1995 HIGHER SCHOOL CERTIFICATE
SOLUTIONS
4 UNIT (ADDITIONAL) MATHEMATICS

QUESTION ONE

(a) Let $u = \ln x$

$\therefore \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{dx}{x}$

$\therefore \displaystyle\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2}$

$= \displaystyle\int u^{-2}\, du$

$= -u^{-1} + c$

$= -\dfrac{1}{u} + c$

$= -\dfrac{1}{\ln x} + c.$

(b) Let $u = x$ and $\dfrac{dv}{dx} = e^x$

$\therefore \dfrac{du}{dx} = 1 \qquad v = e^x$

$\therefore \displaystyle\int xe^x\, dx = xe^x - \int e^x\, dx$

$= xe^x - e^x + c.$

Alternative solution:

$\dfrac{d}{dx}(xe^x) = e^x + xe^x$

$\therefore \quad xe^x = \dfrac{d}{dx}(xe^x) - e^x.$

Integrate both sides with respect to x.

$\therefore \displaystyle\int xe^x\, dx = e^x - xe^x + c$

(c) *Method 1*: Partial fractions

Let $\dfrac{6t+23}{(2t-1)(t+6)} = \dfrac{A}{2t-1} + \dfrac{B}{t+6}$

$\therefore A(t+6) + B(2t-1) \equiv 6t + 23.$

Equating coefficients of like terms:

$$\left.\begin{aligned} A + 2B &= 6 \\ 6A - B &= 23 \end{aligned}\right\} \Rightarrow A = 4,\ B = 1$$

$\therefore \displaystyle\int_1^4 \frac{6t+23}{(2t-1)(t+6)}\, dt$

$= \displaystyle\int_1^4 \frac{4}{2t-1}\, dt + \int_1^4 \frac{dt}{t+6}$

$= \Big[2\ln(2t-1)\Big]_1^4 + \Big[\ln(t+6)\Big]_1^4$

$= 2\ln 7 + 2\ln 1 + \ln 10 - \ln 7$

$= \ln 7 + \ln 10$

$= \ln 70.$

Method 2:

$\displaystyle\int_1^4 \frac{6t+23}{(2t-1)(t+6)}\, dt$

$= \displaystyle\int_1^4 \frac{6t+23}{2t^2+11t-6}\, dt$

$= \displaystyle\int_1^4 \frac{4t+11}{2t^2+11t-6}\, dt + \int_1^4 \frac{2t+12}{2t^2+11t-6}\, dt$

$= \Big[\ln(2t^2+11t-6)\Big]_1^4 + \displaystyle\int_1^4 \frac{2(t+6)}{(2t-1)(t+6)}\, dt$

$= \ln 70 - \ln 7 + \displaystyle\int_1^4 \frac{2}{2t-1}\, dt$

$= \ln 70 - \ln 7 + \Big[\ln(2t-1)\Big]_1^4$

$= \ln 70 - \ln 7 + \ln 7 - \ln 1$

$= \ln 70.$

Method 3: Completing the square

$\displaystyle\int_1^4 \frac{6t+23}{2t^2+11t-6}\, dt$

$= \displaystyle\int_1^4 \frac{6t+23}{2\left(t+\frac{11}{4}\right)^2 - \frac{169}{8}}\, dt$

$= \displaystyle\int_1^4 \frac{3\left[2\left(t+\frac{11}{4}\right)+\frac{13}{6}\right]}{2\left[\left(t+\frac{11}{4}\right)^2 - \frac{169}{16}\right]}\, dt$

$= \dfrac{3}{2}\displaystyle\int_1^4 \frac{2\left(t+\frac{11}{4}\right)+\frac{13}{6}}{\left(t+\frac{11}{4}\right)^2 - \left(\frac{13}{4}\right)^2}\, dt$

$$= \frac{3}{2}\left[\ln\left\{\left(t+\frac{11}{4}\right)^2-\left(\frac{13}{4}\right)^2\right\}\right]_1^4$$

$$+\frac{3}{2}\times\frac{13}{6}\times\frac{1}{2\times\frac{13}{4}}\left[\ln\left(\frac{t+\frac{11}{4}-\frac{13}{4}}{t+\frac{11}{4}+\frac{13}{4}}\right)\right]_1^4$$

$$= \frac{3}{2}\left(\ln 35-\ln\frac{7}{2}\right)+\frac{1}{2}\left(\ln\frac{7}{20}-\ln\frac{1}{14}\right)$$

$$= \ln 70.$$

(d) $\dfrac{d}{dx}(x\sin^{-1}x) = x\times\dfrac{1}{\sqrt{1-x^2}}+\sin^{-1}x$

$$\therefore \quad \sin^{-1}x = \frac{d}{dx}(x\sin^{-1}x)-\frac{x}{\sqrt{1-x^2}}$$

$$\therefore \int \sin^{-1}x\,dx = x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1}x-\int x(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= x\sin^{-1}x-\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}\times-2}+c$$

$$= x\sin^{-1}x+\sqrt{1-x^2}+c.$$

(e) Let $t = \tan\dfrac{x}{2}$

$$\therefore \quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}\left(\tan^2\frac{x}{2}+1\right)$$

$$\therefore \quad \frac{dt}{dx} = \frac{1}{2}(t^2+1)$$

$$\therefore \quad \frac{2\,dt}{1+t^2} = dx$$

$x=\dfrac{\pi}{2} \Rightarrow t=\tan\dfrac{\pi}{4}=1.$

$x=0 \Rightarrow t=0$

$$\therefore \int_0^{\frac{\pi}{2}}\frac{dx}{5+3\sin x+4\cos x}$$

$$= \int_0^1 \frac{2\,dt}{(1+t^2)\left(5+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)\right)}$$

$$= \int_0^1 \frac{2\,dt}{5+5t^2+6t+4-4t^2}$$

$$= \int_0^1 \frac{2\,dt}{(t+3)^2}$$

$$= -2\left[\frac{1}{t+3}\right]_0^1$$

$$= -2\left(\frac{1}{4}-\frac{1}{3}\right)$$

$$= \frac{1}{6}.$$

QUESTION TWO

(a) $w_1+\overline{w}_2 = 8-2i+(-5-3i)$

$= 3-5i.$

(b) (i) $(1-2i)^2 = 1-4i+4i^2$

$= 1-4i-4$

$= -3-4i.$

(ii) $z = \dfrac{5\pm\sqrt{25-4(7+i)}}{2}$

$= \dfrac{5\pm\sqrt{-3-4i}}{2}$

$= \dfrac{5\pm(1-2i)}{2}$ from (i)

$\therefore \quad z = 3-i$ or $2+i.$

OR $z^2-5z+6\dfrac{1}{4} = -7-i+6\dfrac{1}{4}$

$$\left(z-2\frac{1}{2}\right)^2 = \frac{-3-4i}{4}$$

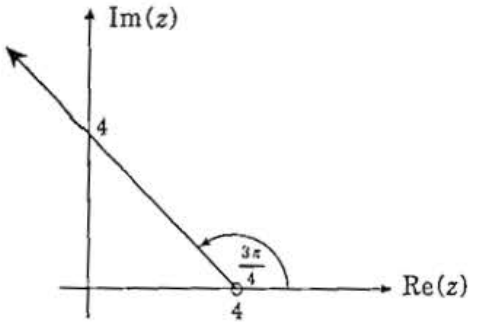
$$\therefore \quad z-2\frac{1}{2} = \pm\sqrt{\frac{-3-4i}{4}}$$

$$\therefore \quad z-2\frac{1}{2} = \pm\frac{1}{2}(1-2i) \text{ from (i)}$$

$$\therefore \quad z = 2\frac{1}{2}\pm\frac{1}{2}(1-2i)$$

$= 3-i$ or $2+i.$

(c) (i)



(Algebraically: Let $z = x+iy$

$$\therefore \quad \arg(x+iy-4) = \arg(x-4+iy)$$

$$= \tan^{-1}\left(\frac{y}{x-4}\right)$$

$$\therefore \quad \tan^{-1}\frac{y}{x-4} = \frac{3\pi}{4}$$

$$\frac{y}{x-4} = -1$$

$\therefore \quad y = 4-x, \quad y > 0.$)

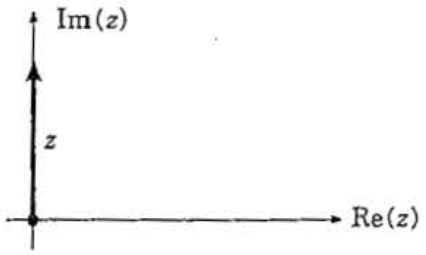
(ii) Let $z = x + iy$

$\therefore \quad y = \sqrt{x^2 + y^2}, \; y \geq 0$

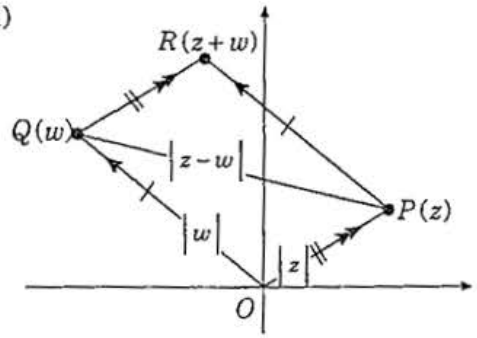
$y^2 = x^2 + y^2$

$x^2 = 0$

$\therefore \; x = 0, \; y \geq 0.$



(d) (i)



Any two sides of a triangle are together greater than the third side, so $PQ < OQ + OP$

$\therefore \quad |z-w| < |w| + |z| \qquad -①$

If POQ is straight, then

$PQ = PO + OQ.$

$\therefore \quad |z-w| = |z| + |w| \qquad -②$

From ① and ②,

$|z-w| \leq |z| + |w|.$

(ii) See diagram above, for point R. $OPRQ$ is a parallelogram.

(iii) $|z-w| = |z+w|$ means that the diagonals are equal, so that $OPRQ$ is a rectangle (or possibly a square).

Method 1:

Let $u = \dfrac{w}{z}$

Then $|u| = \dfrac{|w|}{|z|} = \dfrac{OQ}{OP}$,

and $\arg u = \arg w - \arg z$

$= \angle QOP$

$= \dfrac{\pi}{2}$ since $OPRQ$ is a rectangle.

$\therefore \; u = \dfrac{w}{z}$ is imaginary.

Note: z and w are arbitrary, so OQ and OP, although they must be perpendicular to each other, need not be as shown. It is possible that

$$\arg u = -\frac{\pi}{2}.$$

Method 2:

Now, $|z-w| = |z+w|$ can be rewritten:

$$\left|1 - \frac{w}{z}\right| = \left|1 + \frac{w}{z}\right|$$

This can only happen if $\dfrac{w}{z}$ lies on the imaginary axis so that

$$\mathrm{Re}\left(\frac{w}{z}\right) = 0.$$

Method 3:

Let $z = |z| \operatorname{cis}\theta$

$\therefore \quad w = |w| \operatorname{cis}\left(\theta + \dfrac{\pi}{2}\right)$

$\therefore \quad \dfrac{w}{z} = \dfrac{|w| \operatorname{cis}\left(\theta + \frac{\pi}{2}\right)}{|z| \operatorname{cis}\theta}$

$= \dfrac{|w|}{|z|} \operatorname{cis}\dfrac{\pi}{2}$

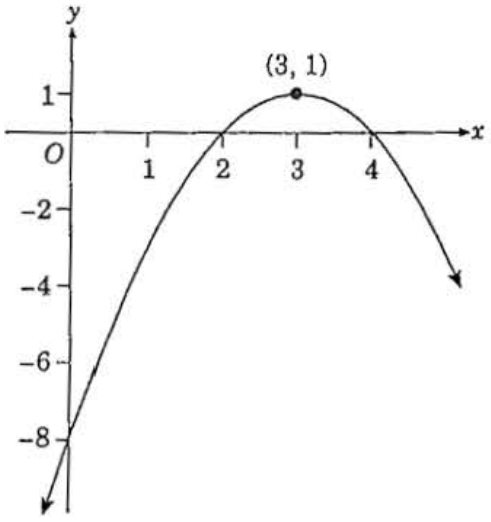
$\therefore \; \dfrac{w}{z}$ is purely imaginary.

QUESTION THREE

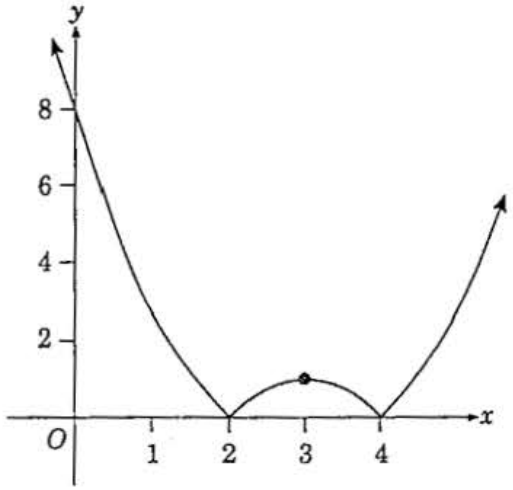
(a) (i) $y = f(x)$

$(-x^2 + 6x - 8) = -(x^2 - 6x + 8)$

$= -(x-2)(x-4)$



(ii) $y = |f(x)|$



(iii) $y^2 = f(x)$

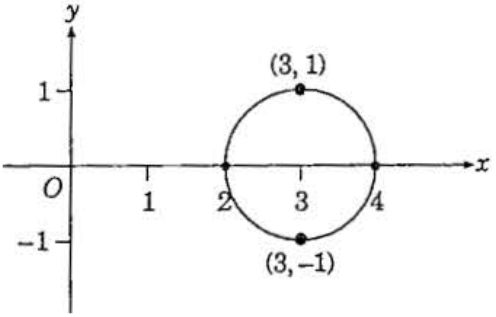
$\therefore \quad y^2 = -x^2 + 6x - 8$

$\therefore \quad x^2 + y^2 - 6x = -8$

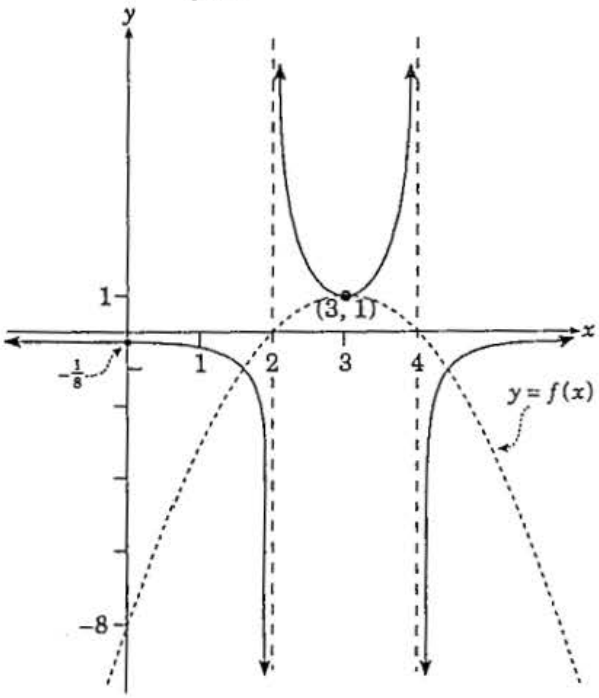
$x^2 - 6x + y^2 = -8 + 9$

$(x-3)^2 + y^2 = 1.$

That is, circle, centre (3, 0) and radius 1 unit .



(iv) $y = \dfrac{1}{f(x)}$

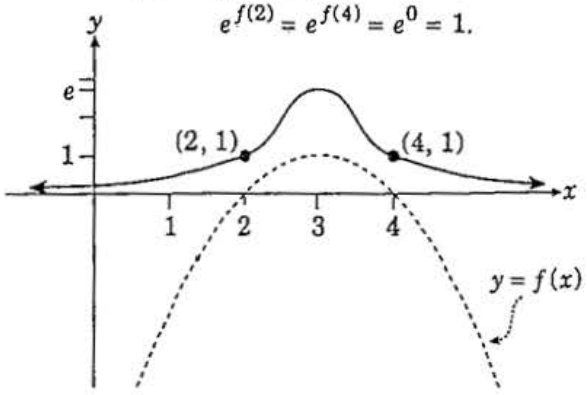


(v) $y = e^{f(x)}$

$f(3) = 1$

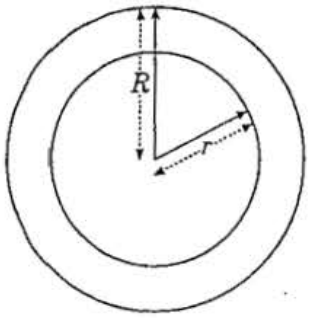
$\therefore \quad e^{f(3)} = e$

$e^{f(2)} = e^{f(4)} = e^0 = 1.$



(b) Let r, R be the inner and outer radii of the annulus, respectively.

$\therefore \quad r = 9 - \sqrt{16 - y^2}$ and $R = 9 + \sqrt{16 - y^2}$.



Area of annulus $= \pi(R^2 - r^2)$

$= \pi(R - r)(R + r)$

$= \pi \times 18 \times 2\sqrt{16 - y^2}$

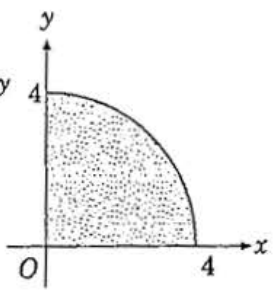
$\therefore$ area of annulus $= 36\pi\sqrt{16 - y^2}$.

(ii) $\therefore \quad V = 36\pi \displaystyle\int_{-4}^{4} \sqrt{16 - y^2}\, dy \quad -①$

$= 36\pi \times 2 \displaystyle\int_{0}^{4} \sqrt{16 - y^2}\, dy \quad -②$

Method 1:

$\displaystyle\int_0^4 \sqrt{16 - y^2}\, dy$ represents the area of a quadrant.



$\therefore \quad \displaystyle\int_0^4 \sqrt{16 - y^2}\, dy = \frac{1}{4} \times \pi \times 4^2$

$= 4\pi.$

Method 2:

Let $y = 4\sin\theta$

$\therefore \quad dy = 4\cos\theta\, d\theta.$

$y = 0, \quad \theta = 0$

$y = 4, \quad \theta = \frac{\pi}{2}$

$\therefore \quad \sqrt{16-y^2} = \sqrt{16(1-\sin^2\theta)}$

$= 4\cos\theta.$

$\therefore \int_0^4 \sqrt{16-y^2}\, dy$

$= \int_0^{\frac{\pi}{2}} 16\cos^2\theta\, d\theta$

$= 16\times\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\, d\theta$

$= 8\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$= 8\left(\frac{\pi}{2}+0\right)-8(0+0)$

$= 4\pi.$

(Could have also used the substitution $y = 4\cos\theta$.)

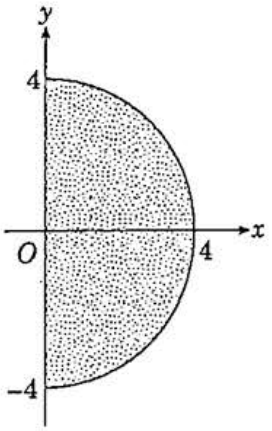
Using either of the above two methods, ② becomes $V = 36\pi\times 2\times 4\pi$.

$\therefore$ Volume $= 288\pi^2$ u^3

Note also that

$\int_{-4}^4 \sqrt{16-y^2}\, dy$

represents the area of a semi-circle.



$\therefore \int_{-4}^4 \sqrt{16-y^2}\, dy$

$= \frac{1}{2}\times\pi\times 4^2$

$= 8\pi.$

$\therefore$ ① becomes:

$V = 36\pi\times 8\pi$

$\therefore$ Volume $= 288\pi^2$ u^3.

QUESTION FOUR

(a) (i) $\cos\left(\frac{4k\pi}{7}\right)+i\sin\left(\frac{4k\pi}{7}\right)=1$

since $z^k = 1$.

$\therefore \cos\frac{4k\pi}{7}+i\sin\frac{4k\pi}{7}$

$= \cos 2n\pi + i\sin 2n\pi,$

where n is an integer.

Equating reals: $\frac{4k\pi}{7} = 2n\pi$

$\therefore \quad k = \frac{7n}{2}.$

Since k a positive integer, smallest k is when $n = 2$, so that $k = 7$.

(ii) *Method 1*:

Since w is a solution of $z^n = 1$, then

$w^n = 1$

$\therefore \quad (w^n)^m = 1$

$\therefore \quad w^{mn} = 1$

that is, $(w^m)^n = 1$ which satisfies $z^n = 1$.

That is, w^m is a solution of $z^n = 1$.

Method 2:

Let $w = \cos\theta + i\sin\theta$ where $w^n = 1$

$\therefore \quad w = \cos\frac{2\ell\pi}{n}+i\sin\frac{2\ell\pi}{n},$

where ℓ, n are integers,

$\therefore \quad w^m = \cos m\theta + i\sin m\theta$

$= \cos\frac{2m\ell\pi}{n}+i\sin\frac{2m\ell\pi}{n}$

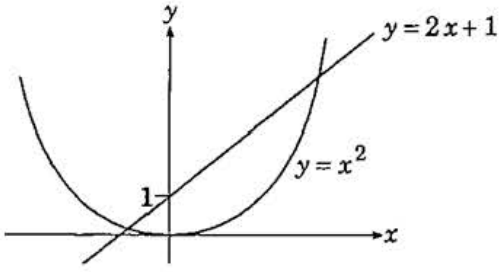
$= \cos\frac{2p\pi}{n}+i\sin\frac{2p\pi}{n},$

where $p = \ell m$, so that p is an integer,

$\therefore \quad (w^m)^n = 1.$

w^m is a solution of $z^n = 1$.

(b) (i) $x^2 > 2x+1$



$x^2 = 2x+1$

$x^2-2x-1=0$

$x = \frac{2\pm\sqrt{4+4}}{2}$

$= 1\pm\sqrt{2}$

$\therefore$ Solutions are $x > 1+\sqrt{2}$ or $x < 1-\sqrt{2}$.

(ii) For $n=5$, $2^5=32$ and $5^2=25$,
that is $2^5>5^2$,
$\therefore$ true for $n=5$.

Let $2^k>k^2$ where k is an integer, $k\geq 5$.
Hence prove, for $k+1$,
that $2^{k+1}>(k+1)^2$.

Method 1:

$2^{k+1}=2^k\times 2$
$>2k^2$ by assumption, $k\geq 5$
$\therefore\ 2^{k+1}>k^2+k^2$.

$\therefore$ Using (b)(i), where $k>\sqrt{2}+1$, (true since $k\geq 5$), we get
$2^{k+1}>k^2+2k+1$, so that
$2^{k+1}>(k+1)^2$, as required.

$\therefore$ Holds true for $n=k+1$ if true for $n=k$.

$\therefore$ Holds true for $n=5$, $n=6$, and so on for all integers greater than or equal to 5.

Method 2:

Required to prove $2^{k+1}-(k+1)^2>0$.
LHS $=2\times 2^k-k^2-2k-1$
$=2\times 2^k-2k^2+k^2-2k-1$
$\therefore$ LHS $=2(2^k-k^2)+k^2-2k-1$.

Note that $2^k-k^2>0$ from our assumption, and that $k^2-2k-1>0$ for $k>\sqrt{2}+1$ from (b)(i).

$\therefore$ LHS >0
i.e. $2^{k+1}>(k+1)^2$ . . .

Method 3:

Assume true for $n=k$,
i.e. $2^k>k^2$ and $2^k-k^2>0$.
Then $2^{k+1}-(k+1)^2$
$=2.2^k-k^2-2k-1$
$>2k^2-k^2-2k-1$
$=k^2-2k-1$
$=(k-1)^2-2$
>0 since $k\geq 5$ . . .

(c) (i) $\sin(2m+1)x$
$=\sin(2mx+x)$
$=\sin 2mx\cos x+\cos 2mx\sin x$
and $\sin(2m-1)x$
$=\sin(2mx-x)$
$=\sin 2mx\cos x-\cos 2mx\sin x$.

$\therefore\ \dfrac{\sin(2m+1)x}{\sin x}-\dfrac{\sin(2m-1)x}{\sin x}$
$=\dfrac{2\cos 2mx\sin x}{\sin x}$
$=2\cos(2mx)$, as required.

Or, using $\sin(A+B)-\sin(A-B)$
$=2\cos A\sin B$,
where $A=2mx$ and $B=x$, we get
$\dfrac{\sin(2mx+x)-\sin(2mx-x)}{\sin x}$
$=\dfrac{2\cos 2mx\sin x}{\sin x}$
$=2\cos 2mx$.

(ii) $\displaystyle\int_0^{\frac{\pi}{2}}\cos(2mx)\,dx=\left[\frac{\sin 2mx}{2m}\right]_0^{\frac{\pi}{2}}$
$=\dfrac{\sin m\pi}{2m}-0$
$=0$,
since $\sin m\pi=0$ for all integer m.

(iii) From (i),
$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin(2m+1)x}{\sin x}\,dx-\int_0^{\frac{\pi}{2}}\frac{\sin(2m-1)x}{\sin x}\,dx$
$=2\displaystyle\int_0^{\frac{\pi}{2}}\cos(2mx)\,dx$
$=2\times 0$ from (ii)
$=0$.

$\therefore\ \displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin(2m+1)x}{\sin x}\,dx$
$=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin(2m-1)x}{\sin x}\,dx$.

(iv) If $m=1$,
$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin(2mx)}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x}\,dx$
$=\displaystyle\int_0^{\frac{\pi}{2}}1\,dx$
$=\left[x\right]_0^{\frac{\pi}{2}}$
$=\dfrac{\pi}{2}$, as required.

(v) Using (iii) with $m=2$, we get
$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin 5x}{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin 3x}{\sin x}\,dx$.

Now, using (iii) and (iv) where $m = 1$, we get

$$\int_0^{\frac{\pi}{2}} \frac{\sin 3x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x}\,dx = \frac{\pi}{2}.$$

$$\therefore \int \frac{\sin 5x}{\sin x}\,dx = \frac{\pi}{2}.$$

QUESTION FIVE

(a) (i) $\sin x + \sin 3x$

$= \sin(2x - x) + \sin(2x + x)$

$= \sin 2x \cos x - \cos 2x \sin x + \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin 2x \cos x.$

(ii) From (i),

$\sin x + \sin 3x + \sin 2x$

$= 2\sin 2x \cos x + \sin 2x$

$= \sin 2x(2\cos x + 1).$

$\therefore$ Consider $\sin 2x(2\cos x + 1) = 0$ for $0 \le x < 2\pi$.

$\therefore \sin 2x = 0, \quad 0 \le 2x < 4\pi,$

or $\cos x = -\frac{1}{2}, \quad 0 \le x < 2\pi.$

$\therefore 2x = 0, \pi, 2\pi, 3\pi,$ or $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}.$

(b) (i) $t_1 + t_2 + t_3 =$ sum of roots

$= 0.$

(ii) $t_1^2 + t_2^2 + t_3^2$

$= (t_1 + t_2 + t_3)^2 - 2(t_1t_2 + t_2t_3 + t_3t_1)$

But $t_1t_2 + t_2t_3 + t_3t_1 = c$

$\therefore \quad t_1^2 + t_2^2 + t_3^2 = 0^2 - 2c$

$= -2c.$

(iii) *Method 1*:

$f(t) = t^3 + ct + d$ has turning points when $f'(t) = 3t^2 + c = 0$, that is,

when $t = \pm\sqrt{\frac{-c}{3}}$.

Let $u = \sqrt{\frac{-c}{3}}$ and $v = -u = -\sqrt{\frac{-c}{3}}$.

$\therefore f(u).f(v)$

$= (u^3 + cu + d)(-u^3 - cu + d)$

$= \left[u(u^2 + c) + d\right]\left[-u(u^2 + c) + d\right]$

$= \left[u\left(-\frac{c}{3} + c\right) + d\right]\left[-u\left(-\frac{c}{3} + c\right) + d\right]$

where $3u^2 + c = 0$, from above

$= \left(u.\frac{2c}{3} + d\right)\left(-u.\frac{2c}{3} + d\right)$

$= d^2 - u^2.\frac{4c^2}{9}$

$= d^2 - \left(-\frac{c}{3}\right).\frac{4c^2}{9}$

$= d^2 + \frac{4c^3}{27}$

< 0, since $f(u).f(v) < 0$

$\therefore 27d^2 + 4c^3 < 0.$

Method 2:

$27d^2 + 4c^3$

$= 27(t_1t_2t_3)^2 + 4c^3$

$= 27t_1^2t_2^2t_3^2 + 4c^3$

$= 27\left(\sqrt[3]{t_1^2t_2^2t_3^2}\right)^3 + 4c^3$

$\le 27\left(\frac{t_1t_2 + t_2t_3 + t_3t_1}{3}\right)^3 + 4c^3$

$= \frac{27}{27}(c)^3 + 4c^3$

$= 5c^3.$

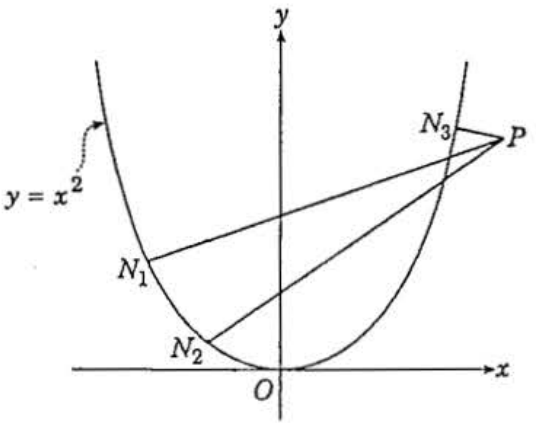
But from (ii), $-2c > 0$

$\therefore \quad c < 0$

so $\quad c^3 < 0$

$\therefore \quad 27d^2 + 4c^3 < 0,$ as required.

(c)



(i) Tangent to $y = x^2$ at (t, t^2) has gradient $2t$.

$\therefore$ Normal has gradient $-\frac{1}{2t}$.

$\therefore$ Equation of normal is

$$y - t^2 = -\frac{1}{2t}(x - t)$$

i.e. $\quad 2ty - 2t^3 = -x + t$

$\therefore \quad 2t^3 + t - 2ty - x = 0$

$$2t^3 + (1 - 2y)t - x = 0$$

$$\therefore \; t^3 + \left(\frac{1-2y}{2}\right)t + \left(\frac{-x}{2}\right) = 0.$$

(ii) If the normals at three distinct points all pass through (x_0, y_0), then there exist three distinct real roots to the equation

$$t^3 + \left(\frac{1-2y_0}{2}\right)t + \left(\frac{-x_0}{2}\right) = 0.$$

By part (b) (iii):

$$27\left(\frac{-x_0}{2}\right)^2 + 4\left(\frac{1-2y_0}{2}\right)^3 < 0$$

$$\therefore \quad 27\left(\frac{x_0}{2}\right)^2 < -4\left(\frac{1-2y_0}{2}\right)^3$$

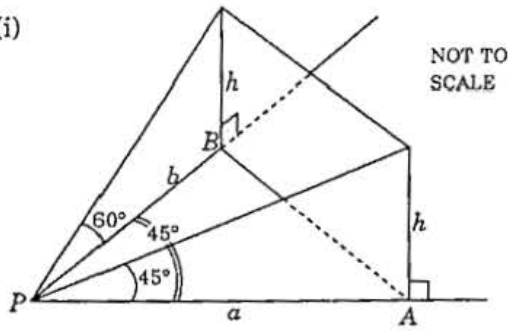
$$27\left(\frac{x_0}{4}\right)^2 < \left(\frac{2y_0 - 1}{2}\right)^3$$

$$3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} < y_0 - \frac{1}{2}$$

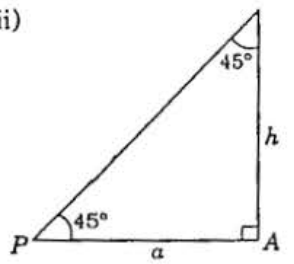
$$\therefore \quad y_0 > 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}.$$

QUESTION SIX

(a) (i)



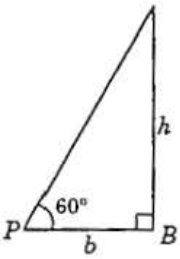
(ii)



$$\frac{a}{h} = \tan 45°$$

$$\therefore \quad a = h$$

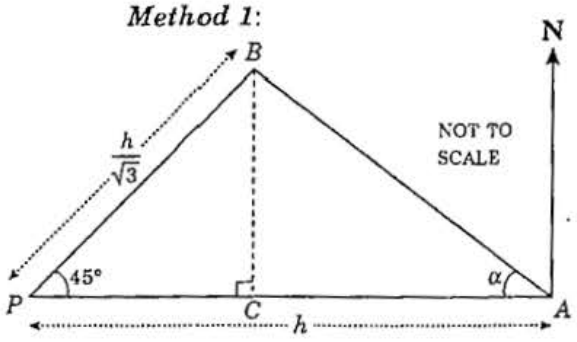
(iii)



$$\frac{h}{b} = \tan 60°$$

$$\therefore \quad b = \frac{h}{\sqrt{3}}$$

Method 1:



In ΔBPC, $PC = \frac{h}{\sqrt{3}} \cos 45° = \frac{h}{\sqrt{6}} = BC.$

$\therefore$ In ΔBCA, $CA = h\left(1 - \frac{1}{\sqrt{6}}\right)$

$$\therefore \quad \tan\alpha = \frac{AC}{CA}$$

$$= \frac{h}{\sqrt{6}} + h\left(1 - \frac{1}{\sqrt{6}}\right)$$

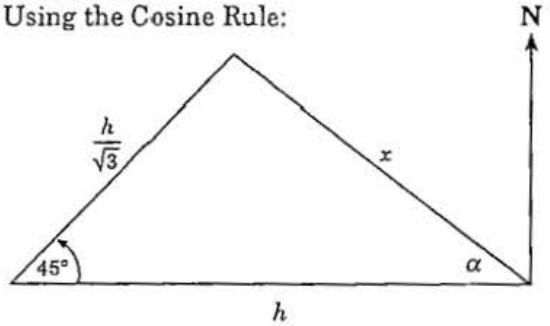
$$= \frac{1}{\sqrt{6} - 1}$$

$\therefore \; \alpha = 35°$ (to nearest degree).

$\therefore$ Bearing is 305° (or N55°W).

Method 2:

Using the Cosine Rule:



$$x^2 = h^2 + \left(\frac{h}{\sqrt{3}}\right)^2 - 2 \times h \times \frac{h}{\sqrt{3}} \cos 45°$$

$$= h^2 + \frac{h^2}{3} - \frac{\sqrt{6}h^2}{3}$$

$$= h^2 \frac{(4 - \sqrt{6})}{3} \quad (= 0{\cdot}5168h^2)$$

$$\therefore \quad x = \sqrt{\frac{4 - \sqrt{6}}{3}}\, h \quad (= 0{\cdot}7189h)$$

$\therefore$ Using the Sine Rule:

$$\frac{x}{\sin 45°} = \frac{\frac{h}{\sqrt{3}}}{\sin\alpha}$$

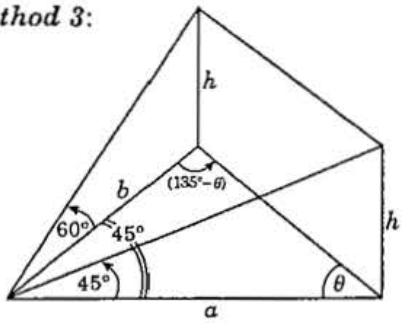
$$\sin\alpha = \frac{\frac{h}{\sqrt{3}} \times \sin 45°}{x}$$

$$\approx 0{\cdot}5679$$

$\therefore \quad \alpha \approx 35°.$

$\therefore$ Bearing is 305° or (N55°W).

Method 3:



$$\frac{\sin\theta}{b} = \frac{\sin(135°-\theta)}{a}$$

$$\therefore \quad \frac{\sin\theta}{\frac{h}{\sqrt{3}}} = \frac{\sin 135°\cos\theta - \cos 135°\sin\theta}{h}$$

$$\therefore \quad \sqrt{3}\sin\theta = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$$

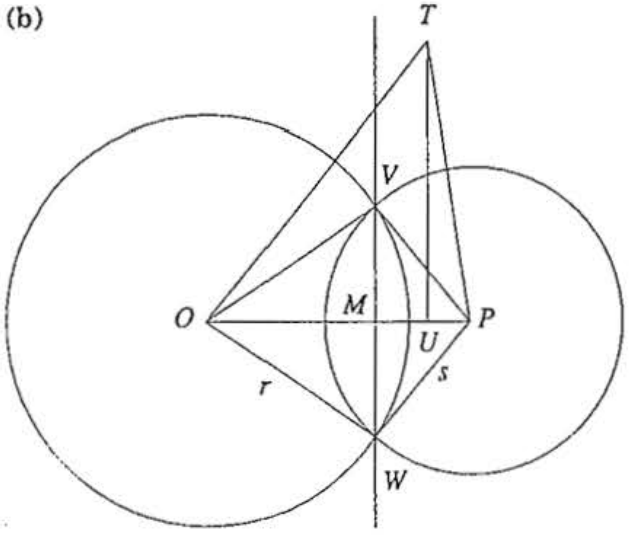
$$\therefore \quad \sqrt{6}\sin\theta = \cos\theta + \sin\theta$$

$$\therefore \quad (\sqrt{6}-1)\sin\theta = \cos\theta$$

$$\therefore \quad \tan\theta = \frac{1}{\sqrt{6}-1} \Rightarrow \theta \approx 35° \text{ (nearest degree)}$$

$\therefore$ Bearing is 305° or (N55°W).

(b)



(i) $\Delta OVP \equiv \Delta OWP$ (SSS), since OP common, $OV = OW$ (equal radii) and $PV = PW$ (equal radii).

Hence, $\angle POV = \angle POW$ (corresp. angles equal in congruent triangles).

$\therefore \Delta OVM \equiv \Delta OWM$ (SAS) since OM common, $OV = OW$ (equal radii), $\angle MOV = \angle MOW$.

$\therefore OP$ and VW are perpendicular.

Hence $\angle VMO = \angle WMO$.

But $\angle VMW = 180°$

$\therefore \quad \angle VMO = \angle WMO = 90°.$

(ii) Using Pythagoras' Theorem in ΔOTU and ΔPTU:

$TU^2 = OT^2 - OU^2 = PT^2 - PU^2$

Hence $OT^2 - PT^2 = OU^2 - PU^2$.

Similarly in Δs OVM, PVM,

$OM^2 - PM^2 = OV^2 - PV^2 = r^2 - s^2$.

(iii) If T is on VW, then $U = M$, so that

$OT^2 - PT^2 = OM^2 - PM^2 = r^2 - s^2$.

To show T lies on VW if

$OT^2 - PT^2 = r^2 - s^2$:

From (ii), $OT^2 - PT^2 = OU^2 - PU^2$

and $OM^2 - PM^2 = r^2 - s^2$

$\therefore OU^2 - PU^2 = OM^2 - PM^2$ —①

Let $OP = d$

$OU = x \Rightarrow PU = d - x$

$OM = y \Rightarrow PM = d - y$

$\therefore$ From ①, $x^2 - (d-x)^2 = y^2 - (d-y)^2$

$\therefore \quad x^2 - d^2 + 2dx - x^2 = y^2 - d^2 + 2dy - y^2$

$\therefore \quad 2dx = 2dy$

$\therefore \quad x = y$, i.e. $U \equiv M$

$\therefore T$ lies on VW.

(iv) A point T lies on AB when

$OT^2 - PT^2 = r^2 - s^2$.

T also lies on CD when

$PT^2 - TQ^2 = s^2 - t^2$.

$\therefore$ When T lies on both AB and CD,

$(OT^2 - PT^2) + (PT^2 - TQ^2) = (r^2 - s^2) + (s^2 - t^2)$

$OT^2 - TQ^2 = r^2 - t^2$

which is the condition for T to lie on EF.

$\therefore AB$, CD, EF are concurrent.

QUESTION SEVEN

(a) (i) Using integration by parts:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x . \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \frac{d}{dx}(-\cos x)\, dx$$

$$= \left[-\cos x . \sin^{n-1} x\right]_0^{\frac{\pi}{2}}$$

$$- (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x . \cos x . -\cos x \, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x)\, dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$\therefore \quad n I_n = (n-1) I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}, \text{ as required.}$$

(ii) From (i),

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times I_{2n-4}, \text{ etc.}$$

Note that $I_0 = \int_0^{\frac{\pi}{2}} (\sin x)^0 \, dx$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= \frac{\pi}{2}$$

and $I_2 = \frac{1}{2} \times \frac{\pi}{2}$,

and $I_4 = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$, from (i)

$$\therefore \quad I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}.$$

Similarly, $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$

$$I_{2n-1} = \frac{2n-2}{2n-1} I_{2n-3}, \text{ etc.}$$

Also, $I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx$

$$= -\left[\cos x\right]_0^{\frac{\pi}{2}} = 1$$

and $I_3 = \frac{2}{3} \times I_1 = \frac{2}{3} \times 1$

and $I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} \times 1$

$$\therefore \quad I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1.$$

(iii) For $0 < x < \frac{\pi}{2}$, $0 < \sin x < 1$.

$\therefore \; 0 < (\sin x)^{k+1} < (\sin x)^k < 1$, for $0 < x < \frac{\pi}{2}$ and $k \geq 0$ and an integer,

$$\therefore \quad \int_0^{\frac{\pi}{2}} (\sin x)^{k+1} \, dx < \int_0^{\frac{\pi}{2}} (\sin x)^k \, dx$$

that is, $I_k > I_{k+1}$.

(iv) Given that $I_{2n} > I_{2n+1}$:

$$\frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} > \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1$$

$$\therefore \frac{\pi}{2} > \frac{(2n)^2 (2n-2)^2 \times \cdots \times 4^2 \times 2^2}{(2n+1)(2n-1)^2 (2n-3)^2 \times \cdots \times 5^2 \times 3^2 \times 1^2} \quad \text{—①}$$

Given that $I_{2n-1} > I_{2n}$:

$$\frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} \times \cdots \times \frac{4}{5} \times \frac{2}{3} \times 1 > \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore \frac{(2n)(2n-2)^2 (2n-4)^2 \times \cdots \times 4^2 \times 2^2}{(2n-1)^2 (2n-3)^2 \times \cdots \times 5^2 \times 3^2 \times 1^2} > \frac{\pi}{2} \quad \text{—②}$$

$② \times \frac{2n}{2n+1}$:

$$\frac{(2n)^2 (2n-2)^2 (2n-4)^2 \times \cdots \times 4^2 \times 2^2}{(2n+1)(2n-1)^2 (2n-3)^2 \times \cdots \times 5^2 \times 3^2 \times 1^2} > \frac{\pi}{2}\left(\frac{2n}{2n+1}\right) \quad \text{—③}$$

From ① and ③, we get:

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2 \times 4^2 \times \cdots \times (2n)^2}{1^2 \times 3^2 \times 5^2 \times \cdots \times (2n-1)^2 (2n+1)} < \frac{\pi}{2}, \text{ as required.}$$

(b) (i) *Method 1*:

$\binom{2n}{k}$ where $k = 0, 1, 2, \ldots, 2n$ represents a line in Pascal's Δ.

The largest value for $\binom{2n}{k}$ on this line is the middle term.

There are $2n+1$ terms, so the middle term is the $(n+1)$th term, that is where $k=n$.

Method 2:

$$\text{Consider } \frac{P_{k+1}}{P_k} = \frac{\binom{2n}{k+1}}{\binom{2n}{k}}$$

$$= \frac{\dfrac{(2n)!}{(k+1)!(2n-k-1)!}}{\dfrac{(2n)!}{k!(2n-k)!}}$$

$$= \frac{k!(2n-k)!}{(k+1)!(2n-k-1)!}$$

$$= \frac{2n-k}{k+1}.$$

If $k<n$, then $2n-k>n$ and $k+1\le n$, so $\dfrac{P_{k+1}}{P_k}>1$, while if $k\ge n$, then $2n-k\le n$ and $k+1>n$, so that $\dfrac{P_{k+1}}{P_k}<1$.

This shows that P_k increases in size until P_n, and then decreases, and hence it follows that P_k is biggest when $k=n$.

(ii) $$P_n = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$

$$= \frac{(2n)!}{n!n!}\times\frac{1}{2^{2n}}$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}, \text{ as required.}$$

(iii) *Method 1*: From (b)(ii),

$$P_n = \frac{1.2.3.4\times\cdots\times(2n-2)(2n-1)(2n)}{2^{2n}.(n!).1.2.3\ldots(n-1)n}$$

$$= \frac{1.2.3.4\ldots(2n-2)(2n-1)(2n)}{2^n\, n!\; 2.4.6\ldots(2n-2)(2n)}$$

$$= \frac{1.3.5\ldots(2n-3)(2n-1)}{2^n\, n!}$$

$$= \frac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots(2n-2)(2n)}$$

$$= \left[\frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{1.3^2.5^2\ldots(2n-3)^2(2n-1)^2}\right]^{-\frac{1}{2}}$$

$$= \left[\frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{1.3^2.5^2\ldots(2n-3)^2(2n-1)^2(2n+1)}\right]^{-\frac{1}{2}} \times\left(\frac{1}{2n+1}\right)^{\frac{1}{2}}.$$

$$= \frac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots(2n-2)(2n)}$$

$$= \left[\frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{1.3^2.5^2\ldots(2n-3)^2(2n-1)^2}\right]^{-\frac{1}{2}}$$

$$= \left[\frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{1.3^2.5^2\ldots(2n-3)^2(2n-1)^2(2n+1)}\right]^{-\frac{1}{2}} \times\left(\frac{1}{2n+1}\right)^{\frac{1}{2}}.$$

Using the result of part (a), it follows that

$$\left(\frac{\pi}{2}\right)^{-\frac{1}{2}} < \left[\frac{2^2.4^2.6^2\ldots(2n-2)^2(2n)^2}{1.3^2.5^2\ldots(2n-3)^2(2n-1)^2(2n+1)^2}\right]^{-\frac{1}{2}} < \left[\frac{\pi}{2}\left(\frac{2n}{2n+1}\right)\right]^{-\frac{1}{2}}$$

and therefore

$$\left[\frac{2}{\pi}.\frac{1}{2n+1}\right]^{\frac{1}{2}} < P_n < \left[\frac{2}{\pi}.\left(\frac{2n+1}{2n}\right)\left(\frac{1}{2n+1}\right)\right]^{\frac{1}{2}}$$

$$\left[\frac{1}{\pi\left(n+\frac{1}{2}\right)}\right]^{\frac{1}{2}} < P_n < \left[\frac{1}{n\pi}\right]^{\frac{1}{2}}.$$

Method 2: From (a)(iv),

$$\sqrt{\frac{\pi}{2}.\frac{2n}{2n+1}} < \frac{2.4\ldots(2n)}{1.3.5\ldots(2n-1)\sqrt{2n+1}} < \sqrt{\frac{\pi}{2}}$$

$$\sqrt{\frac{2(2n+1)}{\pi.2n}} > \frac{1.3.5\ldots(2n-1)\sqrt{2n+1}}{2^n.1.2.3\ldots n} > \sqrt{\frac{2}{\pi}}$$

$$\sqrt{\frac{2(2n+1)}{\pi.2n}} > \frac{1.2.3.4.5\cdots(2n-1)\,2n.\sqrt{2n+1}}{2^n.n!.2.4.6\ldots 2n} > \sqrt{\frac{2}{\pi}}$$

$$\sqrt{\frac{2(2n+1)}{\pi.2n}} > \frac{(2n)!\sqrt{2n+1}}{2^n.n!.2^n.1.2.3\ldots n} > \sqrt{\frac{2}{\pi}}$$

$$\sqrt{\frac{2(2n+1)}{\pi.2n}} > \frac{(2n)!}{2^{2n}(n!)^2}\sqrt{2n+1} > \sqrt{\frac{2}{\pi}}$$

$$\therefore \quad \sqrt{\frac{2(2n+1)}{\pi.2n(2n+1)}} > P_n > \sqrt{\frac{2}{\pi(2n+1)}}$$

$$\frac{1}{\sqrt{\pi n}} > P_n > \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}.$$

QUESTION EIGHT

(a) $(p-q)^2 \geq 0$ where p and q are real numbers.

$$\therefore \quad p^2 - 2pq + q^2 \geq 0$$

$$\therefore \quad 2pq \leq p^2 + q^2$$

$$pq \leq \frac{p^2+q^2}{2}.$$

(b) ***Method 1***:

Suppose $P(x_1, y_1)$ is any point on the line ℓ. Then, using the equation of ℓ,

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1,$$

we see that

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1$$

$$= \frac{1}{a^2}(x_1 x_0) + \frac{1}{b^2}(y_1 y_0)$$

Using the result of (a), and rearranging,

$$1 \leq \frac{1}{a^2}\left(\frac{x_1^2 + x_0^2}{2}\right) + \frac{1}{b^2}\left(\frac{y_1^2 + y_0^2}{2}\right)$$

$$= \frac{x_1^2}{2a^2} + \frac{x_0^2}{2a^2} + \frac{y_1^2}{2b^2} + \frac{y_0^2}{2b^2}$$

$$= \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) + \frac{1}{2}\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right)$$

$$< \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) + \frac{1}{2},$$

since $\quad \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} < 1.$

$$\therefore \quad \frac{1}{2} < \frac{1}{2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right)$$

$$\therefore \quad 1 < \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

Hence $P(x_1, y_1)$ lies outside the ellipse, and hence so does ℓ.

Method 2:

To show that ℓ lies outside the ellipse $\mathcal{E}$, it suffices to show that ℓ and $\mathcal{E}$ never meet.

So consider the simultaneous equations:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1 \qquad \text{—①}$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{—②}$$

From ①,

$$\frac{x_0 x}{a^2} = 1 - \frac{y_0 y}{b^2}$$

$$\therefore \quad x = \frac{a^2}{x_0}\left(1 - \frac{y_0 y}{b^2}\right)$$

Substituting in ②:

$$\frac{1}{a^2}\left[\frac{a^2}{x_0}\left(1 - \frac{y_0 y}{b^2}\right)\right]^2 + \frac{y^2}{b^2} = 1$$

$$\therefore \quad \frac{a^2}{x_0}\left(1 - \frac{2y_0 y}{b^2} + \frac{y_0^2 y^2}{b^4}\right) + \frac{y^2}{b^2} = 1$$

$$\therefore \quad a^2\left(b^4 - 2b^2 y_0 y + y_0^2 y^2\right) + x_0^2 b^2 y^2 = x_0^2 b^4$$

$$\therefore \quad y^2\left[a^2 y_0^2 + x_0^2 b^2\right] + y\left[-2a^2 b^2 y_0\right] + a^2 b^4 - x_0^2 b^4 = 0 \qquad \text{—③}$$

For this equation, the discriminant Δ is given by:

$$\Delta = \left[-2a^2 b^2 y_0\right]^2 - 4\left[a^2 y_0^2 + x_0^2 b^2\right] \times \left[a^2 b^4 - x_0^2 b^4\right]$$

$$= 4a^4 b^4 y_0^2 - 4b^4\left(a^2 y_0^2 + x_0^2 b^2\right) \times \left(a^2 - x_0^2\right)$$

$$= 4b^4\left[a^4 y_0^2 - \left(a^2 y_0^2 + x_0^2 b^2\right) \times \left(a^2 - x_0^2\right)\right]$$

$$= 4b^4\left[a^4 y_0^2 - a^4 y_0^2 + a^2 y_0^2 x_0^2 - a^2 b^2 x_0^2 + x_0^4 b^2\right]$$

$$= 4b^4\left[a^2 y_0^2 x_0^2 + x_0^4 b^2 - a^2 x_0^2 b^2\right]$$

$$= 4a^2 b^6 x_0^2\left[\frac{y_0^2}{b^2} + \frac{x_0^2}{a^2} - 1\right],$$

and $\Delta < 0$ since $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} < 1.$

This means that the equation ③ has no real roots, so the simultaneous equations ① and ② have no real solutions, that is, ℓ and $\mathcal{E}$ do not meet, as required.

(ii) If $Q(x_2, y_2)$ lies on the line ℓ, then its coordinates satisfy the equation of ℓ,

that is, $\dfrac{x_2 x_0}{a^2} + \dfrac{y_2 y_0}{b^2} = 1.$

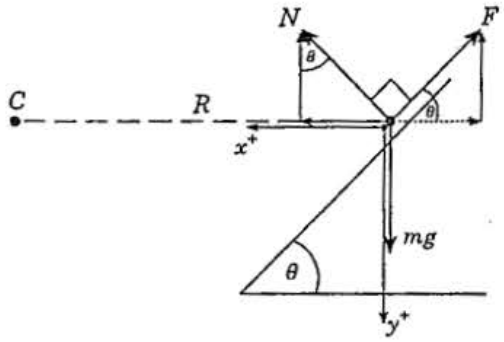
If this is true, then it also means that the point (x_0, y_0) lies on the line

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1.$$

That is, $M(x_0, y_0)$ lies on the chord of contact to $\mathcal{E}$ from any point on ℓ.

Note: In this part, it is essential not to confuse the coordinates of the fixed point $M(x_0, y_0)$, the coordinates of the point $Q(x_2, y_2)$, and the variables x and y, which can take infinitely many values in both the equation of ℓ and the equation of the chords of contact.

(c)



(i) Resolving forces in the directions shown:

Vertically:

$$m\ddot{y} = mg - N\cos\theta - F\sin\theta$$

Radially:

$$m\ddot{x} = \frac{mv^2}{R} = N\sin\theta - F\cos\theta$$

As there is no movement vertically,

$$\ddot{y} = 0,$$

$$\therefore \quad mg = N\cos\theta - F\sin\theta$$

and $\dfrac{mv^2}{R} = N\sin\theta - F\cos\theta$

By division, $\dfrac{v^2}{Rg} = \dfrac{N\sin\theta - F\cos\theta}{N\sin\theta + F\cos\theta}.$

(ii) If the particle is on the verge of slipping up the track, then F attains its maximum value acting down the hill.

That is, $F = -\mu N$

$$\therefore \quad \frac{v_{\max}^2}{Rg} = \frac{N\sin\theta + \mu N\cos\theta}{N\cos\theta - \mu N\sin\theta}$$

$$= \frac{N\left(\frac{\sin\theta}{\cos\theta} + \mu\frac{\cos\theta}{\cos\theta}\right)}{N\left(\frac{\cos\theta}{\cos\theta} - \mu\frac{\sin\theta}{\cos\theta}\right)}$$

$$= \frac{\tan\theta + \mu}{1 - \mu\tan\theta}.$$

(iii) Let $v_{\min}$ be the minimum speed that the particle can travel without sliding down the track.

So, if $v = v_{\min}$, the particle is on the point of sliding down the track.

$$\therefore \quad F = \mu N$$

and $\dfrac{v_{\min}^2}{Rg} = \dfrac{N\sin\theta - \mu N\cos\theta}{N\cos\theta + \mu N\sin\theta}$

$$= \frac{\tan\theta - \mu}{1 + \mu\tan\theta}.$$

If $\mu \geq \tan\theta \quad \Rightarrow \quad \dfrac{v_{\min}^2}{Rg} \leq 0$

$$\therefore \quad v_{\min} = 0.$$

$\therefore$ The particle will not slide down the track for any value of v.