geq 0$ for all x.

(ii) $f''(x) = 60x^3 - 60x$

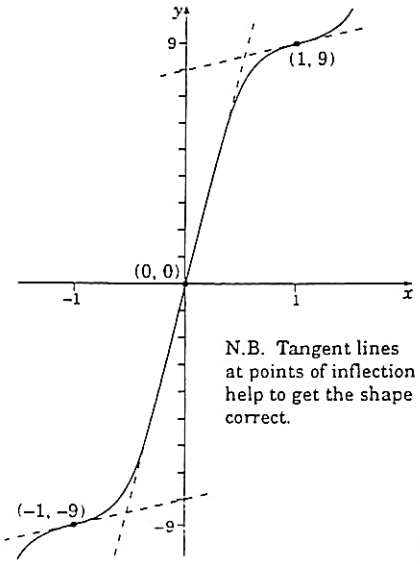
$= 60x\left(x^2 - 1\right)$

$= 60x(x-1)(x+1).$



$f''(x) > 0$ for $x > 1$ and $-1 < x < 0$.

(iii) $f'(x) \geq 1$, so $f(x)$ is monotonic increasing, has no stationary points, and the smallest slope occurs at $x = \pm 1$ where $f'(x) = 1$, that is, at $(-1, -9)$ and $(1, 9)$. These are points of inflection. The other P.O.I. is $(0, 0)$ where the slope is 16.



N.B. Tangent lines at points of inflection help to get the shape correct.

(c) (i) P(A wins on her 1st draw) $= \dfrac{2}{5}$.

(ii) P(B wins on her 1st draw) $= P(\bar{A}B)$

$= \dfrac{3}{5}.\dfrac{3}{5} = \dfrac{9}{25}.$

(iii) P(A wins in less than 4 of her turns)

$= \mathrm{P}(A \text{ or } \bar{A}\bar{B}A \text{ or } \bar{A}\bar{B}\bar{A}\bar{B}A)$

$= \dfrac{2}{5} + \left(\dfrac{3}{5}.\dfrac{2}{5}\right)\dfrac{2}{5} + \left(\dfrac{3}{5}.\dfrac{2}{5}.\dfrac{3}{5}.\dfrac{2}{5}\right).\dfrac{2}{5}$

$= \dfrac{2}{5}\left(1 + \dfrac{6}{25} + \dfrac{36}{625}\right)$

$= \dfrac{1622}{3125}$ (or 0 519 04).

(iv) P(A wins eventually)

$= \dfrac{2}{5}\left[1 + \dfrac{6}{25} + \left(\dfrac{6}{25}\right)^2 + \cdots\right]$

$= \dfrac{2}{5}\left(\dfrac{1}{1 - \frac{6}{25}}\right) \quad \left\{\begin{array}{l}\text{Sum of GP, } a = \frac{2}{5} \\ \text{and } r = \frac{6}{25}.\end{array}\right\}$

$= \dfrac{2}{5} \times \dfrac{25}{19}$

$= \dfrac{10}{19}$ (or 0 5263).

QUESTION 4

(a) (i) $e = \sqrt{1 + \dfrac{b^2}{a^2}}$ $\qquad a^2 = 9,\ b^2 = 7$

$a = 3,\ b = \sqrt{7}.$

$= \sqrt{1 + \dfrac{7}{9}}$

$= \sqrt{\dfrac{16}{9}}$

$= \dfrac{4}{3}.$

Foci have coordinates

$(\pm ae, 0) = \left(\pm 3 \times \dfrac{4}{3}, 0\right) = (\pm 4, 0).$

(ii) Directrices have equations $x = \pm\dfrac{a}{e}$.

That is, $x = \pm\dfrac{3}{\frac{4}{3}}$, that is, $x = \pm\dfrac{9}{4}$.

(iii) For T, substitute $y = 0$ into

$\dfrac{x_0 x}{9} - \dfrac{y_0 y}{7} = 1.$

$\therefore\quad \dfrac{x_0 x}{9} = 1$

$\therefore\quad x = \dfrac{9}{x_0}.$

$\therefore\ T = \left(\dfrac{9}{x_0}, 0\right).$

(iv) From the focus–directrix definition, we have $PS = ePM$ and $PS' = ePM'$.

$\therefore\quad \dfrac{PS}{PS'} = \dfrac{PM}{PM'} = \dfrac{x_0 - \frac{9}{4}}{x_0 - \frac{9}{4}} = \dfrac{4x_0 - 9}{4x_0 - 9}.$

But $TS = 4 - \dfrac{9}{x_0}$

and $TS' = \dfrac{9}{x_0} - (-4) = \dfrac{9}{x_0} + 4.$

$\therefore\quad \dfrac{TS}{TS'} = \dfrac{4 - \frac{9}{x_0}}{4 + \frac{9}{x_0}} = \dfrac{4x_0 - 9}{4x_0 + 9}$

$\therefore\quad \dfrac{PS}{PS'} = \dfrac{TS}{TS'} \quad \left(= \dfrac{4x_0 - 9}{4x_0 + 9}\right).$

(b) (i) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$\therefore\ \cot 2A = \dfrac{1 - \tan^2 A}{2\tan A}.$

(ii) From (i), $2\tan A \cot 2A = 1 - \tan^2 A$.

$\therefore\ \tan^2 A - 2\tan A \cot 2A - 1 = 0.$

Substituting $x = \tan A$ gives

$x^2 + 2x\cot 2A - 1 = 0.$

So $x = \tan A$ is a solution of this equation.

But the product of roots is -1 ($c.a$).

so the other root is given by $x \tan A = -1$.

$\therefore \quad x = -\dfrac{1}{\tan A} = -\cot A.$

OR

Let the roots of $x^2 + 2x\cot 2A - 1 = 0$ be α and β.

$\therefore \quad \alpha + \beta = -2\cot 2A$ and $\alpha\beta = -1$.

Now $\alpha + \beta = -2\left(\dfrac{1-\tan^2 A}{2\tan A}\right)$ (from (i))

$= \tan A - \dfrac{1}{\tan A}$

$= \tan A + (-\cot A).$

Since $\tan A \times (-\cot A) = -1$, we conclude that the two roots are $\tan A$ and $(-\cot A)$.

OR

If $x = \tan A$,

$\text{LHS} = \tan^2 A + 2\tan A \times \dfrac{1-\tan^2 A}{2\tan A} - 1$

$= \tan^2 A + 1 - \tan^2 A - 1$

$= 0$

$= \text{RHS}.$

$\therefore$ $\tan A$ satisfies the equation.

If $x = -\cot A$,

$\text{LHS} = \cot^2 A + 2\cot 2A \times -\cot A - 1$

$= \dfrac{1}{\tan^2 A} - \dfrac{2\left(1-\tan^2 A\right)}{2\tan^2 A} - 1$

$= \dfrac{2-2-2\tan^2 A}{2\tan^2 A} - 1$

$= 0$

$= \text{RHS}.$

$\therefore$ $-\cot A$ satisfies the equation.

(iii) $\tan\dfrac{\pi}{8}$ and $-\cot\dfrac{\pi}{8}$ are the solutions of

$x^2 + 2x\cot\dfrac{\pi}{4} - 1 = 0.$

That is, $x^2 + 2x - 1 = 0 \quad \left(\text{since } \cot\dfrac{\pi}{4} = 1\right).$

That is, $(x+1)^2 = 2$

$\therefore \quad x = -1 \pm \sqrt{2}.$

$\therefore \tan\dfrac{\pi}{8} = \sqrt{2} - 1$ (the other solution is $-$ve),

and $-\cot\dfrac{\pi}{8} = -1 - \sqrt{2}$ (useful for (iv)).

(iv) $\tan\dfrac{\pi}{16} - \cot\dfrac{\pi}{16}$ is the sum of the roots of the equation $x^2 - 2x\cot\dfrac{\pi}{8} - 1 = 0$.

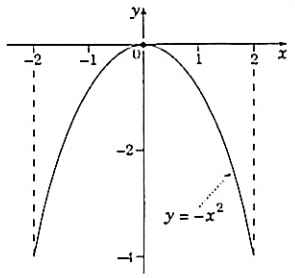
$\therefore \quad \tan\dfrac{\pi}{16} - \cot\dfrac{\pi}{16} = -2\cot\dfrac{\pi}{8} \quad \left(-\dfrac{b}{a}\right)$

$= -2\left(\sqrt{2} - 1\right)$

$\left(\text{since } \cot\dfrac{\pi}{8} = \sqrt{2} - 1, \text{ from (iii)}\right).$

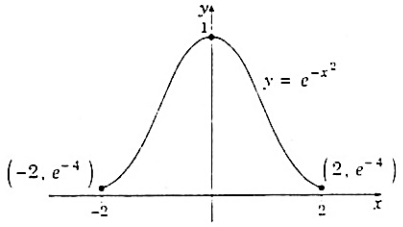
QUESTION 5

(a) (i)

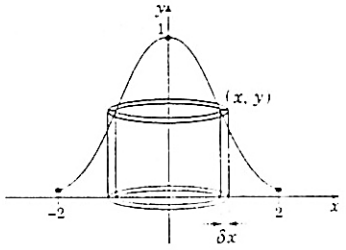


(ii) e^{-x^2} is an even function. The graph is smooth and symmetric about the y axis, going through $(0, 1)$ and $\left(\pm 2, e^{-4}\right)$.

Note that $e^{-4} \approx 0{\cdot}02$.



(iii)



The volume of a typical shell is

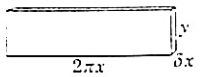
$\delta V = \pi y\left[(x+\delta x)^2 - x^2\right]$

$= \pi e^{-x^2}\left[2x\delta x + (\delta x)^2\right]$

$\approx 2\pi x e^{-x^2}\delta x$ (ignoring the $(\delta x)^2$ term).

or:

Stretch the shell into a rectangular sheet.



$\therefore \quad \delta V = 2\pi x \times y \times \delta x$

$= 2\pi x e^{-x^2}\delta x.$

The total volume is

$V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{2} 2\pi x\left(e^{-x^2}\right)\delta x$

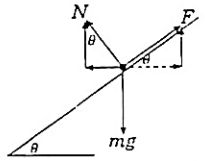
$= \pi\int_0^2 2x e^{-x^2}\,dx$

$= \pi\left[-e^{-x^2}\right]_0^2$

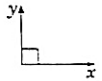
$= -\pi\left(e^{-4} - e^0\right)$

$= \pi\left(1 - e^{-4}\right).$

(b)



Resolving in the two directions:



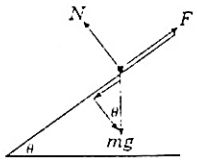
$x \rightarrow$: $\quad F\cos\theta - N\sin\theta = 0 \quad$ — ①

$y \uparrow$: $\quad F\sin\theta + N\cos\theta - mg = 0 \quad$ — ②

From ①: $\quad F\cos\theta = N\sin\theta.$

$\therefore \quad \dfrac{F}{N} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta.$

OR



Resolving in the two directions:



$x \nearrow$: $\quad F - mg\sin\theta = 0$

$y \nwarrow$: $\quad N - mg\cos\theta = 0.$

$\therefore \quad \dfrac{F}{N} = \dfrac{mg\sin\theta}{mg\cos\theta} = \tan\theta.$

(c) (i) $\quad P(z) = z^4 + bz^2 + d$

$\therefore \quad P'(z) = 4z^3 + 2bz$

$\therefore \quad P'(a) = 4a^3 + 2ba$

and $P'(-a) = 4(-a)^3 + 2b(-a)$

$= -\left(4a^3 + 2ba\right)$

$= -P'(a).$

$\therefore$ $P'(z)$ is an odd function.

(ii) We can use the principle that α is a double root (at least) of a polynomial $P(z)$ if and only if $P(\alpha) = 0$ and $P'(\alpha) = 0$.

Since in this case α is a double root, it follows that both $P(\alpha) = 0$ and $P'(\alpha) = 0$.

Since $P(z)$ is even: $\quad P(-\alpha) = P(\alpha) = 0.$

Since $P'(z)$ is odd: $\quad P'(-\alpha) = -P'(\alpha) = 0.$

$\therefore$ $-\alpha$ is a double root (using the principle again).

OR

$P(-\alpha) = (-\alpha)^4 + b(-\alpha)^2 + d$

$= \alpha^4 + b\alpha^2 + d$

$= P(\alpha) = 0.$

So $-\alpha$ is a root. If the fourth root is β, then, since the roots sum to zero, we have $\beta - \alpha + \alpha + \alpha = 0$.

$\therefore \quad \beta = -\alpha.$

So $-\alpha$ is a double root.

(iii) ***Method 1*** (using coefficients)

The roots of $P'(z)$ are α, $-\alpha$ and 0.

The sum of the product of the roots, two at a time, is $\dfrac{2b}{4} = \dfrac{b}{2}$.

$\therefore \quad \alpha\times(-\alpha) + \alpha\times 0 + (-\alpha)\times 0 = \dfrac{b}{2}.$

That is, $\quad \alpha^2 = -\dfrac{b}{2}.$

The product of the roots of $P(z)$ is d.

$\therefore \quad \alpha\times\alpha\times(-\alpha)\times(-\alpha) = d.$

That is, $\quad \alpha^4 = d,$

but $\quad \alpha^4 = \left(\alpha^2\right)^2$

$= \left(-\dfrac{b}{2}\right)^2$

$= \dfrac{b^2}{4}.$

$\therefore \quad d = \dfrac{b^2}{4}.$

Method 2

The alternative substitution method uses $P'(\alpha) = 0$ and $P(\alpha) = 0$.

Combinations of coefficient and substitution methods also work.

Method 3

$P(z) = z^4 + bz^2 + d$

$= u^2 + bu + d, \quad \text{where } u = z^2$

$= Q(u).$

Since α is a double root of $P(z)$, α^2 is a double root of $Q(u)$.

This means that $Q(u)$ is a perfect square.

(continued)

But $Q(u) = \left(u + \frac{b}{2}\right)^2 + d - \frac{b^2}{4}$,

and so $d - \frac{b^2}{4} = 0$.

$\therefore \quad d = \frac{b^2}{4}$.

(iv) From (iii): $b = -2\alpha^2$
$= -2 \times \left(\sqrt{3}\,i\right)^2$, if $\alpha = \sqrt{3}\,i$,
$= -2 \times -3$
$= 6$.

(v) Using $\alpha^2 = -\frac{b}{2}$,

if $b > 0$, then roots are imaginary $\left(\pm\sqrt{\frac{b}{2}}\,i\right)$,

if $b < 0$, then roots are real $\left(\pm\sqrt{-\frac{b}{2}}\right)$.

N.B. $b \neq 0$, since $d \neq 0$.

$\therefore$ For real roots, $b < 0$.

QUESTION 6

(a) (i) $1 - x^2 + x^4 - \cdots + x^{4n}$
$= 1 - x^2 + \left(x^2\right)^2 - \cdots + \left(x^2\right)^{2n}$.

This is the sum of a GP with $a = 1$, $r = -x^2$ and $N = 2n + 1$.

$\therefore \quad S_N = \frac{a\left(1 - r^N\right)}{1 - r}$

$= \frac{1 \times \left[1 - \left(-x^2\right)^{2n+1}\right]}{1 - \left(-x^2\right)}$

$= \frac{1 - \left(x^2\right)^{2n+1}}{1 + x^2}$.

$\left(\text{since } (-x^2)^{2n+1} = -(x^2)^{2n+1}\right.$
$\left.\text{because } 2n + 1 \text{ is odd}\right)$

$= \frac{1 - x^{4n+2}}{1 - x^2}$.

(ii) As $x^2 \geq 0$ for all x, $1 - x^2 \geq 1$.

$\therefore \quad 0 < \frac{1}{1 - x^2} \leq 1 \qquad -①$

Also, $x^{4n-2} = \left(x^{2n-1}\right)^2 \geq 0$, for all x.

Multiply inequality ① by x^{4n-2}, giving

$0 \leq \frac{x^{4n-2}}{1 - x^2} \leq x^{4n-2}$.

N.B. $<$ becomes $\leq$ because x^{4n-2} might be zero, and so we might have multiplied by zero.

Adding $\frac{1}{1 - x^2}$:

$\frac{1}{1 + x^2} \leq \frac{1}{1 + x^2} + \frac{x^{4n+2}}{1 + x^2}$

$\leq \frac{1}{1 + x^2} + x^{4n+2}$.

$\therefore \quad \frac{1}{1 + x^2} \leq 1 - x^2 + x^4 - \cdots + x^{4n}$

$\leq \frac{1}{1 + x^2} + x^{4n+2}$

(using (i)).

(iii) As $f(x) = \frac{1}{1 + x^2}$, $g(x) = \frac{1 + x^{4n+2}}{1 + x^2}$,

$h(x) = \frac{1}{1 + x^2} + x^{4n+2}$ are continuous functions for all x, then if $y \geq 0$,

$\int_0^y \frac{1}{1 + x^2}\,dx \leq \int_0^y \frac{1 + x^{4n+2}}{1 + x^2}\,dx$

$\leq \int_0^y \frac{1}{1 - x^2} + x^{4n-2}\,dx$.

$\therefore \tan^{-1} y \leq y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots + \frac{y^{4n+1}}{4n+1}$

$\leq \tan^{-1} y + \frac{y^{4n+3}}{4n+3}$.

But if $0 \leq y \leq 1$, then $y^{4n+3} \leq 1$.

$\therefore \tan^{-1} y + \frac{y^{4n+3}}{4n+3} \leq \tan^{-1} y + \frac{1}{4n+3}$.

$\therefore \tan^{-1} y \leq y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots + \frac{y^{4n+1}}{4n-1}$

$\leq \tan^{-1} y + \frac{1}{4n+3}$.

(iv) Substitute $y = 1$, $n = 250$.

$\therefore \tan^{-1} 1 \leq 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{1001}$

$\leq \tan^{-1} 1 + \frac{1}{1003}$.

i.e. $\frac{\pi}{4} < 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{1001} < \frac{\pi}{4} + \frac{1}{1003}$.

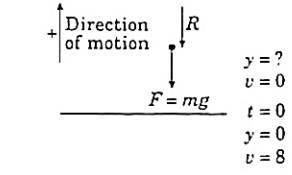
N.B. $\leq$ has been replaced by $<$ since the outside expressions are irrational and the inside expression is rational, so equality is ruled out.

$\therefore 0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots - \frac{1}{1001}\right) - \frac{\pi}{4} < \frac{1}{1003}$.

But $1003 > 1000$, so $\frac{1}{1003} < \frac{1}{1000}$.

$\therefore 0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots - \frac{1}{1001}\right) - \frac{\pi}{4} < \frac{1}{1000}$.

(b)



$m\ddot{y} = -(F + R)$

$2\ddot{y} = -20 - \frac{v^2}{5}$

$\therefore \quad \ddot{y} = -10 - \frac{v^2}{10}$.

(i) $\ddot{y} = v\frac{dv}{dy}$.

$\therefore \quad v\frac{dv}{dy} = -10 - \frac{v^2}{10}$

$\frac{dv}{dy} = \frac{-10}{v} - \frac{v}{10}$

$\frac{dy}{dv} = \frac{-10v}{v^2 + 100}$

$\therefore \quad \left[\,y\,\right]_0^y = \int_8^v \frac{-10v}{v^2 + 100}\,dv$

$y = -5\left[\ln\left(v^2 + 100\right)\right]_8^v$

$-\frac{y}{5} = \ln\left(v^2 + 100\right) - \ln 164$

$-\frac{y}{5} = \ln\left(\frac{v^2 + 100}{164}\right)$

$\therefore \quad \frac{v^2 + 100}{164} = e^{-\frac{y}{5}}$

$\therefore \quad v^2 = 164\,e^{-\frac{y}{5}} - 100$.

(ii) Max. height is reached when $v = 0$.

Subst. into $y = -5\ln\left(\frac{v^2 - 100}{164}\right)$

$\therefore \quad y = -5\ln\left(\frac{100}{164}\right) = 5\ln\left(\frac{164}{100}\right)$.

$\therefore$ Max. height $= 5\ln 1{\cdot}64$ metres.

(iii) $\ddot{y} = \frac{dv}{dt}$.

$\therefore \quad \frac{dv}{dt} = -10 - \frac{v^2}{10}$

$\frac{dt}{dv} = \frac{-10}{v^2 - 100}$

$\therefore \quad t = \int_8^0 \frac{-10}{v^2 - 100}\,dv$

$= -10 \cdot \frac{1}{10}\left[\tan^{-1}\frac{v}{10}\right]_8^0$

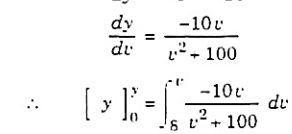
$= \tan^{-1}\left[\frac{v}{10}\right]_0^8$

$= \tan^{-1}\left(\frac{4}{5}\right)$ ($\approx 0{\cdot}67$ seconds).

$\therefore$ Reaches max. height after $\tan^{-1}\left(\frac{4}{5}\right)$ seconds.

(iv) As the ball falls, the resistance changes direction and so we must use a modified equation of motion. The syllabus recommends using a change of coordinate system. In the method shown below, the origin (mentioned in the question) becomes $5\ln 1{\cdot}64$ (from (ii)). It is also possible to use the original coordinate system, in which case the origin remains the origin. Both methods are given.

Method 1 (change of coordinate system)



$my = F - R$

$2y = 20 - \frac{v^2}{5}$

$\therefore \quad \ddot{y} = 10 - \frac{v^2}{10}$.

$v\frac{dv}{dy} = 10 - \frac{v^2}{10}$

$\frac{dv}{dy} = \frac{10}{v} - \frac{v}{10}$

$\frac{dy}{dv} = \frac{10v}{100 - v^2}$

$\therefore \quad \left[\,y\,\right]_0^{5\ln 1.64} = \int_0^v \frac{10v}{100 - v^2}\,dv$.

(N.B. v is at the original origin now.)

$5\ln 1{\cdot}64 = -5\left[\ln\left(100 - v^2\right)\right]_0^v$

$-\ln 1{\cdot}64 = \ln\left(100 - v^2\right) - \ln 100$

$\ln\left(100 - v^2\right) = \log\left(\frac{100}{1{\cdot}64}\right)$.

$\therefore \quad 100 - v^2 = \frac{100}{1{\cdot}64}$

$v^2 = \frac{64}{1{\cdot}64}$

$v = \frac{8}{\sqrt{1{\cdot}64}}$ (since $v > 0$)

$= 6{\cdot}25$ m/s.

$\therefore$ Ball is travelling at 6·25 m/s approx. when it returns to the origin.

(continued)

Method 2 (using original coordinate system)

$$\ddot{y} = \frac{v^2}{10} - 10.$$

$$v\frac{dv}{dy} = \frac{v^2 - 100}{10}$$

$$\frac{dy}{dv} = \frac{10v}{v^2 - 100}$$

$$y = 5\ln\left(100 - v^2\right) + C. \quad \left(\text{N.B. } v^2 < 100\right)$$

$v = 0$: $y = 5\ln 1{\cdot}64$.

$\therefore \quad C = 5\ln 1{\cdot}64 - 5\ln 100$

$= 5\ln(0{\cdot}0164)$.

$\therefore \quad y = 5\ln\left[\left(100 - v^2\right) \times 0{\cdot}0164\right]$.

At the origin:

$y = 0 \Rightarrow \left(100 - v^2\right) \times 0{\cdot}0164 = 1$.

$\therefore \quad v^2 = 100 - \dfrac{1}{0{\cdot}0164}$

$= 100\left(1 - \dfrac{1}{1{\cdot}64}\right)$

$= 100\dfrac{0{\cdot}64}{1{\cdot}64}$.

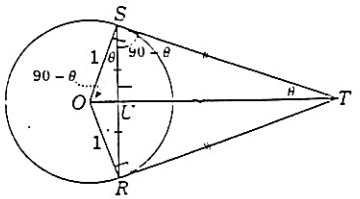
$\therefore \quad v = -\dfrac{10 \times 0{\cdot}8}{\sqrt{1{\cdot}64}}$

$= \dfrac{-8}{\sqrt{1\,64}}$.

Thus the ball is travelling at $\dfrac{8}{\sqrt{1\,64}}$ m/s when it returns to the origin.

QUESTION 7

(a)



Let $\angle STU = \theta$

$\angle SUT = 90°$ ($OT \perp SR$)

$\therefore \quad \angle UST = 90° - \theta$ ($\angle$ sum of $\Delta SUT = 180°$)

$\angle OST = 90°$ (radius $OS \perp$ tangent ST)

$\therefore \quad \angle OSU = \theta$

$\therefore \quad \angle SOU = 90° - \theta$ ($\angle$ sum of $\Delta SOU = 180°$)

$\therefore \quad \Delta SOU \,|||\, \Delta TOS$ (equiangular)

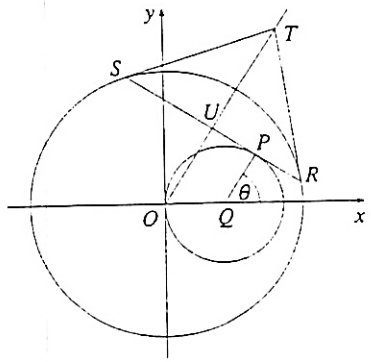
$\therefore \quad \dfrac{OU}{OS} = \dfrac{OS}{OT}$ (corres. sides of similar Δs in same ratio)

$\therefore \quad OU \cdot OT = OS^2$.

But OS = radius of circle = 1 unit.

$\therefore \quad OU \cdot OT = 1$.

(b)



(i) $OT \perp RS$ (data).

$QP \perp RS$ (tangent and a radius at the point of contact are perpendicular).

$\therefore \quad OT \| QP$ (as both are perpendicular to RS).

(ii) $m_{QP} = \dfrac{r\sin\theta - 0}{(r + r\cos\theta) - r}$ $\left\{\begin{matrix} Q(r, 0); \\ P(r + r\cos\theta, r\sin\theta)\end{matrix}\right\}$

$= \tan\theta$.

$\therefore m_{RS} = -\cot\theta \quad \left(m_{QP} \cdot m_{RS} = -1\right)$.

RS: $\quad y - r\sin\theta = -\cot\theta\left[x - (r + r\cos\theta)\right]$

$\therefore y\sin\theta - r\sin^2\theta = -x\cos\theta + r\cos\theta(1 + \cos\theta)$ (multiplying by $\sin\theta$)

$\therefore \quad x\cos\theta + y\sin\theta = r\left(\sin^2\theta + \cos\theta + \cos^2\theta\right)$,

that is, $x\cos\theta + y\sin\theta = r(1 + \cos\theta)$, as required.

OR

OU: $\quad y = x\tan\theta$.

OU meets RS when

$x\cos\theta + x\tan\theta \cdot \sin\theta = r(1 + \cos\theta)$.

Multiply by $x\cos\theta$:

$$x\left(\cos^2\theta + \sin^2\theta\right) = r\cos\theta(1 + \cos\theta)$$

$$x = r\cos\theta(1 + \cos\theta)$$

$$\therefore \quad y = r\sin\theta(1 + \cos\theta).$$

$\therefore \quad OU^2 = r^2\cos^2\theta(1 + \cos\theta)^2 + r^2\sin^2\theta(1 + \cos\theta)^2$

$= r^2(1 + \cos\theta)^2\left(\cos^2\theta + \sin^2\theta\right)$.

$\therefore \quad OU = r(1 + \cos\theta)$.

(iv) As $OU \cdot OT = 1$, then $OT = \dfrac{1}{r(\cos\theta + 1)}$.

Now, $\angle TOQ = \theta$ (corres. $\angle$s equal, $OT \| QP$).

$\therefore$ Coord's of T are given by

$x = OT\cos\theta, \quad y = OT\sin\theta$

$\therefore \quad x = \dfrac{\cos\theta}{r(\cos\theta + 1)}, \quad y = \dfrac{\sin\theta}{r(\cos\theta + 1)}$.

If T lies on the curve $r^2y^2 + 2rx = 1$, then the coord's satisfy the equation.

LHS $= r^2\left(\dfrac{\sin\theta}{r(\cos\theta + 1)}\right)^2 + 2r\left(\dfrac{\cos\theta}{r(\cos\theta + 1)}\right)$

$= \dfrac{\sin^2\theta}{(\cos\theta + 1)^2} + \dfrac{2\cos\theta(\cos\theta + 1)}{(\cos\theta + 1)^2}$

$= \dfrac{1 - \cos^2\theta + 2\cos^2\theta + 2\cos\theta}{(\cos\theta + 1)^2}$

$= \dfrac{1 + 2\cos\theta + \cos^2\theta}{(\cos\theta + 1)^2}$

$= \dfrac{(1 + \cos\theta)^2}{(\cos\theta + 1)^2}$

$= 1$

= RHS.

$\therefore \quad T$ lies on the curve $r^2y^2 + 2rx = 1$.

(c) (i) $\quad x^2 = 4ay \quad -①$

$x^2 + y^2 + 2gx + 2fy + c = 0 \quad -②$

Substitute for $y = \dfrac{x^2}{4a}$ from ① into ②:

$x^2 + \left(\dfrac{x^2}{4a}\right)^2 + 2gx + \dfrac{2fx^2}{4a} + c = 0.$

$\therefore \quad \dfrac{x^4}{16a^2} + \left(1 + \dfrac{f}{2a}\right)x^2 + 2gx + c = 0.$

$\left(\text{Or } x^4 + (16a^2 + 8af)x^2 + 32a^2gx + 16a^2c = 0.\right)$

(ii) The four roots of this quartic are $\alpha, \alpha, \beta, \gamma$, where α is the x value of the point J, (double root), β is the x value of the point K, and γ is the x value of point L.

The sum of the roots $= 0$ $\left(\text{coeff. of } x^3\right)$

$\therefore \quad 2\alpha + \beta + \gamma = 0.$

The x value of M, the midpoint of KL, is $\left(\dfrac{\beta + \gamma}{2}\right)$.

The x value of the midpoint of JM is

$$\frac{\alpha + \left(\dfrac{\beta + \gamma}{2}\right)}{2} = \frac{1}{4}(2\alpha + \beta + \gamma)$$

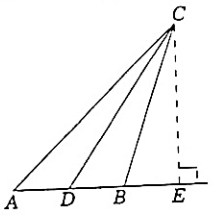
$$= \frac{1}{4} \times 0$$

$$= 0.$$

$\therefore$ The midpoint of JM has x value 0, which is the x value of N.

$\therefore \quad N$ is the midpoint, and $JN = NM$.

(iii) The result follows from repeated use of the fact that a median divides a triangle into two equal areas.



If D is the midpoint of AB, then Δs ACD and DCB have the same base ($AD = DB$) and the same common height (CE), and therefore the same area.

Alternatively, the areas are given by

$$\frac{1}{2} \times AD \times DC \times \sin\angle ADC,$$

and $\dfrac{1}{2} \times DB \times DC \times \sin\angle BDC$.

But $\sin\angle ADC = \sin\angle BDC$ (supplementary $\angle$s),

so if $AD = DB$, the areas are equal.

So area ΔJKN = area ΔNKM ($JN = NM$, from (ii)),

$\therefore$ area $\Delta JKM = 2 \times$ area ΔJKN.

Also, area ΔJKM = area ΔJML ($KM = ML$, given),

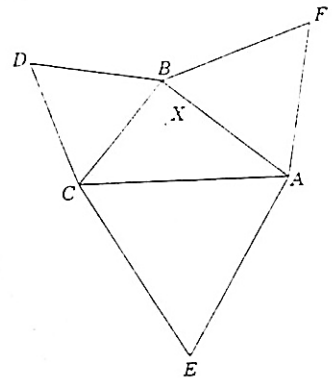
$\therefore$ area $\Delta JKL = 2 \times$ area ΔJKM

$= 4 \times$ area ΔJKN,

$\therefore$ area $\Delta JKN = \dfrac{1}{4}$ area ΔJKL, as required.

QUESTION 8

(a)



(i) $\quad \angle BCE = \angle BCA + \angle ACE$

$= \angle BCA + 60°$.

$\angle DCA = \angle DCB - \angle BCA$

$= 60° - \angle BCA$.

$\therefore \quad \angle BCE = \angle DCA$.

(ii) $BC = DC$ (data, ΔBCD equilateral).
$CE = CA$ (ΔACE equilateral).
$\angle BCE = \angle DCA$ (proved in (i)).
$\therefore\ \Delta BCE \equiv \Delta DCA$ (SAS).

(iii) $\angle CBE = \angle CDA$ (from congruent Δs BCE, DCA).
$\angle CBE$, $\angle CDA$ are equal angles standing on CX and so may be considered as $\angle$s at the circumference of a circle through B, D, C, X,
that is, $BDCX$ is a cyclic quadrilateral.

(iv) From (iii):
$\angle BXD = \angle BCD$
$= \frac{\pi}{3}$ ($\angle$s at circumference on BD).
$\angle DXC = \angle DBC$
$= \frac{\pi}{3}$ ($\angle$s at circumference on DC).
$\angle CXE = \angle CDB$
$= \frac{\pi}{3}$ (ext. $\angle$ of a cyclic quad. is equal to the int. opp. $\angle$).
$\angle EXA = \angle BXD$
$= \frac{\pi}{3}$ (vertically opp. $\angle$s).

(v) Reflex angle $BXA = 240^\circ$ (sum of four angles in (iv)).
$\therefore\ \angle BXA = 120^\circ$
$\therefore\ \angle BXA$, $\angle BFA$ are supplementary ($\angle BFA = 60^\circ$)
$\therefore\ BXAF$ is a cyclic quadrilateral.
$\therefore\ \angle BXF = \angle BAF = 60^\circ$ ($\angle BAF$ is in equilateral ΔBAF and both $\angle$s are at the circumference standing on arc BF).
As $\angle BXC = 120^\circ$ and $\angle BXF = 60^\circ$
$\therefore\ C, X, F$ are collinear.

(vi) $BE = AD$ (corresponding sides, $\Delta BCE \equiv \Delta DCA$).
Also, $CB = DB$ (equilateral ΔDBC)
$FB = AB$ (equilateral ΔFBA)
$\angle FBC = \angle ABD$ (both $60^\circ + \angle ABC$).
$\therefore\ \Delta FBC \equiv \Delta ABD$ (SAS)
$\therefore\ FC = AD$ (corresponding sides)
$\therefore\ FC = BE = AD$.

N.B. This last argument repeats parts (i) and (ii), with ΔBFA replacing ΔBDC.
There is a symmetry in the diagram—any result involving D, C, F and A will also be true if D and F, and C and A are interchanged. Therefore, having said that $BE = AD$, we can immediately conclude that $BE = FC$ (symmetry, F replacing D, C replacing A).

(b) (i) $\angle UOT = \frac{\pi}{3}$ (equilateral ΔUOT).
Multiplication of t by $\omega\left(= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ means a rotation, in an anticlockwise direction about 0, through $\frac{\pi}{3}$,
so that $u = \omega t$.

(ii) $\angle ROS = \frac{\pi}{3}$ (equilateral ΔORS).
To get from S to R means a similar rotation of $\left(-\frac{\pi}{3}\right)$.
$\therefore\quad r = s\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$
$= s\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$
$= s\bar{\omega}$ where $\bar{\omega} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$
OR
$s = \omega r$ (as for (i)).
$\therefore\quad r = \frac{s}{\omega}$ $\left(= s\bar{\omega}$ since $\frac{1}{\omega} = \bar{\omega}\right)$.

(iii) $\vec{RT} = t - r$
$\vec{SU} = u - s$.
$\omega\vec{RT} = \omega t - \omega r$
$= u - s = \vec{SU}$.
$\therefore\quad |\vec{SU}| = |\omega\vec{RT}|$
$= |\omega|\,|\vec{RT}|$
$= |\vec{RT}|$, since $|\omega| = 1$.
That is, $SU = RT$, as required.
N.B. This is the same result as in part (a), even though it looks different and the methods are different.

END OF 4 UNIT (ADDITIONAL) MATHEMATICS SOLUTIONS