# 1996 HIGHER SCHOOL CERTIFICATE SOLUTIONS

# 4 UNIT (ADDITIONAL) MATHEMATICS

**QUESTION 1**

(a) $$\int_1^3 \frac{4}{(2+x)^2}\,dx = \int_1^3 4(2+x)^{-2}\,dx$$
$$= -4\left[\frac{1}{2+x}\right]_1^3$$
$$= -4\left(\frac{1}{5}-\frac{1}{3}\right)$$
$$= \frac{8}{15}.$$

**OR**

Put $u = 2+x$, so that $\frac{du}{dx} = 1$.

$x = 1,\ u = 3$
$x = 3,\ u = 5$

$$\therefore \int_1^3 \frac{4}{(2+x)^2}\,dx = \int_3^5 \frac{4}{u^2}\,du$$
$$= \left[\frac{-4}{u}\right]_3^5$$
$$= -\frac{4}{5}+\frac{4}{3}$$
$$= \frac{8}{15}.$$

(b) ***Method 1***

Put $u = \tan\theta$, so that $\frac{du}{d\theta} = \sec^2\theta$.

$$\therefore \int \sec^2\theta \tan\theta\,d\theta = \int u\,du$$
$$= \frac{1}{2}u^2 + C_1$$
$$= \frac{1}{2}\tan^2\theta + C_1.$$

***Method 2***

Put $u = \sec\theta$, so that $\frac{du}{d\theta} = \sec\theta\tan\theta$.

$$\therefore \int \sec\theta.\sec\theta\tan\theta\,d\theta = \int u\,du$$
$$= \frac{1}{2}u^2 + C_2$$
$$= \frac{1}{2}\sec^2\theta + C_2.$$

***Method 3***

Let $I_n = \int \sec^2\theta\tan\theta\,d\theta$.

Let $u = \tan\theta, \quad v = \tan\theta$

$\frac{du}{d\theta} = \sec^2\theta, \quad \frac{dv}{d\theta} = \sec^2\theta$

$\therefore \quad I_n = \tan^2\theta - I_n$

$2I_n = \tan^2\theta$

$I_n = \frac{1}{2}\tan^2\theta + C_1.$

(c) ***Method 1***

$$\frac{5t^2+3}{t(t^2+1)} \equiv \frac{a}{t} + \frac{bt+c}{t^2+1}$$

$\therefore \quad 5t^2+3 = a(t^2+1) + t(bt+c)$

i.e. $\quad 5t^2+3 = (a+b)t^2 + ct + a.$

Equating like coefficients:

$c = 0$
$a = 3$
$a+b = 5$
$\therefore \ b = 2$

$$\therefore \int \frac{5t^2+3}{t(t^2+1)}\,dt = \int \frac{3}{t}\,dt + \int \frac{2t}{t^2+1}\,dt$$
$$= 3\log_e|t| + \log_e(t^2+1)$$

***Method 2***

$$\int \frac{5t^2+3}{t(t^2+1)}\,dt = \int \frac{3t^2+1}{t^3+t}\,dt + \int \frac{2t^2+2}{t^3+t}\,dt$$
$$= \int \frac{3t^2+1}{t^3+t}\,dt + \int \frac{2(t^2+1)}{t(t^2+1)}$$
$$= \log_e|t^3+1| + 2\log_e|t| +$$

***Method 3***

$$\int \frac{5t^2+3}{t(t^2+1)}\,dt = \int \frac{5t^4+3t^2}{t^5+t^3}\,dt$$
$$= \log_e|t^5+t^3| + C.$$

***Method 4***

$$\int \frac{5t^2+3}{t(t^2+1)}\,dt = \int \frac{2t^2}{t(t^2+1)}\,dt + \int \frac{3t^2+3}{t(t^2+1)}\,dt$$

$$= \int \frac{2t}{t^2+1}\,dt + \int \frac{3(t^2+1)}{t(t^2+1)}\,dt$$

$$= \int \frac{2t}{t^2+1}\,dt + \int \frac{3}{t}\,dt$$

$$= \log_e(t^2+1) + 3\log_e|t| + C.$$

**(d)** Let $u = \tan^{-1}x$ and $dv = x\,dx$.

$$\frac{du}{dx} = \frac{1}{1+x^2} \qquad \therefore \frac{dv}{dx} = x$$

$$v = \frac{1}{2}x^2.$$

$$\therefore \int x\tan^{-1}x\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \qquad \text{—①}$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C_1.$$

**OR** after ①, proceed as follows:

Put $x = \tan\theta$, so that $\frac{dx}{d\theta} = \sec^2\theta$.

$$\therefore \int \frac{x^2}{1+x^2}\,dx = \int \frac{\tan^2\theta}{1+\tan^2\theta} \times \sec^2\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C_2$$

$$= x - \tan^{-1}x + C_2$$

$$\therefore \int x\tan^{-1}x\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x + C_2\right)$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C_1.$$

**(e)** $x = 2\sin\theta$

$$\therefore \frac{dx}{d\theta} = 2\cos\theta \quad \text{or} \quad dx = 2\cos\theta\,d\theta$$

| $x = -1,\ 2\sin\theta = -1$ | $x = \sqrt{3},\ 2\sin\theta = \sqrt{3}$ |
|---|---|
| $\sin\theta = -\frac{1}{2}$ | $\sin\theta = \frac{\sqrt{3}}{2}$ |
| $\theta = -\frac{\pi}{6}$ | $\theta = \frac{\pi}{3}$ |

$$\therefore \int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta\,d\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\,d\theta$$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos 2\theta)\,d\theta$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 2\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right) - 2\left(\left(-\frac{\pi}{6}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$= \pi - \sqrt{3}.$$

**OR**

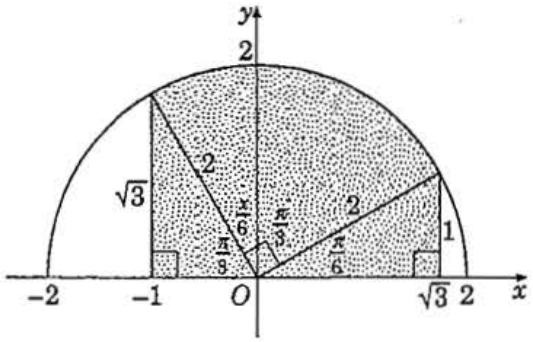
$$\text{Let } I = \int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$$

$$= -\int_{-1}^{\sqrt{3}} \frac{4-x^2-4}{\sqrt{4-x^2}}\,dx$$

$$= -\int_{-1}^{\sqrt{3}} \left(\frac{4-x^2}{\sqrt{4-x^2}} - \frac{4}{\sqrt{4-x^2}}\right)dx$$

$$= -\int_{-1}^{\sqrt{3}} \sqrt{4-x^2}\,dx + 4\int_{-1}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx.$$

The first integral can be evaluated by considering the shaded area in the following diagram.




$$\therefore\ I = -\left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{1}{2}\times 2^2\times\frac{\pi}{2}\right] + 4\left[\sin^{-1}\frac{x}{2}\right]_{-1}^{\sqrt{3}}$$

$$= -\sqrt{3}-\pi+4\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$$

$$= \pi-\sqrt{3}.$$

## QUESTION 2

(a) $z = c-i$, $c$ real.

(i) $\overline{(iz)} = \overline{i(c-i)}$

$= \overline{ci+1}$

$= 1-ci.$

(ii) $\frac{1}{z} = \frac{1}{c-i}\times\frac{c+i}{c+i}$

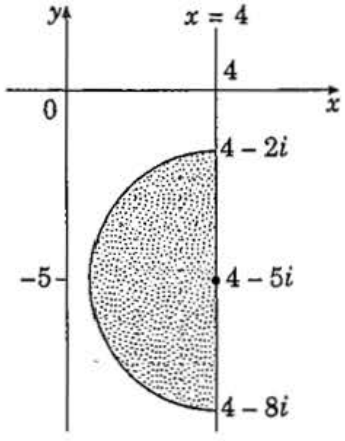
$= \frac{c+i}{c^2+1}$

$= \frac{c}{c^2+1}+\frac{i}{c^2+1}.$

(b) $|z-4+5i| \le 3$

$|z-(4-5i)| \le 3,$

i.e. circle, centre $4-5i$.




(c) (i) To prove

$(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta,\quad n\ge 1.$

When $n=1$, LHS $= (\cos\theta+i\sin\theta)^1$

RHS $= \cos\theta+i\sin\theta$

$=$ LHS.

Let $k$ be an integer for which the result is true then $(\cos\theta+i\sin\theta)^k = \cos k\theta+i\sin k\theta.$

When $n = k+1$,

$(\cos\theta+i\sin\theta)^{k+1} = (\cos k\theta+i\sin k\theta)(\cos\theta+i\sin$

$= \cos k\theta.\cos\theta+\cos k\theta.\sin\theta.i$

$+\cos\theta\sin k\theta.i-\sin k\theta.\sin$

$= \cos(k\theta+\theta)+i\sin(k\theta+\theta)$

$= \cos(k+1)\theta+i\sin(k+1)\theta.$

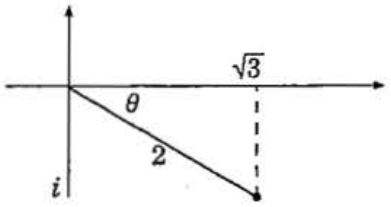
That is, if the result is true for $n=k$, then it is t for $n=k+1$.

Since the result is true for $n=1$, and for $n=k$ whenever it is true for $n=k$, it is therefore tru for all positive integral values of $n$.

(ii) $w = \sqrt{3}-i$

$\dot{w} = 2\cos\frac{\pi}{6}-i2\sin\frac{\pi}{6}$

$= 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right).$



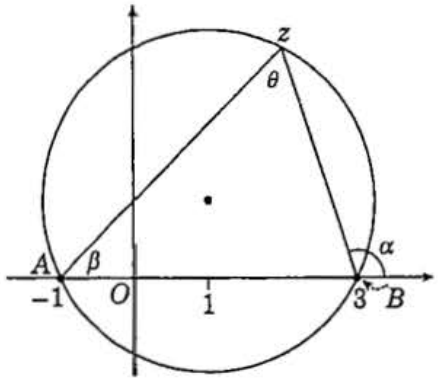

$\theta =$

(iii) $w^5 = 2^5\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)$

$= 2^5\left(-\frac{\sqrt{3}}{2}-i.\frac{1}{2}\right)$

$= -16\sqrt{3}-16i.$

(d)



(i) $\arg(z-3)=\alpha$

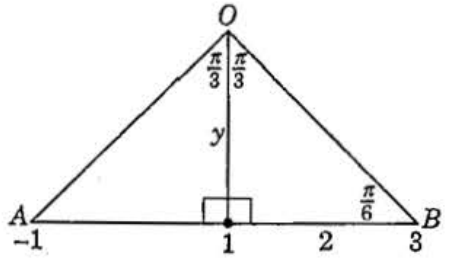
$\arg(z+1)=\beta.$

$\alpha-\beta=\theta,$

since the exterior angle of a triangle is equal to the sum of the interior opposite angles.

But we are told $\alpha-\beta=\frac{\pi}{3}$, $\therefore \theta=\frac{\pi}{3}$.

(ii) The centre will lie on the line $x=1$, since the perpendicular bisector of the chord $AB$ passes through the centre.




$\angle AOB=2\theta$ (angle at the centre is twice the angle at the circumference)

$\therefore \quad \angle AOB=\frac{2\pi}{3}$

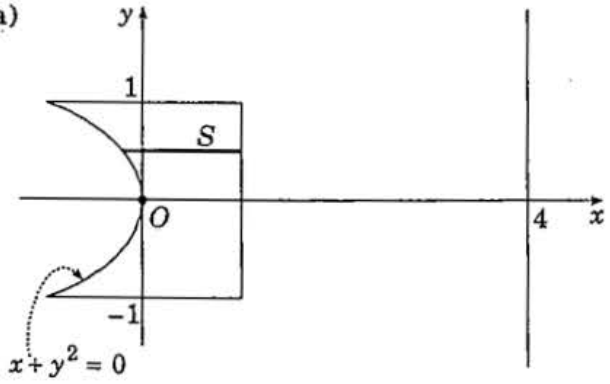
$\therefore \quad \frac{y}{2}=\tan\frac{\pi}{6}$

$\therefore \quad y=\frac{2}{\sqrt{3}}.$

Centre is $\left(1, \frac{2}{\sqrt{3}}\right)$.

**QUESTION 3**

(a)




(i) $x+y^2=0,\ x=-y^2$

$\therefore$ At $S$, area of annulus is

$$\pi\left(R^2-r^2\right),$$

where $R=4+y^2,\ r=3.$

$$\therefore \text{ Area } =\pi\left\{\left(4+y^2\right)^2-3^2\right\}$$

$$=\pi\left(16+8y^2+y^4-9\right)$$

$$=\pi\left(y^4+8y^2+7\right).$$

(ii) $$\text{Volume } =\pi\int_{-1}^{1}\left(y^4+8y^2+7\right)dy$$

$$=2\int_0^1 \pi\left(y^4+8y^2+7\right)dy$$

$$=2\pi\left[\frac{y^5}{5}+\frac{8y^3}{3}+7y\right]_0^1$$

$$=2\pi\left(\frac{1}{5}+\frac{8}{3}+7\right)$$

$$=2\pi\left(\frac{3+40+105}{15}\right)$$

$$=\frac{2\pi\times 148}{15}$$

$$=\frac{296\pi}{15}$$

$\approx 62$ cubic units.

(b) (i) ***Method* 1**

$$\int_0^{\frac{\pi}{2}}(\sin x)^{2k}\cos x\,dx=\left[\frac{(\sin x)^{2k+1}}{2k+1}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\left(\sin\frac{\pi}{2}\right)^{2k+1}}{2k+1}-\frac{(\sin 0)^{2k+1}}{2k+1}$$

$$=\frac{1}{2k+1}-0$$

$$=\frac{1}{2k+1}.$$

***Method* 2** Using integration by parts:

Let $I=\int_0^{\frac{\pi}{2}}(\sin x)^{2k}\cos x\,dx$

$\therefore \quad I=\left[(\sin x)^{2k+1}\right]_0^{\frac{\pi}{2}}-2kI$

$\therefore \quad (2k+1)I=1-0$

$\therefore \quad I=\frac{1}{2k+1}.$

(ii) $(\cos x)^{2n}=\left(1-\sin^2 x\right)^n,$

so $(\cos x)^{2n+1}=(\cos x)^{2n}\cos x$

$$=\left(1-\sin^2 x\right)^n\cos x$$

$$=\left[\binom{n}{0}-\binom{n}{1}\sin^2 x+\binom{n}{2}\sin^4 x-\cdots+\binom{n}{n}(-1)^n\sin^{2n}x\right]\cos x$$

$$=\left[1+\binom{n}{1}\left(-\sin^2 x\right)+\binom{n}{2}\left(-\sin^2 x\right)^2+\cdots+\binom{n}{n}\left(-\sin^2 x\right)^n\right]\cos x$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(-\sin^2 x\right)^k \cos x$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sin^{2k} x \cos x$$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}\, dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x\, dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k - \frac{1}{2k+1}, \text{ from (i).}$$

**OR** Let $u = \sin x$, $du = \cos x\, dx$.

$$\therefore \quad I = \int_0^1 \left(1-u^2\right)^n\, du$$

$$= \int_0^1 \left[1 - \binom{n}{1} u^2 + \binom{n}{2} u^4 - \cdots + (-1)^n \binom{n}{n} u^{2n}\right] du$$

$$= \int_0^1 \sum_{k=0}^{n} \binom{n}{k} \left(-u^2\right)^k du$$

$$= \int_0^1 \sum_{k=0}^{n} \binom{n}{k} (-1)^k u^{2k}\, du$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 u^{2k}\, du$$

$$= \left[\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1} u^{2k+1}\right]_0^1$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k}.$$

(iii) Putting $n = 2$ in (ii),

$$\int_0^{\frac{\pi}{2}} \cos^5 x\, dx = \sum_{k=0}^{2} (-1)^k \frac{1}{2k+1} \binom{2}{k}$$

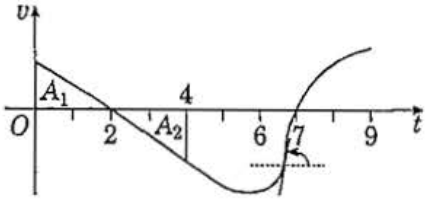
$$= \binom{2}{0} - \frac{1}{3}\binom{2}{1} + \frac{1}{5}\binom{2}{2}$$

$$= 1 - \frac{2}{3} + \frac{1}{5}$$

$$= \frac{15-10+3}{15}$$

$$= \frac{8}{15}.$$

(c)




(i) Acceleration is greatest when $dv/dt$ is greatest; that is, when the tangent h. the greatest slope. This occurs at $6{\cdot}5 \le t < 7$ (from the above graph).

(ii) Since the velocity–time graph is linear between $t = 0$ and $t = 4$, the areas $A_1$ and $A_2$, as shown, are equal.

The displacement between $t = 0$ and $t = 4$ is

$$x = \int_0^4 v\, dt$$

$$= \int_0^2 v\, dt + \int_2^4 v\, dt$$

$$= A_1 - A_2$$

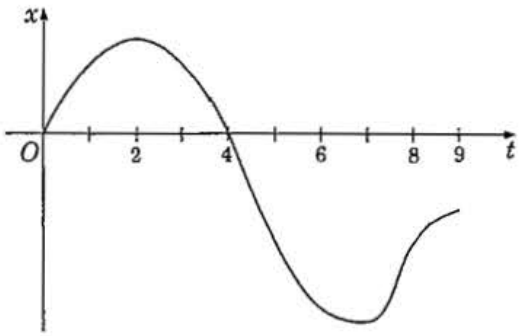
$$= 0.$$

$\therefore$ The particle first returns to $x = 0$ at $t = 4$.

(iii) For time $t = 0$ to $t = 5$, the graph is parabolic; and $x = 0$ when $t = 0, 4$.

When $t = 2$, $x$ has a maximum value.

When $t = 7$, $x$ has a minimum value.




## QUESTION 4

(a) $1 + x + x^2 + x^3 + \cdots + x^n = \dfrac{x^{n+1}-1}{x-1}$

$$\therefore \quad 1 + 2x + 3x^2 + \cdots + nx^{n-1}$$

$$= \frac{(x-1)(n+1)x^n - (x^{n+1}-1)}{(x-1)^2}.$$

Put $x = 2$,

then $1 + 2\times 2 + 3\times 4 + 4\times 8 + \cdots + n.2^{n-1}$

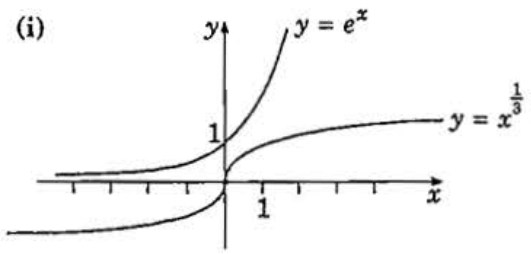
$$= \frac{(2-1)(n+1)2^n - (2^{n+1}-1)}{(2-1)^2}$$

$$= n.2^n + 2^n - 2^{n+1} + 1$$

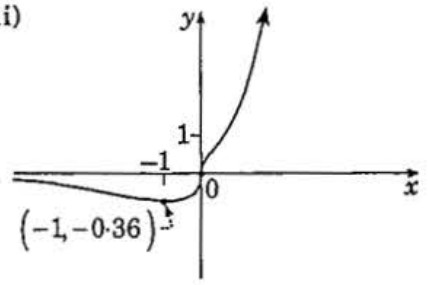
$$= 2^n(n+1-2) + 1$$

$$= 2^n(n-1) + 1.$$

(b) (i)




(ii)




(iii) There will be only one solution for $m \le 0$.

(c) Let $p$ be the probability that the number 20 is drawn in any game. Then $p = 0{\cdot}2$.
Let $q = 1 - p = 0{\cdot}8$.

(i) Probability of getting 20 in *exactly* two games is ${}^5C_2\, p^2 q^3 = \dfrac{5\times4}{1\times2}\, 0{\cdot}02^2 \times 0{\cdot}8^3$
$= 0{\cdot}2048.$

(ii) Probability of no 20s and one 20 is
$$\binom{5}{0}p^0q^5 + \binom{5}{1}pq^4 = 0{\cdot}8^5 + 5\times0{\cdot}2\times0{\cdot}8^4$$
$$= 0{\cdot}327\,68 + 0{\cdot}4096$$
$$= 0{\cdot}737\,28.$$
$\therefore$ Probability of *at least* two 20s $= 1 - 0{\cdot}737\,28$
$= 0{\cdot}262\,72.$

(iii) There are $\binom{20}{4}$ unordered selections of four from twenty, and there are $\binom{j}{4}$ unordered selections of four from $j$.

Hence, $\dfrac{\binom{j}{4}}{\binom{20}{4}} = \dfrac{(j)(j-1)(j-2)(j-3)}{20\times19\times18\times17}.$

(iv) Since $j$ is the *largest* of the four numbers, the other three numbers must be selected from $1, 2, 3, \ldots, j-1$.
This can be done in $\binom{j-1}{3}$ ways.
There are $\binom{20}{4}$ ways of selecting four from twenty,
$\therefore$ the required probability is $\dfrac{\binom{j-1}{3}}{\binom{20}{4}}.$

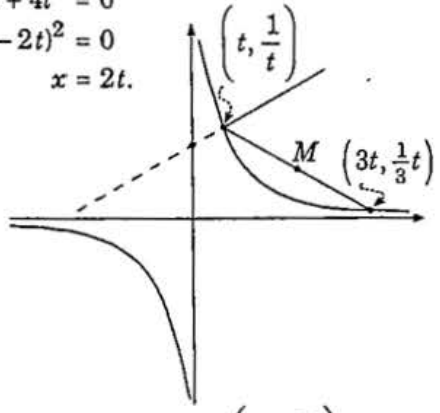
**QUESTION 5**

(a) $y = \dfrac{1}{x}$

(i) The midpoint of $\left(t, \frac{1}{t}\right)$ and $\left(3t, \frac{1}{3t}\right)$ is $\left(\frac{1}{2}(t+3t), \frac{1}{2}\left(\frac{1}{t}+\frac{1}{3t}\right)\right) = \left(2t, \frac{4}{6t}\right)$;
that is, $x = 2t$, $y = \dfrac{2}{3t}$.
Then $3xy = 3.2t.\dfrac{2}{3t}$
$= 4.$
$\therefore$ The midpoint lies on the curve $3xy = 4$. —①

(ii) The line joining $\left(t, \frac{1}{t}\right)$ and $\left(3t, \frac{1}{3t}\right)$ has gradient
$$\frac{\frac{1}{t}-\frac{1}{3t}}{t-3t} = \frac{\frac{3-1}{3t}}{-2t}$$
$$= \frac{2}{-6t^2}$$
$$= -\frac{1}{3t^2}.$$
$\therefore$ The equation of the line is
$$y - \frac{1}{t} = -\frac{1}{3t^2}(x-t)$$
$$= -\frac{x}{3t^2} + \frac{1}{3t}$$
$\therefore$ $$y = -\frac{x}{3t^2} + \frac{4}{3t}.$$ —②

Solving ① and ② simultaneously:
$$3x\left(\frac{-x}{3t^2} + \frac{4}{3t}\right) = 4$$
$$3x(-x+4t) = 12t^2$$
$$-3x^2 + 12xt = 12t^2$$
$$3x^2 - 12xt + 12t^2 = 0$$
$$x^2 - 4xt + 4t^2 = 0$$
$$(x-2t)^2 = 0$$
$$x = 2t.$$
$$y = \frac{-2t}{3t^2} + \frac{4}{3t}$$
$$= \frac{-2}{3t} + \frac{4}{3t}$$
$$= \frac{2}{3t}.$$




That is, only one point of contact $\left(2t, \dfrac{2}{3t}\right)$.
$\therefore$ The line is a tangent to $3xy = 4$.

**OR** $3xy = 4$

$$y = \frac{4}{3x}$$

$$y' = \frac{-4}{3x^2}.$$

At $x = 2t$, $y' = -\dfrac{1}{3t^2}$ (gradient of the tangent).

The gradient of the line passing through $\left(t, \frac{1}{t}\right)$ and $\left(3t, \frac{1}{3t}\right)$ is

$$m = \frac{\frac{1}{3t} - \frac{1}{t}}{3t - t} = -\frac{1}{3t^2}.$$

$\therefore$ The line passing through $\left(t, \frac{1}{t}\right)$ and $\left(3t, \frac{1}{3t}\right)$ is tangent to $3xy = 4$ at $\left(2t, \frac{2}{3t}\right)$.

**(iii)** $y = \dfrac{1}{x} = x^{-1}$

$$\frac{dy}{dx} = -1x^{-2}$$

$$= -\frac{1}{x^2}$$

$$= -\frac{1}{t^2} \text{ at } \left(t, \frac{1}{t}\right).$$

$\therefore$ The normal has gradient $t^2$.

Its equation is $y - \dfrac{1}{t} = t^2(x - t)$

$$ty - 1 = t^3x - t^4$$

$$t^4 - t^3x + ty - 1 = 0.$$

**(iv)** If $x = 0$, $y = h$ lies on the normal, then $t^4 + ht - 1 = 0$. We want to check how many roots this equation has.

Let $P(t) = t^4 + ht - 1$

$\therefore$ $P(\infty) > 0$.

$P(-\infty) > 0$

$P(0) = -1$

$P(1) = 1 + h - 1$

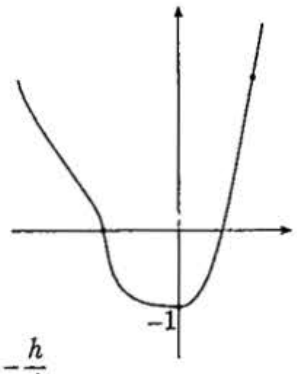
$= h$

$P(-1) = 1 - h - 1$

$= -h.$

$P'(t) = 4t^3 + h$

$= 0$ when $t^3 = -\dfrac{h}{4}$




$\therefore$ $t = -\left(\dfrac{h}{4}\right)^{\frac{1}{3}}$, so only one turning point.

$P''(t) = 12t^2 > 0$

$\therefore$ minimum turning point below $t$ axis.

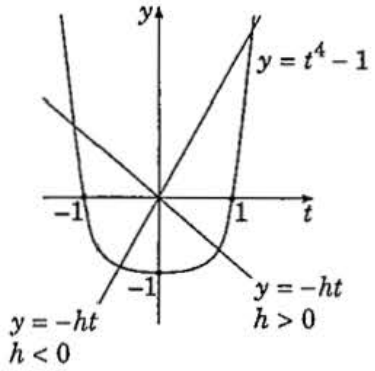
Hence only two real roots, so only two points with intersection $R(0, h)$.

**OR** Solving

$$t^4 + th - 1 = 0$$

$$t^4 - 1 = -ht.$$

A quick sketch shows that the graph of the straight line $y = -ht$ must cross the graph of $y = t^4 - 1$ in two places.




$\therefore$ $t^4 + th - 1 = 0$ has exactly two solutions, regardless of the value of $h$.

**(b)** $x^4 + ax^3 + bx^2 + cx + d = 0$

**(i)** For polynomials with real coefficients, complex roots occur in conjugate pairs. That is, $(a + ib)(a - ib) = a^2 + b^2$ is a factor, and all roots are either real or in conjugate pairs.

**(ii)** So $(x - ki)(x + ki) = x^2 + k^2$ is a factor.

$$P(x) = x^4 + ax^3 + bx^2 + cx + d$$

$$P(ki) = k^4 - ak^3 i - bk^2 + cki + d$$

$$= 0$$

Equating imaginary parts:

$$-ak^3 + ck = 0$$

$\therefore$ $$k^2 = \frac{c}{a}.$$

**OR** Let the roots be $ki$, $-ki$, $\alpha$, $\beta$.

$$\sum \alpha = \alpha + \beta = -a.$$

$$\sum \alpha\beta\gamma = ki\alpha\beta - ki\alpha\beta + k^2\alpha + k^2\beta$$

$$= k^2(\alpha + \beta)$$

$$= -c.$$

$\therefore$ $$-ak^2 = -c$$

$$k^2 = \frac{c}{a}.$$

**OR** $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$$P(x) = \left(x^2 + k^2\right)\left(x^2 + Ax + B\right)$$

Equating coefficients of $x^3$: $a = A$

Equating coefficients of $x$: $c = Ak^2$

$\therefore$ $c = ak^2$.

(iii) From $P(ki) = 0$, equating real parts:

$$k^4 - bk^2 + d = 0.$$

Since $k^2 = \frac{c}{a}$,

then $\frac{c^2}{a^2} - b\frac{c}{a} + d = 0$

$\therefore \quad c^2 - abc + a^2d = 0$

i.e. $c^2 + a^2d = abc.$

(iv) ***Method 1***

If $b = 0$, the equation is $x^4 + ax^3 + cx + d = 0$.

Let the roots be $ki$, $-ki$, $2$, $\alpha$.

$$\sum \alpha = 2 + \alpha = -a$$

$\therefore \quad \alpha = -(a+2)$

$$\sum \alpha\beta = k^2 + 2\alpha = 0$$

$\therefore \quad k^2 = -2a\alpha$

Substituting $k^2 = \frac{c}{a}$ from (ii):

$$c = -2\alpha$$

$\therefore \quad c = 2a(a+2)$

$\therefore$ $c$ is even (as $a$ is an integer).

***Method 2***

$P(2) = 0 \Rightarrow 16 + 8a + 4b + 2c + d = 0.$

Since $b = 0$: $16 + 8a + 2c + d = 0.$

Now $c^2 + a^2d = abc,$

$b = 0 \Rightarrow \quad d = -\frac{c^2}{a^2}$

$\therefore \quad 16 + 8a + 2c - \frac{c^2}{a^2} = 0$

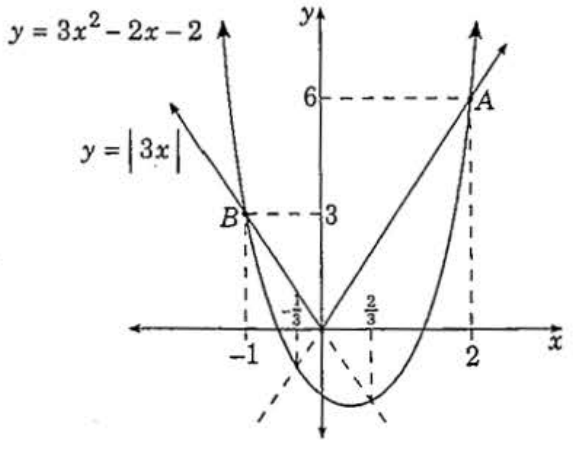
$c^2 - 2a^2c - 16a^2 - 8a^3 = 0.$

The quadratic formula gives

$c = 2a(a+2)$ or $c = -4a.$

## QUESTION 6

(a)




$3x^2 - 2x - 2 \le |3x|$

To find $A$, suppose $x \ge 0$,

then $3x^2 - 2x - 2 = 3x$

$3x^2 - 5x - 2 = 0$

$(x-2)(3x+1) = 0$

$\therefore \quad x = 2$ or $-\frac{1}{3}$,

so $A$ has $x$ coordinate 2.

To find $B$, suppose $x < 0$,

then $3x^2 - 2x - 2 = -3x$

$3x^2 + x - 2 = 0$

$(x+1)(3x-2) = 0$

$\therefore \quad x = -1$ or $\frac{2}{3}$,

so $B$ has $x$ coordinate $-1$.

So the solution of the inequality is $-1 \le x \le 2$.

**OR** If $x \ge 0$, then $3x^2 - 2x - 2 \le 3x$

i.e. $(x-2)(3x+1) \le 0$

$\therefore \quad 0 \le x \le 2.$

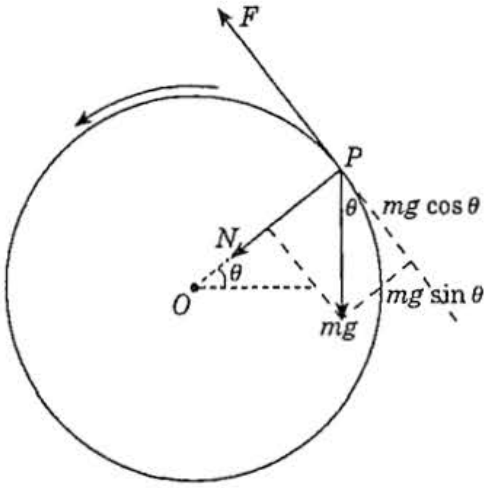
If $x < 0$, then $3x^2 - 2x - 2 \le -3x$

i.e. $(x+1)(3x-2) \le 0$

$\therefore \quad -1 \le x < 0.$

Combining these gives $-1 \le x \le 2$.

(b)




Resolving radially, $N + mg\sin\theta = mr\omega^2$.

Resolving tangentially, $F - mg\cos\theta = 0$.

So $N = m(r\omega^2 - g\sin\theta)$ and $F = mg\cos\theta$,

$\therefore \quad \frac{F}{N} = \frac{g\cos\theta}{r\omega^2 - g\sin\theta}.$

(c) (i) $\angle PTQ = \theta$ (corresponding angles, $PL \| Ox$)

$\gamma = \angle PTQ + \angle TPQ$ (exterior angle of triangle equals sum of interior opposite angles)

$= \theta + \phi$

$= 2\theta$ (since $\phi = \theta$).

**OR** $\gamma + \angle LPQ = \pi$ (cointerior angles, $PL \| TQ$)

$\theta + \phi + \angle LPQ = \pi$ (adjacent angles in straight line)

so $\gamma = \theta + \phi$ (subtracting)

$= 2\theta$, since $\theta = \phi$ (given).

**(ii)** Gradient of $PT$ is $\tan\theta$. Also, gradient of $PT$ is gradient of curve at $P(at^2, 2at)$,

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}.$$

$$\therefore \text{ Gradient of } PT = \frac{1}{t}$$

$$\therefore \quad \tan\theta = \frac{1}{t}.$$

**OR**

$$y^2 = 4ax$$

$$\therefore \quad 2y = 4a\frac{dx}{dy}$$

$$\therefore \quad \frac{dy}{dx} = \frac{4a}{2y} = \frac{4a}{4at} = \frac{1}{t}$$

$$\therefore \quad \tan\theta = \frac{1}{t}.$$

**(iii)** The gradient of $PQ$ is given by

$$\begin{aligned} m &= \tan\gamma \\ &= \tan 2\theta \\ &= \frac{2\tan\theta}{1-\tan^2\theta} \\ &= \frac{\frac{2}{t}}{1-\frac{1}{t^2}} \\ &= \frac{2t}{t^2-1}. \end{aligned}$$

$\therefore$ The equation of $PQ$ is $y - 2at = \dfrac{2t}{t^2-1}(x - at^2)$.

That is, $(t^2-1)y - (t^2-1)2at = 2t(x - at^2)$

$$(t^2-1)y - 2at^3 + 2at = 2tx - 2at^3$$

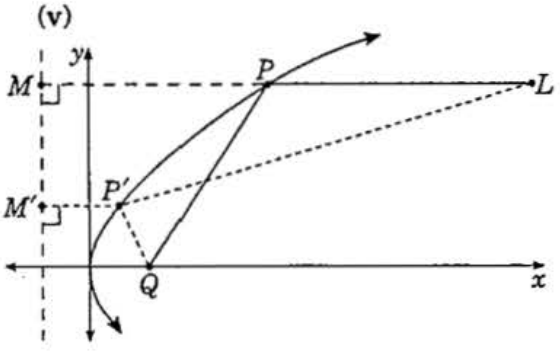
$$2tx = (t^2-1)y + 2at.$$

**(iv)** Substituting $y = 0$ in the equation of $PQ$, $2tx = 2at$

$$x = a$$

$\therefore$ $Q$ is $(a, 0)$, the focus.

**(v)**




Since $PQ = PM$, the path length is $LP + PM$ which is simply $LM$, because $L$, $P$, and $M$ are collinear. Let $P'$ be any other point on the parabola. Since $P'Q = P'M'$, the new path length is $LP' + P'M'$ which is longer, because the perpendicular $LM$ is the shortest distance from $L$ to the directrix.

## QUESTION 7

**(a)** **(i)** At $x = 1$, $\dfrac{d^2x}{dt^2} = 3$.

So the particle starts from rest with positive acceleration, and therefore moves in the positive $x$ direction.

**(ii)** $$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \frac{5-2x}{x^3}.$$

That is, $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 5x^{-3} - 2x^{-2}$

$$\therefore \quad \tfrac{1}{2}v^2 = -\tfrac{5}{2}x^{-2} + 2x^{-1} + C.$$

When $x = 1$, $v = 0$.

$$\therefore \quad 0 = -\frac{5}{2} + 2 + C; \quad C = \frac{1}{2}$$

$$\therefore \quad \tfrac{1}{2}v^2 = \frac{-5}{2x^2} + \frac{2}{x} + \frac{1}{2}$$

$$v^2 = \frac{x^2 + 4x - 5}{2x^2}$$

Since $x^2 + 4x - 5 = (x+5)(x-1) \neq 0$ for $x > 1$,

$\therefore$ $v \neq 0$ for $x > 1$, and so $v$ remains positive.

$$\therefore \quad v = \frac{\sqrt{x^2+4x-5}}{x} \text{ for } x \geq 1.$$

**(iii)** For $x > \frac{5}{2}$ the acceleration is negative and so the particle will slow down.

Now $v = \sqrt{1 + \dfrac{4}{x} - \dfrac{5}{x^2}}$.

$\therefore$ As $x \to \infty$, $v \to 1$.

That is, the velocity decreases and approaches 1 as $x$ increases.

**(b)** **(i)** $f(x) = \ln x - ax + b, \qquad x > 0.$

$$f'(x) = \frac{1}{x} - a$$

$= 0$ when $\dfrac{1}{x} = a$, that is $x = \dfrac{1}{a}$.

$$f''(x) = -x^{-2}$$

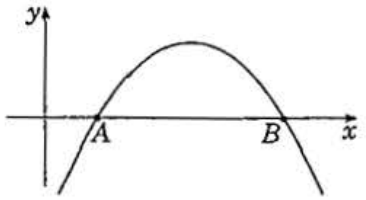
$$= -\frac{1}{x^2}$$

$< 0$ for all $x > 0$.

$\therefore$ The single stationary point is a maximum.

(ii) Let $\alpha$, $\beta$ be the $x$ coordinates of $A$, $B$ respectively.

If $y = \ln x$ and $y = ax - b$ intersect at $A$ and $B$, then $f(x) = 0$ at $\alpha$ and $\beta$, and the curve $y = f(x)$ crosses the $x$ axis at $\alpha$ and $\beta$.



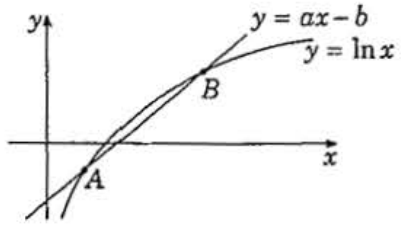

Since, from (i), $f$ has a single maximum turning point, $f(x) > 0$ for $\alpha < x < \beta$.

That is, $\ln x > ax - b$ for $x$ in the interval from $\alpha$ to $\beta$.

$\therefore$ The curve $y = \ln x$ lies above $y = ax - b$ for $\alpha < x < \beta$.

That is, the chord $AB$ lies below the curve $y = \ln x$.

OR




Since $y = \ln x$ is concave down for all $x > 0$, if a straight line intersects at two distinct points $A$ and $B$, the chord $AB$ must lie below the curve.

(iii) Using $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $v = x$, $\frac{dv}{dx} = 1$,

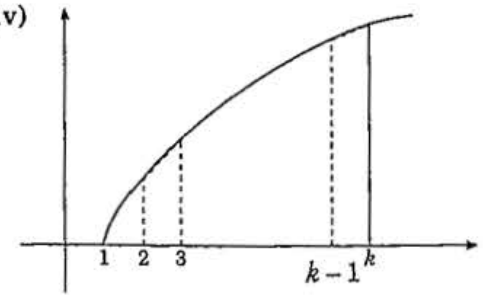
we have $\int_1^k \ln x\, dx = \left[x \ln x\right]_1^k - \int_1^k x \times \frac{1}{x}\, dx$

$= \left[x \ln x\right]_1^k - \left[x\right]_1^k$

$= k \ln k - \ln 1 - k + 1$

$= k \ln k - k + 1$, since $\ln 1 = 0$.

(iv)




By the trapezoidal rule,

$$\int_1^k \ln x\, dx \approx \tfrac{1}{2}(\ln 1 + \ln 2) + (\ln 2 + \ln 3) + \cdots + \tfrac{1}{2}[\ln(k-1) + \ln k]$$

$$= \tfrac{1}{2}\ln k + \ln 2 + \ln 3 + \cdots + \ln(k-1)$$

$$= \tfrac{1}{2}\ln k + \ln[2 \times 3 \times 4 \times \cdots \times (k-1)]$$

$$= \tfrac{1}{2}\ln k + \ln[(k-1)!].$$

(v) By part (ii), the trapezia lie below the curve.

$\therefore$ Area of trapezia < area under the curve

$\therefore \ \tfrac{1}{2}\ln k + \ln(k-1)! < k \ln k - k + 1.$

$\ln k! < k \ln k - k + 1 + \tfrac{1}{2}\ln k$

$\ln k! < \ln k^k - k + 1 + \ln \sqrt{k}$

$\therefore \quad k! < k^k . e^{-k} . e^1 . \sqrt{k}$

i.e. $\quad k! < e\sqrt{k}\left(\frac{k}{e}\right)^k.$

OR

$\frac{1}{2}\ln k + \ln(k-1)! < k \ln k - k + 1$

$\therefore \ \sqrt{k}(k-1)! < k^k . e^{-k} . e^1$

$\therefore \quad k! < k^k . e^{-k} . e . \sqrt{k},$

i.e. $\quad k! < e\sqrt{k}\left(\frac{k}{e}\right)^k.$

## QUESTION 8

(a) (i) If $z = w^k$, then

$z^9 - 1 = (w^k)^9 - 1$

$= w^{9k} - 1$

$= \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^{9k} - 1$

$= \cos\left(\frac{2\pi}{9} \times 9k\right) + i\sin\left(\frac{2\pi}{9} \times 9k\right) - 1$ (de Moivre's theorem)

$= \cos 2\pi k + i \sin 2\pi k - 1$

$= 1 + i \times 0 - 1$

$= 0.$

$\therefore \ w^k$ is a solution of $z^9 - 1 = 0$.

OR $\quad w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$

$\therefore \quad w^9 = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^9$

$= \cos 2\pi + i \sin 2\pi$ (de Moivre's theorem)

$= 1.$

$\therefore \ w$ is a root of $z^9 - 1 = 0$.

Now $\left(w^k\right)^9 - 1 = \left(w^9\right)^k - 1$

$= 1 - 1 = 0.$

$\therefore \ w^k$ is also a root of $z^9 - 1 = 0$.

(ii) ***Method 1***

$(1-w)(1+w+w^2+\cdots+w^8) = 1-w^9 = 0$,
since $w$ is a solution of $1-w^9=0$.
But since $1-w \neq 0$, the other factor must be zero,
so $\quad 1+w+w^2+\cdots+w^8 = 0$
$\therefore \quad w+w^2+\cdots+w^8 = -1.$

***Method 2***

Roots of $x^9-1=0$ are $1, w, \ldots, w^8$.

$$\text{Sum of roots} = -\frac{\text{coeff. of } z^8}{\text{coeff. of } z^9} = -\frac{b}{a} = 0.$$

$\therefore \quad 1+w+\cdots+w^8 = 0.$

***Method 3***

Consider $1+w+w^2+\cdots+w^8$.
This is the sum of a GP with
$a=1,\ r=w,\ n=9.$

$$\therefore\ 1+w+w^2+\cdots+w^8 = \frac{w^9-1}{w-1} = 0 \quad (w \neq 1).$$

(iii) ***Method 1***

From (ii): $\quad w+w^8+w^2+w^7+w^3+w^6$
$\qquad +w^4+w^5 = -1$
$\therefore \quad w+w^{-1}+w^2+w^{-2}+w^3$
$\qquad +w^{-3}+w^4+w^{-4} = -1 \qquad$ —①

By de Moivre's theorem, and the fact that
$\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$,

$$w^k+w^{-k} = \cos\frac{2\pi k}{9}+i\sin\frac{2\pi k}{9} + \cos\left(-\frac{2\pi k}{9}\right)+i\sin\left(-\frac{2\pi k}{9}\right)$$
$$= 2\cos\frac{2\pi k}{9}. \qquad —②$$

Therefore, from ①, we conclude that

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}+2\cos\frac{6\pi}{9}+2\cos\frac{8\pi}{9} = -1.$$

Using $\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$,

we get $\quad 4\cos\frac{3\pi}{9}\cos\frac{\pi}{9}+4\cos\frac{7\pi}{9}\cos\frac{\pi}{9} = -1$

$$4\cos\frac{\pi}{9}\left(\cos\frac{3\pi}{9}+\cos\frac{7\pi}{9}\right) = -1$$

$$8\cos\frac{\pi}{9}\cos\frac{5\pi}{9}+\cos\frac{2\pi}{9} = -1.$$

But $\quad \cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$

$$\therefore \quad \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}.$$

***Method 2***

$\cos\frac{\pi}{9} = -\cos\frac{8\pi}{9}$, and

$\cos x = \frac{1}{2}[\cos x + i\sin x + \cos(-x) + i\sin(-x)]$
(as in ② above)

$$\therefore\ \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$
$$= -\frac{1}{8}\left[\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}+\cos\left(\frac{-8\pi}{9}\right)+i\sin\left(\frac{-8\pi}{9}\right)\right]$$
$$\times\left[\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}+\cos\left(\frac{-2\pi}{9}\right)+i\sin\left(\frac{-2\pi}{9}\right)\right]$$
$$\times\left[\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}+\cos\left(\frac{-4\pi}{9}\right)+i\sin\left(\frac{-4\pi}{9}\right)\right]$$
$$= -\frac{1}{8}\left(w^4+w^5\right)\left(w+w^8\right)\left(w^2+w^7\right)$$
$$= -\frac{1}{8}\left(w^4+w^5\right)\left(w^3+w^{10}+w^8+w^{15}\right)$$
$$= -\frac{1}{8}\left(w^7+w^8+w^{14}+w^{15}+w^{12}+w^{13}+w^{19}+w^{20}\right)$$
$$= -\frac{1}{8}\left(w^7+w^8+w^5+w^6+w^3+w^4+w+w^2\right)$$
$$= -\frac{1}{8}\left(w+w^2+w^3+\cdots+w^8\right)$$
$$= \frac{1}{8}, \text{ from (ii).}$$

***Method 3***

From (i),

$$z^9-1 = \left(z-z^0\right)\left(z-w\right)\left(z-w^8\right)\left(z-w^2\right)\left(z-w^7\right)$$
$$\left(z-w^3\right)\left(z-w^6\right)\left(z-w^4\right)\left(z-w^5\right)$$
$$= (z-1)\left[z^2-\left(w+w^8\right)z+1\right]$$
$$\left[z^2-\left(w^2+w^7\right)z+1\right]\left[z^2-\left(w^3+w^6\right)z+1\right]$$
$$\left[z^2-\left(w^4+w^5\right)z+1\right]. \qquad —③$$

Put $z=i$, so that $z^2+1=0$.

$$\therefore\ i-1 = (i-1)\left[-i\left(w+w^8\right)\right]\left[-i\left(w^2+w^7\right)\right]$$
$$\times\left[-i\left(w^3+w^6\right)\right]\left[-i\left(w^4+w^5\right)\right]$$

so that $\left(w+w^8\right)\left(w^2+w^7\right)\left(w^3+w^6\right)\left(w^4+w^5\right)=1$

$$\therefore \quad \left(2\cos\frac{2\pi}{9}\right)\left(2\cos\frac{4\pi}{9}\right)\left(2\cos\frac{6\pi}{9}\right)\left(2\cos\frac{8\pi}{9}\right)=1.$$

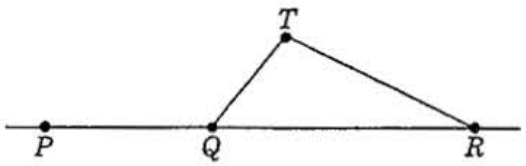
Note that $\cos\frac{6\pi}{9} = -\cos\frac{3\pi}{9} = -\frac{1}{2}$

and $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$,

$$\therefore\ 2\cos\frac{2\pi}{9}\times 2\cos\frac{4\pi}{9}\times 2\times -\frac{1}{2}\times -2\cos\frac{\pi}{9} = 1,$$

i.e. $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}.$

**(b)**




$QT + QR > RT$ (triangle inequality)

$\therefore\ QT + PR - PQ > RT$ $(QR = PR - PQ)$

$\therefore\ QT - QP > RT - RP$

**(c)** (i) $AK = AN$, $BK = BL$, $CL = CM$, $DM = DN$ (two tangents drawn to a circle from an external point are equal)

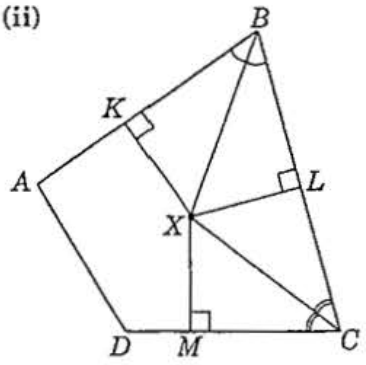
Now $AB + DC = AK + KB + DM + MC$

$= AN + BL + DN + CL$ (using above results)

$= AN + ND + BL + LC$

$= AD + BC.$

(ii)




Draw perpendiculars to $AB$, $BC$, and $CD$ from $X$ and call the points of intersection $K$, $L$, and $M$ respectively.

Now $\Delta XKB \equiv \Delta XLB$ ($\angle KBX = \angle LBX$, $\angle XKB = \angle XLB = 90°$, $XB$ common, AAS)

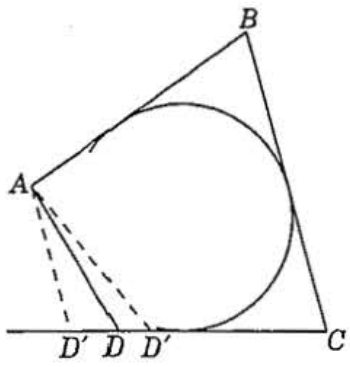
and so $XK = XL$ (corresponding sides of congruent triangles).

Similarly $\Delta XLC \equiv \Delta XMC$

and $XL = XM$.

Thus a circle, centre $X$, passing through $L$ also passes through $K$ and $M$, and since $AB$, $BC$, and $CD$ are perpendicular to $XK$, $XL$, and $XM$ respectively, these lines are tangents to the circle.

(iii)




From (ii), there exists a circle to which $AB$, $BC$, and $CD$ are tangent. Let $D'$ be a point on $DC$ such that $AD'$ is also a tangent to this circle.

There are three possibilities:

($\alpha$) $D' = D$;

($\beta$) $D'$ lies on $CD$ produced;

($\gamma$) $D'$ lies between $C$ and $D$.

Using the result in (b), we get:

Case ($\beta$): $DA - DC > D'A - D'C$

i.e. $D'A - D'C < DA - DC$

Case ($\gamma$): $D'A - D'C > DA - DC$

Using the result in (i), we get

$$AB + D'C = AD' + BC,$$

and the stated condition is

$$AB + CD = AD + BC.$$

Subtracting, we get

$$D'C - DC = D'A - DA$$

i.e. $D'A - D'C = DA - DC.$

This is incompatible with cases ($\beta$) and ($\gamma$), hence $D = D'$, and so $AD$ is also a tangent to the circle.

**END OF 4 UNIT (ADDITIONAL) MATHEMATICS SOLUTIONS**