1994 HIGHER SCHOOL CERTIFICATE SOLUTIONS

4 UNIT MATHEMATICS

QUESTION ONE

(a) (i) $\displaystyle\int \frac{4x-12}{x^2-6x+13}\,dx$

$= \displaystyle 2\int \frac{2x-6}{x^2-6x+13}\,dx$

$= 2\log\left|x^2-6x+13\right| + C.$

(ii) $\displaystyle\int \frac{1}{x^2-6x+13}\,dx$

$= \displaystyle\int \frac{dx}{x^2-6x+13}$

$= \displaystyle\int \frac{dx}{(x-3)^2+2^2}$

$= \displaystyle\frac{1}{2}\tan^{-1}\frac{x-3}{2} + C.$

(b) $\displaystyle\int_{\frac{3}{2}}^{3} \sqrt{9-u^2}\,du$ Let $u = 3\sin\theta$

$du = 3\cos\theta\,d\theta.$

When $u = \frac{3}{2}$, $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$.

When $u = 3$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$.

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\,.3\cos\theta\,d\theta$

$= \displaystyle 9\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\,d\theta$

$= \displaystyle 9\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

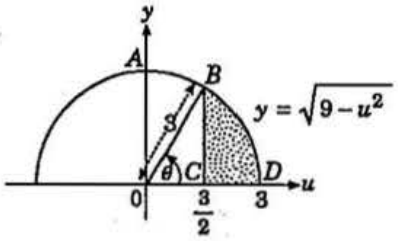
$= \displaystyle\frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= \displaystyle\frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \displaystyle\frac{9}{2}\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{6}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right] = \frac{9}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right).$

OR

The integral is equal to the area shaded in the diagram below:



Let $\theta = \angle BOC$, then $\cos\theta = \frac{\frac{3}{2}}{3} = \frac{1}{2}$

$\theta = \frac{\pi}{3}.$

Hence $\displaystyle\int_{\frac{3}{2}}^{3} \sqrt{9-u^2}$

= Area BCD

= Area of quarter-circle OAD − area of sector OAB − area of ΔOBC

$= \displaystyle\frac{1}{4}.\pi.3^2 - \frac{1}{2}.3^2.\frac{\pi}{6} - \frac{1}{2}.\frac{3}{2}.\frac{3\sqrt{3}}{2}$

$= \displaystyle\frac{9}{2}\left(\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$

$= \displaystyle\frac{9}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right).$

(c) $\displaystyle\frac{11-2t}{(2t-1)(3-t)} = \frac{A}{2t-1} + \frac{B}{3-t}$

$= \displaystyle\frac{A(3-t)+B(2t-1)}{(2t-1)(3-t)}$

$\therefore\ A(3-t)+B(2t-1) = 11-2t.$

When $t = 3$, $11-6 = B.5$

$B = 1.$

When $t = 0$, $11 = 3A - B$

$3A = 12$, $A = 4.$

$$\therefore \int_1^2 \frac{11-2t}{(2t-1)(3-t)}\,dt$$

$$= \int_1^2 \frac{4}{2t-1}+\frac{1}{3-t}\,dt$$

$$= \Big[2\ln|2t-1|-\ln|3-t|\Big]_1^2$$

$$= 2\ln 3-\ln 1-(2\ln 1-\ln 2)$$

$$= \ln 9+\ln 2$$

$$= \ln 18.$$

(d)

Let $I = \int_0^{\pi} e^{2x}\sin x\,dx$

$$= -\int_0^{\pi} e^{2x}\,d(\cos x)$$

$$= \Big[-e^{2x}\cos x\Big]_0^{\pi}+\int_0^{\pi} 2e^{2x}\cos x\,dx$$

$$= \Big(e^{2\pi}+1\Big)+2\int_0^{\pi} e^{2x}\,d(\sin x)$$

$$= \Big(e^{2\pi}+1\Big)+2\Big[e^{2x}\sin x\Big]_0^{\pi}-2\int_0^{\pi} 2e^{2x}\sin x\,dx$$

$$= \Big(e^{2\pi}+1\Big)+2(0)-4I$$

$\therefore 5I = \Big(e^{2\pi}+1\Big)$

$\therefore \quad I = \frac{1}{5}\Big(e^{2\pi}+1\Big).$

QUESTION TWO

(a) $z = a+ib.$

(i) $4i-z = 4i-a-ib$

$= -a+4i-ib$

$\therefore \quad \mathrm{Im}(4i-z) = 4-b.$

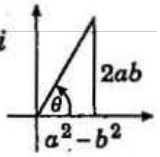
(ii) $3iz = 3ia-3b$

$= -3b+3ai$

$\overline{(3iz)} = -3b-3ai.$

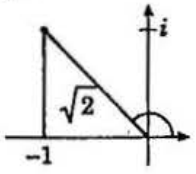
(iii) $z^2 = a^2-b^2+2abi$

$\tan\theta = \frac{2ab}{a^2-b^2}.$



(b) (i) $-1+i$

$= \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right).$

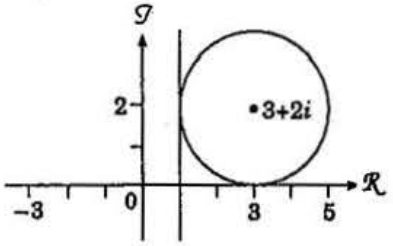


(ii) $(-1+i)^n = \left(\sqrt{2}\right)^n\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)^n$

$= 2^{\frac{n}{2}}\left(\cos\frac{3n\pi}{4}+i\sin\frac{3n\pi}{4}\right).$

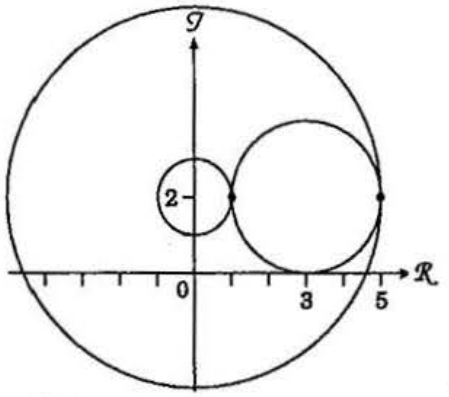
(c) (i) (α) $|z-(3+2i)| = 2$ is a circle, centre $3+2i$, radius 2.

(β) $|z+3| = |z-5|$ is the perpendicular bisector of the interval joining -3 and 5, that is, the line $x = 1$.



(ii) The line is a tangent to the circle. Hence the only point of intersection is $z = 1+2i$.

(iii)



$|z-2i| = k$ represents a circle, centre $2i$, radius k. Hence the equations

$|z-(3+2i)| = 2$ and $|z-2i| = k$

have exactly one solution for z when $k = 1, 5$.

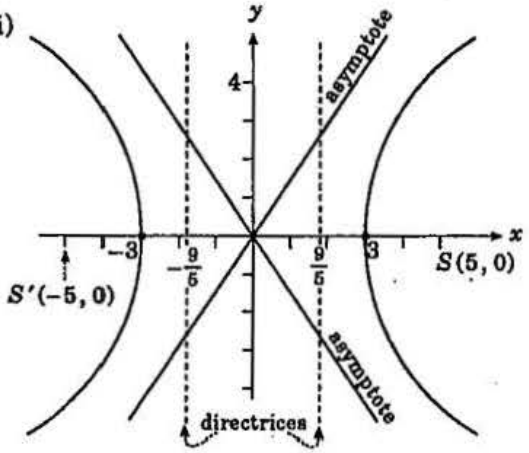
QUESTION THREE

(a) (i) $16x^2-9y^2 = 144$

$\frac{x^2}{9}-\frac{y^2}{16} = 1.$

Hence the asymptotes are $y = \pm\frac{4}{3}x.$

(ii)



foci S, S'

(iii) $x = 3\sec\theta,\ y = 4\tan\theta$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta$$

$$\left(\text{OR } 3(\cos\theta)^{-2}\sin\theta = 3\sin\theta\sec^2\theta\right)$$

$$\frac{dy}{d\theta} = 4\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{4}{3\sin\theta}.$$

OR

Using implicit differentiation,

$$32x - 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{16x}{9y}$$

$$= \frac{16\times 3\sec\theta}{9\times 4\tan\theta}$$

$$= \frac{4}{3\sin\theta}.$$

(iv) Eqn. of tangent at P is:

$$y - 4\tan\theta = \frac{4}{3\sin\theta}(x - 3\sec\theta)$$

$$(3\sin\theta)y - 12\tan\theta\sin\theta = 4x - 12\sec\theta$$

$$(3\sin\theta)y - 12\sin^2\theta\sec\theta = 4x - 12\sec\theta$$

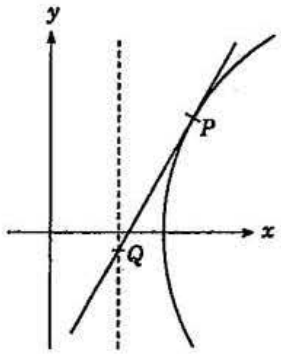
$$4x = (3\sin\theta)y + 12\sec\theta(1-\sin^2\theta)$$

$$= (3\sin\theta)y + 12\sec\theta\cos^2\theta.$$

That is, eqn. of tangent is

$$4x = (3\sin\theta)y + 12\cos\theta.$$

(v)



For $0 < \theta < \frac{\pi}{2}$, P is in the first quadrant, so the nearer directrix is $x = \frac{9}{5}$.

Substituting: $4\times\frac{9}{5} = (3\sin\theta)y + 12\cos\theta$

$$y = \frac{\frac{36}{5} - 12\cos\theta}{3\sin\theta}$$

$$= \frac{12 - 20\cos\theta}{5\sin\theta}.$$

(vi) $$m_{SP} = \frac{4\tan\theta - 0}{3\sec\theta - 5}$$

$$= \frac{4\sin\theta}{3 - 5\cos\theta}.$$

$$m_{SQ} = \frac{\frac{12 - 20\cos\theta}{5\sin\theta} - 0}{\frac{9}{5} - 5}$$

$$= \frac{12 - 20\cos\theta}{9\sin\theta - 25\sin\theta}$$

$$= \frac{12 - 20\cos\theta}{-16\sin\theta}$$

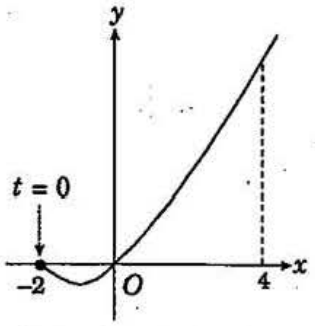
$$= \frac{3 - 5\cos\theta}{-4\sin\theta}.$$

(vii) $$m_{SP}\cdot m_{SQ} = \left(\frac{4\sin\theta}{3-5\cos\theta}\right)\left(\frac{3-5\cos\theta}{-4\sin\theta}\right)$$

$$= -1.$$

$\therefore \angle PSQ$ is a right angle.

(b) (i)



Point O, $x = 0$, $y = 0$ at O,

$\therefore\ t^2 + t - 2 = 0$ and $t^3 - t = 0$

$(t+2)(t-1) = 0$, $\quad t(t^2-1) = 0$

$t = 1, -2$, $\quad \therefore\ t = 0, t = \pm 1$,

that is, for both to be satisfied, $t = 1$.

OR

Since $t \geq 0$,

$t = -2, -1$ can be disregarded.

Point E, $x = 4$, $y > 0$

$$t^2 + t - 2 = 4$$

$$t^2 + t - 6 = 0$$

$$(t+3)(t-2) = 0$$

$$\therefore \quad t = 2, -3.$$

But $t \geq 0$, $\quad \therefore\ t \neq -3$,

$\therefore\ t = 2$.

(ii) $V = 2\pi\int_0^4 x f(x)\,dx$, where $f(x) = y$, $x = t^2 + t - 2$, $\frac{dx}{dt} = 2t + 1$.

$$= 2\pi\int_1^2 (t^2+t-2)(t^3-t)(2t+1)\,dt.$$

QUESTION FOUR

(a) $P(z) = z^3+3z+2i$

$= (z-\alpha)^2(z-\beta)$

$P'(z) = 3z^2+3$

$P'(\alpha) = 0$ since α is a double root of $P(z)$,

$\therefore\ 3(\alpha^2+1) = 0$

$\therefore\ \alpha = \pm i$

$P(i) = i^3+3i+2i$

$= -i+5i \neq 0$

$P(-i) = (-i)^3+3(-i)+2i \qquad (-i^3 = i)$

$= i-3i+2i$

$= 0.$

$\therefore\ (z+i)$ is a factor and is the double root,
that is, $(z+i)^2$ is a factor,
that is, $z^2+2iz-1$ is a factor.

$$z^2+2iz-1 \overline{)\ z^3+3z+2i\ } \quad z-2i$$

$$z^3+2iz^2-z$$

$$-2iz^2+4z+2i$$

$$-2iz^2+4z+2i$$

$\therefore\ \alpha = -i$

$\beta = 2i.$

OR

The product of the roots is $-2i$.
But the roots are $\alpha,\ \alpha,\ \beta$,
$\therefore\ \alpha^2\beta = -2i.$
The sum of the roots is zero,

$\therefore\ 2\alpha+\beta = 0$

so $\beta = -2\alpha \qquad$ —①

$\therefore\ \alpha^2(-2\alpha) = -2i$

$-2\alpha^3 = -2i$

$\alpha^3 = i. \qquad$ —②

Also $\alpha\beta+\alpha\beta+\alpha^2 = 3$

$\therefore\ 2\alpha\beta+\alpha^2 = 3.$

From ①:

$2\alpha(-2\alpha)+\alpha^2 = 3$

$\therefore\ \alpha^2 = -1 \qquad$ —③

$$\alpha = \frac{\alpha^3}{\alpha^2}$$

$$= \frac{i}{-1} \qquad \text{(from ②, ③)}$$

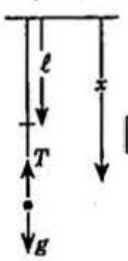
$\therefore\ \alpha = -i.$

Hence $\beta = -2\alpha$

$= 2i.$

Note Results ①, ③, ② can also be derived by equating coefficients of z^2, z and the constant term respectively.

(b)



(i) When $x \le \ell$, the only force acting is gravity,

$\therefore\ M\ddot{x} = Mg$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g.$$

When $x > \ell$ the forces acting are gravity (downwards) and the tension (upwards) in the cable proportional to $(x-\ell)$,

$\therefore\ M\ddot{x} = Mg - Mgk(x-\ell)$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x-\ell).$$

(ii) When $x \le \ell$, $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = g,$

$\therefore\ \dfrac{1}{2}v^2 = gx + C.$

When $x = 0,\ v = 0,\ \therefore\ C = 0.$

$\therefore\ \dfrac{1}{2}v^2 = gx$ or $v^2 = 2gx.$

So when $x = \ell,\ v^2 = 2g\ell.$

(iii) Note that the equation

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x-\ell)$$

holds for $x \ge \ell$, since when $x = \ell$, the second term on the right-hand side is zero. Integrating this equation with respect to x gives

$$\frac{1}{2}v^2 = gx - gk\left(\frac{1}{2}x^2 - \ell x\right) + C$$

Now when $x = \ell,\ v^2 = 2g\ell,$

$\therefore\ g\ell = g\ell - gk\left(\dfrac{1}{2}\ell^2 - \ell^2\right) + C$

$\therefore\ C = -\dfrac{1}{2}gk\ell^2.$

Hence $\dfrac{1}{2}v^2 = gx - gk\left(\dfrac{1}{2}x^2 - \ell x\right) - \dfrac{1}{2}gk\ell^2$

$$v^2 = 2gx - gk\left(x^2 - 2\ell x + \ell^2\right)$$

$$= 2gx - gk(x-\ell)^2.$$

(iv) When $v = 0$,

$$2gx - gk(x-\ell)^2 = 0$$

that is, $2x - kx^2 + 2k\ell x - k\ell^2 = 0$

$$kx^2 - 2x(1+k\ell) + k\ell^2 = 0$$

$$x = \frac{2(1+k\ell) \pm \sqrt{4(1+k\ell)^2 - 4k^2\ell^2}}{2k}$$

$$= \frac{2}{2k} + \frac{2k\ell}{2k} \pm \frac{2}{2k}\sqrt{1+2k\ell+k^2\ell^2-k^2\ell^2}$$

$$= \frac{1}{k} + \ell \pm \sqrt{\frac{1}{k^2}+\frac{2k\ell}{k^2}}$$

$$= \ell + \frac{1}{k} \pm \sqrt{\frac{2\ell}{k}+\frac{1}{k^2}}.$$

$\ell + \frac{1}{k}$

$+$

Alex goes past equilibrium, stops, then rises again so that the first stop, $x > \ell + \frac{1}{k}$, that is, first stop at $x = \ell + \frac{1}{k} + \sqrt{\frac{2\ell}{k}+\frac{1}{k^2}}$.

(v) If $\frac{1}{k} = \frac{\ell}{4} \rightarrow k = \frac{4}{\ell}$

$$x = \ell + \frac{\ell}{4} + \sqrt{\frac{2\ell^2}{4}+\frac{\ell^2}{16}}$$

$$= \ell + \frac{\ell}{4} + \ell \cdot \frac{3}{4}$$

$$= 2\ell.$$

That is, first stop for $x = 2\ell$,
that is, O needs to be at least 2ℓ metres above the stopping place (ground).

QUESTION FIVE

(a) (i)
$$\begin{array}{r} -x-4 \\ -x+5 \overline{)\ x^2 - x - 2} \\ \underline{-x^2+5x} \\ 4x-2 \\ \underline{-4x+20} \\ 18 \end{array}$$

$$\Rightarrow \quad x^2 - x - 2 = (5-x)(-x-4)+18$$

$$\therefore \quad f(x) = \frac{(x-2)(x+1)}{5-x}, \quad x \neq 5$$

$$= \frac{x^2-x-2}{5-x}$$

$$= -x-4+\frac{18}{5-x}$$

OR

$$-x-4+\frac{18}{5-x} = \frac{(5-x)(-x-4)+18}{5-x}, \quad x \neq 5$$

$$= \frac{x^2-x-2}{5-x}$$

$$= \frac{(x-2)(x+1)}{5-x}$$

$$= f(x).$$

(ii) As $x \rightarrow \infty$, $\frac{18}{5-x} \rightarrow 0$ from negative side.

As $x \rightarrow -\infty$, $\frac{18}{5-x} \rightarrow 0$ from positive side.

That is, as $x \rightarrow \pm\infty$, the term $\frac{18}{5-x}$ can be neglected, and $f(x) \rightarrow -x-4$.

(iii) Critical values to consider are 2, –1, 5.

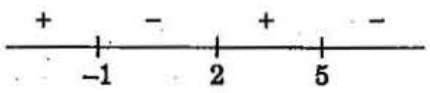
Consider $\frac{(x-2)(x+1)}{5-x}$

$x < -1$, $\frac{-\times-}{+} = +$

$-1 < x < 2$, $\frac{-\times+}{+} = -$

$2 < x < 5$, $\frac{+\times+}{+} = +$

$x > 5$, $\frac{+\times+}{-} = -$



$\therefore$ positive when $x < -1$, $2 < x < 5$,
negative when $-1 < x < 2$, $x > 5$.

(iv) $$f(x) = -x-4+\frac{18}{5-x}$$

$$\therefore \quad f'(x) = -1+\frac{18}{(5-x)^2}.$$

$f'(x) = 0$ when $\frac{18}{(5-x)^2} = 1$,

that is, when $(5-x)^2 = 18$

that is, $(x-5)^2 = 18$

that is, when $x - 5 = \pm\sqrt{18} = \pm 3\sqrt{2}$.

So $f'(x) = 0$ when $x = 5+3\sqrt{2}$
and when $x = 5-3\sqrt{2}$.

These are the stationary points.

(v) As $x \rightarrow 5^-$, $f(x) \rightarrow +\infty$
As $x \rightarrow 5^+$, $f(x) \rightarrow -\infty$
} from part (ii).

When $y = 0$, $x = 2, -1$.

When $x = 0$, $y = -\frac{2}{5}$.

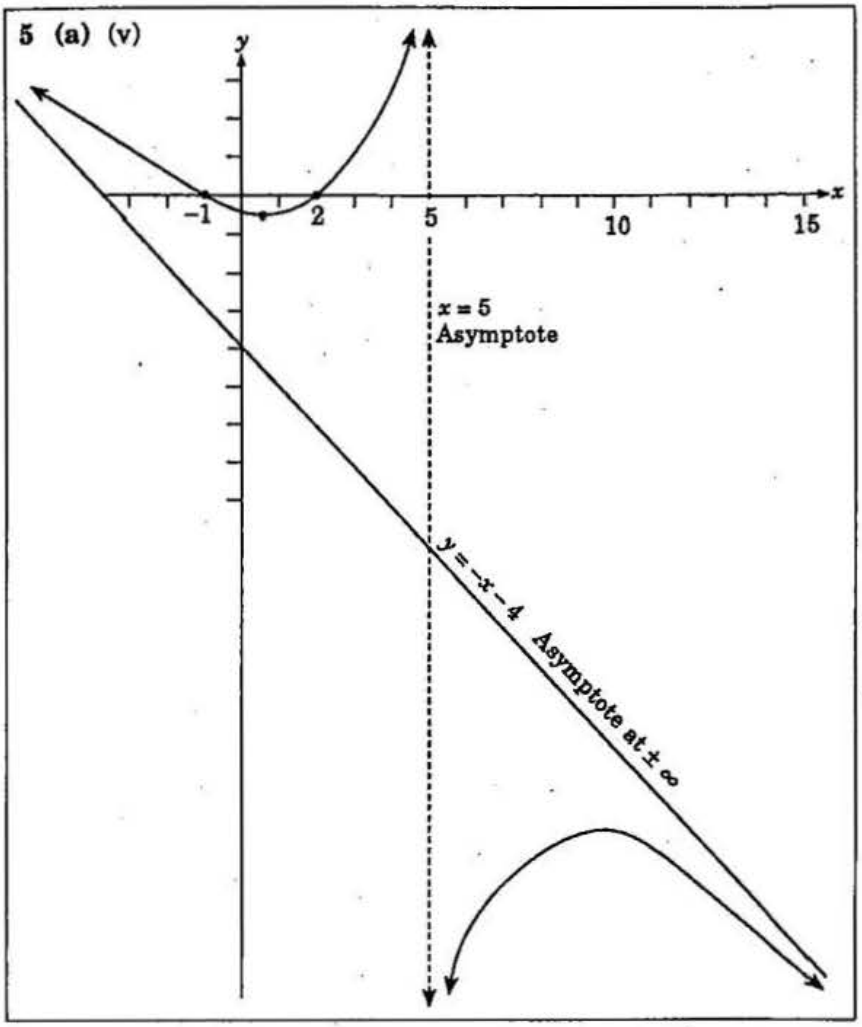
Note to diagram on page 11:

We can check that $f(x)$ never crosses $y = -x-4$ by attempting to solve

$-x-4 = -x-4+\frac{18}{5-x}$, that is $\frac{18}{5-x} = 0$.

This has no solution because the numerator is always positive.

5 (a) (v)



(iii) $r = \dfrac{w^2-w-2}{5-w}$

$= \dfrac{(w-2)(w+1)}{5-w}$.

This is the function in part (a). w and r must both be positive integers.

From the graph, the only possible values are

$w = 3$, gives $r = 2$.

$w = 4$, gives $r = 10$.

$\therefore$ There could be 2 red and 3 white, OR 10 red and 4 white jellybeans.

QUESTION SIX

(a) (i) $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \dfrac{\pi}{2}$.

Now e^x is positive and monotonic increasing for all x.

$\therefore$ for $0 < x < \dfrac{\pi}{2}$, $e^{\sin x} > e^{\frac{2x}{\pi}}$

since $e^{\sin x}$ and $e^{\frac{2x}{\pi}}$ are both positive,

$\therefore \dfrac{1}{e^{\sin x}} < \dfrac{1}{e^{\frac{2x}{\pi}}}$ for $0 < x < \dfrac{\pi}{2}$,

that is, $e^{-\sin x} < e^{-\frac{2x}{\pi}}$.

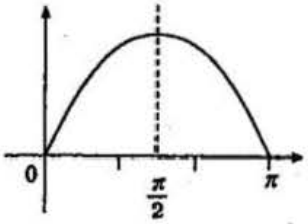
For $0 < x < \dfrac{\pi}{2}$, the graphs of $y = e^{-\sin x}$ and $y = e^{-\frac{2x}{\pi}}$ lie above the x axis,

$\therefore$ since $e^{-\sin x} < e^{-\frac{2x}{\pi}}$,

the area under the curve $y = e^{-\sin x}$ is less than the area under the curve $y = e^{-\frac{2x}{\pi}}$ for $0 < x < \dfrac{\pi}{2}$,

that is, $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}\,dx$.

(ii) The values of $\sin x$ are symmetric about $x = \dfrac{\pi}{2}$, so the graph of $e^{-\sin x}$ from $\pi/2$ to π is the reflection in $\pi/2$ of the graph from 0 to $\pi/2$. Thus the areas are the same.



(b) (i) P(white, white, white)

$= \dfrac{w}{w+r} \times \dfrac{w-1}{w+r-1} \times \dfrac{w-2}{w+r-2}$ —①

(ii) When there are $(w+1)$ white to begin with, P(white, white, white).

$= \dfrac{w+1}{w+r+1} \times \dfrac{w}{w+r} \times \dfrac{w-1}{w+r-1}$ —②

② is twice ①:

i.e. $\dfrac{w+1}{w+r+1} \times \dfrac{w}{w+r} \times \dfrac{w-1}{w+r-1}$

$= 2\dfrac{w}{w+r} \times \dfrac{w-1}{w+r-1} \times \dfrac{w-2}{w+r-2}$

$\therefore (w+1)(w+r-2) = 2(w-2)(w+r+1)$

$w^2 + wr - 2w + w + r - 2$

$= 2(w^2 + wr + w - 2w - 2r - 2)$

$= 2w^2 + 2wr + 2w - 4w - 4r - 4$

$w^2 + wr - w - 5r - 2 = 0$

$w^2 - w - 2 = 5r - rw$

$= r(5-w)$

$r = \dfrac{w^2-w-2}{5-w}$.

OR

Using the substitution $u = \pi - x$,

$$\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\,dx = -\int_{\frac{\pi}{2}}^{0} e^{-\sin(\pi-u)}\,du$$

$$= \int_0^{\frac{\pi}{2}} e^{-\sin u}\,du$$

$$= \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx.$$

(iii) So $\displaystyle\int_0^{\pi} e^{-\sin x}\,dx = 2\int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx$

$$< 2\int_0^{\frac{\pi}{2}} e^{\frac{-2x}{\pi}}\,dx$$

$$= 2\left[\frac{1}{\frac{-2}{\pi}} e^{\frac{-2x}{\pi}}\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\frac{\pi}{2}\left(e^{\frac{-2\pi}{2\pi}} - e^0\right)\right]$$

$$= -\pi\left(e^{-1} - 1\right)$$

$$= -\pi\left(\frac{1}{e} - 1\right)$$

$$= \pi(e-1).$$

(b) (i) $\angle BPC = \angle BAC$ (same segment)

$= \angle YAX$ (vert. opp.)

$= \alpha$.

(ii) $\angle BPC = \angle OPN$ (vert. opp.)

$= \angle OMN$ (same segment)

$= \alpha$.

(iii) In quadrilateral, $\angle YAX = \angle YMX = \alpha$.

But $\angle$s YAX, YMX both stand on the same side of interval XY.

By the converse of 'the angles in the same segment' theorem, $XYAM$ is a cyclic quadrilateral.

Note More details of this converse can be found in 1993 Question 6 solutions (p. 12).

(iv) $\angle XYM = \angle XAM$ (angles in same segment of circle, X, Y, A, M)

$\angle XAM + \angle PAB = 180°$ (XAB is a str. line)

But $\angle PCB + \angle PAB = 180°$ (opp. $\angle$s of cyclic quad.)

$\therefore \quad \angle XAM = \angle PCB.$

Hence $\angle XYM = \angle PCB.$

(v) $\angle XYM = \angle ZCO$ (same as $\angle BCP$)

Now as previously,

$\angle YOZ = \angle NOM$ (vert. opp.)

$\angle NOM = \angle NPM$ (same segment)

$\angle NPM = \angle APB$ (vert. opp.)

$\angle APB = \angle ACB$ (same segment)

$= \angle YCZ$ (same as $\angle ACB$).

$\therefore \angle YOZ = \angle YCZ$

$\therefore$ $YZCO$ is a cyclic quad.

Hence $\angle ZYC = \angle ZOC$

and $\angle CYO = \angle CZO$.

Therefore $\angle XYM + \angle ZYC + \angle CYO$

$= \angle ZCO + \angle ZOC + \angle CZO$

$= 180°$ (sum of the $\angle$s of a Δ is 180°).

Hence XYZ is a straight line.

QUESTION SEVEN

(a) (i) Let $f(x) = \sin x - x + \frac{x^3}{6}$

Then $f'(x) = \cos x - 1 + \frac{3x^2}{6}$

$= \cos x - 1 + \frac{1}{2}x^2$

$f''(x) = -\sin x + x$

$f'''(x) = -\cos x + 1.$

Now $f'''(x) \geq 0$ for $x > 0$ (since $\cos x \leq 1$ for all x);

and $f''(0) = 0$,

$\therefore$ by result quoted in question applied to $f''(x)$, $f''(x) \geq 0$ for $x > 0$;

also $f'(0) = 1 - 1 + 0 = 0.$

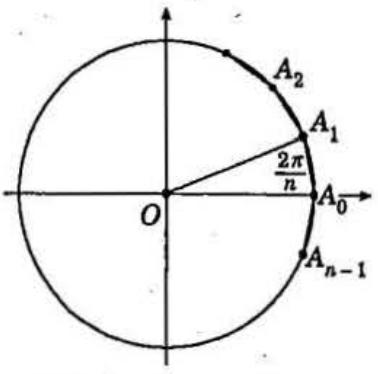
$\therefore$ By result quoted in question applied to $f'(x)$, $f'(x) \geq 0$ for $x > 0$;

and since $f(0) = 0$ by result quoted in question applied to $f(x)$,

$f(x) \geq 0$ for $x > 0$,

that is, $\sin x \geq 1 - \frac{x^2}{6}$ for $x > 0$.

(ii)



N.B. Circle of radius 1.

Let O be the origin. The size of the angle A_0OA_1 is $\frac{2\pi}{n}$.

Area of $\mathcal{P}$ = n × area of ΔA_0A_1O.

Using the formula $\frac{1}{2}ab\sin c$ for the area of ΔABC,

$$\text{area of } \Delta A_0A_1O = \frac{1}{2}\times 1\times 1\times \sin\frac{2\pi}{n}$$
$$= \frac{1}{2}\sin\frac{2\pi}{n}$$
$$\therefore \quad \text{area of } \mathcal{P} = \frac{n}{2}\sin\frac{2\pi}{n}.$$

(iii) $$\frac{\text{Area of } \mathcal{P}}{\text{Area of circle}} = \frac{\frac{n}{2}\sin\frac{2\pi}{n}}{\pi} = \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}.$$

We want $$\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} > 0\cdot 99$$

since $$\frac{\sin x}{x} \geq 1-\frac{x^2}{6}.$$

This will be true if

$$1-\frac{\left(\frac{2\pi}{n}\right)^2}{6} > 0\cdot 99$$

that is, if $$\frac{\left(\frac{2\pi}{n}\right)^2}{6} < 0\cdot 01$$

$$\left(\frac{2\pi}{n}\right)^2 < 0\cdot 06$$

$$n^2 > \frac{4\pi^2}{0\cdot 06}$$

that is, $$n > \frac{2\pi}{\sqrt{0\cdot 06}} = 25\cdot 65$$

$$\therefore \quad n \geq 26.$$

(b) (i) $$t_n + \frac{1}{2n} = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}+\frac{1}{2n}$$
$$= \left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}\right)+\frac{1}{n}$$
$$= \frac{1}{n}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}.$$

(ii) $$t_n + \frac{1}{2n} = \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}$$
= sum of upper rectangles drawn on the graph
$$> \int_n^{2n}\frac{dx}{x} \quad = \ln x\Big]_n^{2n}$$
$$= \ln 2n - \ln n = \ln\frac{2n}{n} = \ln 2.$$

(iii) Let $n = 1$

$$t_1 = \frac{1}{2},\ s_1 = 1-\frac{1}{2} = \frac{1}{2}.$$

Assume $s_k = t_k$.

Required to show $s_{k+1} = t_{k+1}$.

Now $$s_{k+1} = 1-\frac{1}{2}+\frac{1}{3}+\cdots-\frac{1}{2k}+\frac{1}{2(k+1)-1}-\frac{1}{2(k+1)}$$
$$= 1-\frac{1}{2}+\frac{1}{3}+\cdots-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$
$$= s_k+\frac{1}{2k+1}-\frac{1}{2k+2} \qquad \text{—①}$$

From part (i), for $n = k+1$

$$t_{k+1}+\frac{1}{2(k+1)}$$
$$= \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2(k+1)-1}$$
$$= \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}$$
$$= t_k+\frac{1}{2k+1}$$
$$\therefore \quad t_{k+1} = t_k+\frac{1}{2k+1}-\frac{1}{2k+2} \qquad \text{—②}$$

From ① and ②, if $s_k = t_k$, then $s_{k+1} = t_{k+1}$

∴ By the principle of mathematical induction, $s_n = t_n$ for $n = 1, 2, 3, \cdots$.

(iv) $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{9999}-\frac{1}{10\,000}$$
$$= s_{5000}$$
$$= t_{5000} \text{ by (iii)},$$

and $$\ln 2-\frac{1}{2n} < t_n < \ln 2 \qquad \text{[by (ii)]}$$

$$\therefore \quad \ln 2-\frac{1}{10\,000} < t_{5000} < \ln 2$$
$$\ln 2-0\cdot 0001 < t_{5000} < \ln 2$$
$$0\cdot 693\,147-0\cdot 0001 < t_{5000} < 0\cdot 693\,147$$

$\therefore\ t_{5000} = 0\cdot 693$ (to 3 dec. places).

QUESTION EIGHT

(a) (i) $$(a-b)^2 \geq 0$$
$$a^2-2ab+b^2 \geq 0$$
$$a^2+b^2 \geq 2ab.$$

(ii) From (i): $a^2+b^2 \geq 2ab$.

Similarly $b^2+c^2 \geq 2bc$,
$c^2+a^2 \geq 2ca$.

Adding, $2a^2+2b^2+2c^2 \geq 2ab+2bc+2ca$
$$a^2+b^2+c^2 \geq ab+bc+ca.$$

(iii) Since $a > 0,\ b > 0,\ c > 0,$
it follows that $(a+b+c) > 0.$

Also, by (ii), $(a^2+b^2+c^2-ab-bc-ca) \geq 0$
Hence
$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \geq 0$$
that is, $a^3+b^3+c^3-3abc \geq 0$
$$a^3+b^3+c^3 \geq 3abc. \quad \text{—(A)}$$

(iv) Let $a^3 = x,\ b^3 = y,\ c^3 = z.$
Then $a = x^{\frac{1}{3}},\ b = y^{\frac{1}{3}},\ c = z^{\frac{1}{3}}.$
Since $x > 0,\ y > 0,\ z > 0,$ we also have $a > 0,\ b > 0,\ c > 0,$ and so the inequality (A) holds.
Substituting in (A) gives:
$$x+y+z \geq 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$$
$$= 3(xyz)^{\frac{1}{3}}.$$

(v) $8 = (1+x)(1+y)(1+z)$
$= 1+(x+y+z)+(xy+yz+zx)+xyz.$
Now $x+y+z \geq 3(xyz)^{\frac{1}{3}},$
and we can apply this result to xy, yz and $zx,$ since $xy > 0,\ yz > 0,$ and $zx > 0.$
$\therefore\ xy+yz+zx \geq 3(x^2y^2z^2)^{\frac{1}{3}}.$
Hence $8 \geq 1+3(xyz)^{\frac{1}{3}}+3(xyz)^{\frac{2}{3}}+xyz.$
Let $u = xyz.$
Then $8 \geq 1+3u+3u^2+u^3$
$= (1+u)^3,$
$\therefore\ 2 \geq 1+u$
$\therefore\ u \leq 1$
that is, $xyz \leq 1.$

OR

Suppose $xyz > 1,$
then $x+y+z \geq 3(xyz)^{\frac{1}{3}} = 3,$
and $xy+yz+zx \geq 3(x^2y^2z^2)^{\frac{1}{3}} = 3,$
$\therefore\ (1+x)(1+y)(1+z)$
$= 1+x+y+z+xy+yz+zx+xyz$
$> 1+3+3+1 = 8.$
Hence if $(1+x)(1+y)(1+z) = 8,\ xyz \leq 1.$

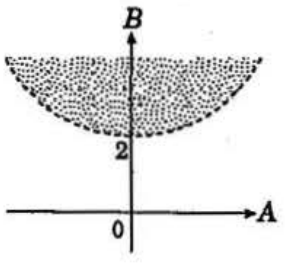
(b) (i) $P(x) = x^4+Ax^3+Bx^2+Ax+1,$
$P(0) = 1$
$\therefore\ \alpha \neq 0,$ by the factor theorem.

(ii) $Q(x) = x^2+\frac{1}{x^2}+A\left(x+\frac{1}{x}\right)+B$
$$= \frac{1}{x^2}\left\{x^4+1+Ax^2\left(x+\frac{1}{x}\right)+Bx^2\right\}$$
$$= \frac{1}{x^2}\left(x^4+1+Ax^3+Ax+Bx^2\right)$$
$$= \frac{P(x)}{x^2}.$$
Since $\alpha \neq 0,\ Q(\alpha) = \frac{P(\alpha)}{\alpha^2} = \frac{0}{\alpha^2} = 0,$
$\therefore\ x = \alpha$ is a root of $Q(x).$

(iii) $Q(x) = \left(x+\frac{1}{x}\right)^2+A\left(x+\frac{1}{x}\right)+B-2$
$= u^2+Au+(B-2)$
$= R(u).$

OR
$$R(u) = \left(x+\frac{1}{x}\right)^2+A\left(x+\frac{1}{x}\right)+B-2$$
$$= x^2+\frac{1}{x^2}+2+A\left(x+\frac{1}{x}\right)+B-2$$
$$= Q(x).$$

(iv) Now $u = x+\frac{1}{x}$
$\Rightarrow\ x^2-ux+1 = 0,$
which has no solutions in x if
$$u^2-4 < 0$$
$$-2 < u < 2.$$
If $R(u)$ has no real roots, then $Q(x)$ has no real roots, so $P(x)$ has no real roots.
$R(u)$ has no roots when
$$A^2-4(B-2) < 0$$
$$A^2-4B+8 < 0$$
$$B > \frac{1}{4}A^2+2.$$



$\therefore$ The curved segment c is $B > \frac{1}{4}A^2+2.$

However, it may also be possible for $R(u)$ to have real roots, and yet $P(x)$ and $Q(x)$ to have no real roots.

Suppose that β is a real root of $R(u)$, that is, $R(\beta) = 0$.

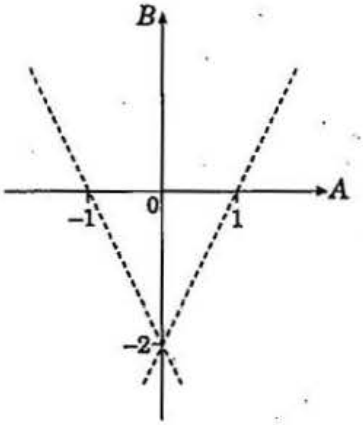
To find roots of $P(x)$, we have to solve $x + \frac{1}{x} = \beta$, that is, $x^2 - \beta x + 1 = 0$.

This has no real roots if $\beta^2 - 4 < 0$, that is, $\beta^2 < 4$, that is, $-2 < \beta < 2$.

The boundary of the region in which $P(x)$ has no real roots must be given by the end-points of this interval, that is, by $\beta = 2$ and $\beta = -2$.

When $\beta = 2$, $R(\beta) = 4 + 2A + B - 2 = 0$
that is, $B = -2A - 2$.

When $\beta = -2$, $R(\beta) = 4 - 2A + B - 2 = 0$
that is, $B = 2A - 2$.



The only one of these two lines which lies in the first quadrant of the A–B plane is $B = 2A - 2$.

Hence this identifies the line ℓ in the diagram.

Notes (1) The line ℓ and the curve c do not belong to the region $\mathcal{D}$, as the dashed lines indicate.

(2) This solution does not actually prove that the region in which $R(u)$ has real roots and $P(x)$ has no real roots lies above the line ℓ (for $A \geq 0$). This fact can be verified by considering the special case where $A = 0$ and $B = 0$.

Then $P(x)$ is $x^4 + 1$, which is known to have no real roots. (The roots are the complex fourth roots of -1.)

OR

For $P(x)$ to have no real roots, the roots of $R(u)$ must satisfy $|\beta| < 2$.

Now the roots of $R(u)$ are

$$\frac{1}{2}\left(-A \pm \sqrt{A^2 - 4(B-2)}\right).$$

Since $A > 0$, the root with the largest absolute value is

$$\frac{1}{2}\left(-A - \sqrt{A^2 - 4B + 8}\right).$$

Therefore, we must have

$$\left|\frac{1}{2}\left(-A - \sqrt{A^2 - 4B + 8}\right)\right| < 2,$$

that is, $A + \sqrt{A^2 - 4B + 8} < 4$

$$0 \leq \sqrt{A^2 - 4B + 8} < 4 - A$$

(Note that this means $A < 4$.)

$$\therefore \quad A^2 - 4B + 8 < (4 - A)^2$$
$$= 16 - 8A + A^2,$$

$$\therefore \quad -4B < 8 - 8A,$$

that is, $B > 2A - 2$.

T is the intersection of ℓ and c.

Hence T is given by $B = \frac{1}{4}A^2 + 2$
and $B = 2A - 2$,

that is, $\frac{1}{4}A^2 + 2 = 2A - 2$,

$$\frac{1}{4}A^2 - 2A + 4 = 0$$
$$A^2 - 8A + 16 = 0$$
$$(A-4)^2 = 0,$$

that is, $A = 4$ and $B = 8 - 2 = 6$.

So T is the point $(4, 6)$.

Note This is the only solution to the simultaneous equations, indicating that the line $B = 2A - 2$ is a tangent to the curve $B = \frac{1}{4}A^2 + 2$.