1993 HIGHER SCHOOL CERTIFICATE SOLUTIONS

4 UNIT MATHEMATICS

QUESTION ONE

(a) $\displaystyle\int_3^8 \frac{x\,dx}{(x+1)-\sqrt{x+1}}$

$x+1=u^2$, $dx=2u\,du$, $x=3,\ u=2$, $x=8,\ u=3$

$$=\int_2^3 \frac{(u^2-1)2u\,du}{u^2-u}$$

$$=\int_2^3 \frac{2(u^2-1)\,du}{u-1}$$

$$=\int_2^3 2(u+1)\,du$$

$$=\left[(u+1)^2\right]_2^3$$

$$=7.$$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$

$t=\tan\frac{\theta}{2}$, $dt=\frac{1}{2}\sec^2\frac{\theta}{2}d\theta$, $\theta=0,\ t=0$, $\theta=\frac{\pi}{2},\ t=1$, $d\theta=2\cos^2\frac{\theta}{2}dt=\frac{2}{1+t^2}\,dt$

$$=\int_0^1 \frac{\frac{2}{1+t^2}\,dt}{2+\frac{1-t^2}{1+t^2}}$$

$$=\int_0^1 \frac{2\,dt}{2(1+t^2)+1-t^2}$$

$$=\int_0^1 \frac{2\,dt}{3+t^2}$$

$$=\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}}-\tan^{-1}0\right)$$

$$=\frac{2}{\sqrt{3}}\cdot\frac{\pi}{6}$$

$$=\frac{\pi}{3\sqrt{3}}.$$

(c) $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1}x\,dx$

$$=\left[x\sin^{-1}x\right]_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}} x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\left(\frac{1}{2}\sin^{-1}\frac{1}{2}-\frac{1}{2}\sin^{-1}0\right)$$

$$+\frac{1}{2}\int_0^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}\,d(1-x^2)$$

$$=\frac{1}{2}\cdot\frac{\pi}{6}+\left[(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$$

$$=\frac{\pi}{12}+\left(\sqrt{\frac{3}{4}}-1\right)$$

$$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1.$$

OR Let $u=\sin^{-1}x$, $x=\sin u$, $dx=\cos u\,du$; $x=0,\ u=0$, $x=\frac{1}{2},\ u=\frac{\pi}{6}$

$$\int_0^{\frac{\pi}{6}} u\cos u\,du$$

$$=\left[u\sin u\right]_0^{\frac{\pi}{6}}-\int_0^{\frac{\pi}{6}}\sin u\,du$$

$$=\frac{\pi}{6}\sin\frac{\pi}{6}+\left[\cos u\right]_0^{\frac{\pi}{6}}$$

$$=\frac{\pi}{6}\cdot\frac{1}{2}+\cos\frac{\pi}{6}-\cos 0$$

$$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1.$$

(d) (i) $\dfrac{4x+3}{(x^2+1)(x+2)}=\dfrac{ax+b}{x^2+1}+\dfrac{c}{x+2}$

$\therefore\ 4x+3=(ax+b)(x+2)+c(x^2+1)$.

Let $x=-2$: $-5=5c$ $\therefore\ c=-1$

Equate coeffs. of x^2: $0=a-1$ $\therefore\ a=1$

OR $x=0$: $2b-1=3$, $b=2$

Equate constants:

$3 = 2b - 1$ $\quad x = 1$

$b = 2$ $\quad (a+2)(3) - (2) = 7$

$\therefore\ a = 1,\ b = 2,\ c = -1.$ $\quad 3(a+2) = 9$

$a + 2 = 3$

$a = 1$

(ii) $\displaystyle\int \frac{4x+3}{(x^2+1)(x+2)}\,dx$

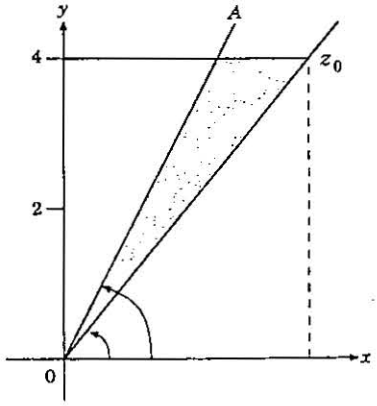
$$= \int \left(\frac{x+2}{x^2+1} - \frac{1}{x+2} \right) dx$$

$$= \int \left(\frac{x}{x^2+1} + \frac{2}{x^2+1} - \frac{1}{x+2} \right) dx$$

$$= \frac{1}{2}\log\left| x^2+1 \right| + 2\tan^{-1}x - \log\left| x+2 \right| + C.$$

QUESTION TWO

(a) (i)



(ii) $z_0 = 4\sqrt{3} + 4i$ has maximum modulus.

(b) (i) De Moivre's Theorem states:

$\left(\cos\theta + i\sin\theta\right)^3 = \cos 3\theta + i\sin 3\theta.$

Binomial Theorem expansion gives:

$$\left(\cos\theta + i\sin\theta\right)^3$$

$$= \cos^3\theta + 3\cos^2\theta.i\sin\theta + 3\cos\theta.i^2\sin^2\theta + i^3\sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + 3\cos^2\theta\sin\theta i - \sin^3\theta.i.$$

Equating real parts:

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$

(ii) Equating imaginary parts:

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$

OR

(b) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta.$

$\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta.$

(c) $\displaystyle \mathrm{Re}\left(\frac{z-4}{z}\right) = 0$

Now $\displaystyle \frac{z-4}{z} = \frac{x+iy-4}{x+iy}$

$$= \frac{x+iy-4}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x^2 - 4x + y^2 + 4yi}{x^2+y^2}$$

$\displaystyle \therefore \mathrm{Re}\left(\frac{z-4}{z}\right) = \frac{x^2-4x+y^2}{x^2+y^2} = 0$ [(0, 0) is not part of it.]

i.e. $x^2 - 4x + y^2 = 0,$

[i.e. $(x-2)^2 + y^2 = 4.$
The locus is a circle centre (2, 0) and radius 2, but excluding (0, 0).]

OR $\displaystyle \mathrm{Re}\left(1 - \frac{4}{z}\right) = 0$

$\displaystyle \therefore \quad \mathrm{Re}\left(\frac{4}{z}\right) = 1$

$\displaystyle \therefore \quad \mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{4} \quad (\text{so } z \neq 0)$

i.e. $\displaystyle \mathrm{Re}\left(\frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right) = \frac{1}{4}$

i.e. $\displaystyle \frac{x}{x^2+y^2} = \frac{1}{4} \quad [x \neq 0,\ y \neq 0]$

i.e. $x^2 - 4x + y^2 = 0,$ but not (0, 0).

OR

If the real part of $\displaystyle \frac{z-4}{z}$ is zero, then we may write $\displaystyle \frac{z-4}{z} = ki$, where $k \neq 0$ is a constant.

$\therefore\ z - 4 = kzi$ or $x + iy - 4 = ki(x+iy)$

$= kix - ky.$

Equating real and imaginary parts,

$x - 4 = -ky$ —①

$y = kx.$ —②

From ②: $\displaystyle k = \frac{y}{x}$ $\quad [k \neq 0 \text{ so } x \neq y \neq 0]$

Substitute into ①, giving $\displaystyle x - 4 = \frac{-y^2}{x}$,

i.e. $x^2 - 4x + y^2 = 0,$ except (0, 0).

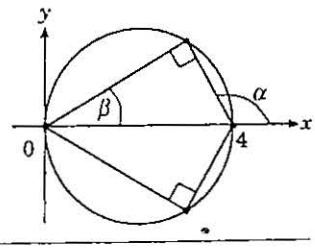
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If the real part of $\displaystyle \frac{z-4}{z} = 0$, then

$\displaystyle \arg\left(\frac{z-4}{z}\right) = \pm\frac{\pi}{2}$ and $z \neq 0,$

$\therefore\ \arg(z-4) - \arg z$

$= \pm\frac{\pi}{2}.$



If $\alpha = \arg(z-4)$ and $\beta = \arg z$,
then $\alpha - \beta = \pm\frac{\pi}{2}$ means that the locus is a circle,
with diameter 4, centre (0, 2),
i.e. $(x-2)^2 + y^2 = 4$, excluding (0, 0).
OR
If the real part of the complex number is zero, then the sum of the number and its conjugate is zero.

Hence $\frac{z-4}{z} + \frac{\bar{z}-4}{\bar{z}} = 0 \qquad [z \neq 0]$

$\therefore \quad \bar{z}(z-4) + z(\bar{z}-4) = 0$

i.e. $z\bar{z} - 4\bar{z} + z\bar{z} - 4z = 0$

i.e. $2(x^2+y^2) - 4(z+\bar{z}) = 0$

$\therefore \quad 2(x^2+y^2) - 4 \cdot 2x = 0$

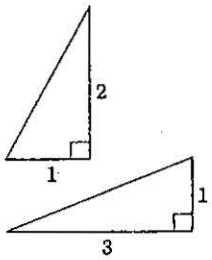
$\therefore \quad x^2 + y^2 - 4x = 0$, excluding (0, 0).

(d) Let $\arg z_1 = \tan^{-1} 2$

$\therefore \quad z_1 = 1 + 2i$

and let $\arg z_2 = \tan^{-1}\frac{1}{3}$

$\therefore \quad z_2 = 3 + i,$



then $\frac{z_1}{z_2} = \frac{1+2i}{3+i}$

$= \frac{1+2i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{5+5i}{10}$

$= \frac{1}{2} + \frac{i}{2},$

$\therefore \quad \arg\frac{z_1}{z_2} = \tan^{-1}\frac{\frac{1}{2}}{\frac{1}{2}}$

$= \tan^{-1} 1$

$= \frac{\pi}{4},$

$\therefore \arg z_1 - \arg z_2$

$= \arg(1+2i) - \arg(3+i)$

$= \tan^{-1} 2 - \tan^{-1}\frac{1}{3}$

$= \arg\frac{z_1}{z_2} = \frac{\pi}{4}.$

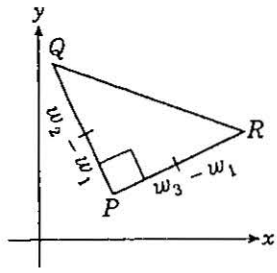
N.B. It was not sufficient to give the following:
Let $A = \tan^{-1} 2$ and $B = \tan^{-1}\frac{1}{3}$.
Then $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \frac{2 - \frac{1}{3}}{1 + 2 \times \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1,$

$\therefore \quad A - B = \frac{\pi}{4},$

and hence $\tan^{-1} 2 - \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$,
because this did not identify z_1 and z_2.

(e)



ΔPQR is right angled at $P(w_1)$ and is isosceles, $PQ = PR$. The reason is that multiplication of $(w_3 - w_1)$ by i rotates this length 90° anti-clockwise, and does not change the length, since:

$$\begin{aligned} |w_2 - w_1| &= |i(w_3 - w_1)| \\ &= |i| \times |w_3 - w_1| \\ &= |w_3 - w_1|. \end{aligned}$$

QUESTION THREE

(a) (i) $x = ct \qquad y = \frac{c}{t}$

$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = -\frac{c}{t^2}$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{t^2}.$

At P, $\frac{dy}{dx} = -\frac{1}{t^2}$,

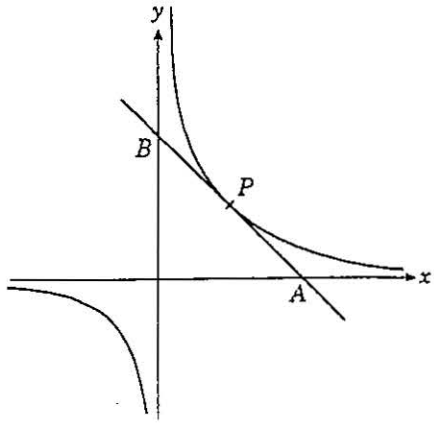
$\therefore$ Eqn. of tangent at P is

$$\left(y - \frac{c}{t}\right) = -\frac{1}{t^2}(x - ct)$$

$$t^2 y - ct = -x + ct$$

$$x + t^2 y = 2ct.$$

(ii)



At A, $y = 0$, $\therefore \quad x = 2ct.$

At B, $x = 0$, $\therefore \quad t^2 y = 2ct$

$$y = \frac{2c}{t}.$$

$$\therefore \quad PA^2 = (ct-2ct)^2 + \left(\frac{c}{t}-0\right)^2$$

$$= c^2t^2 + \frac{c^2}{t^2}.$$

$$PB^2 = (ct-0)^2 + \left(\frac{c}{t}-\frac{2c}{t}\right)^2$$

$$= c^2t^2 + \frac{c^2}{t^2},$$

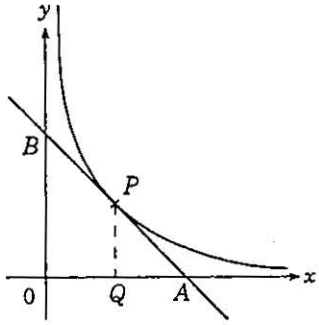
$$\therefore \quad PA = PB.$$

OR

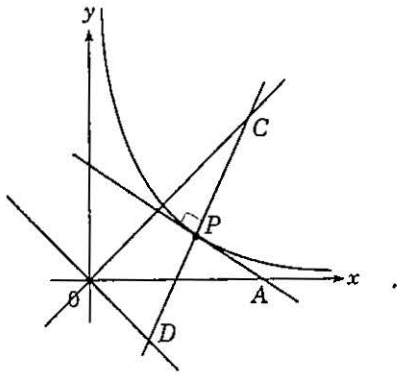
The abscissae of B, P, A are $O, ct, 2ct$,

$\therefore \quad AQ = QO,$
so $\quad AP = PB.$

[Equal Intercept Theorem]



(iii)



Eqn. of PC is $t^3x - ty = c(t^4-1)$ [given]
C lies on $y = x$,

$$\therefore \quad t^3x - tx = c(t^4-1)$$

$$x = \frac{c(t^4-1)}{t(t^2-1)}$$

$$= \frac{c(t^2-1)}{t}$$

$$\therefore \quad c = \left(\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right).$$

D lies on $y = -x$,

$$\therefore \quad t^3x + tx = c(t^4-1)$$

$$x = \frac{c(t^2-1)}{t}$$

$$\therefore \quad D = \left(\frac{c(t^2-1)}{t}, \frac{c(t^2-1)}{t}\right),$$

$$\therefore \quad PC^2 = \left[ct - \frac{c(t^2+1)}{t}\right]^2 + \left[\frac{c}{t} - \frac{c(t^2+1)}{t}\right]^2$$

$$= \left[\frac{ct^2 - ct^2 - c)}{t}\right]^2 + \left[\frac{c - ct^2 - c)}{t}\right]^2$$

$$= \frac{c^2}{t^2} + c^2t^2.$$

$$PD^2 = \left[ct - \frac{c(t^2-1)}{t}\right]^2 + \left[\frac{c}{t} - \frac{c(t^2-1)}{t}\right]^2$$

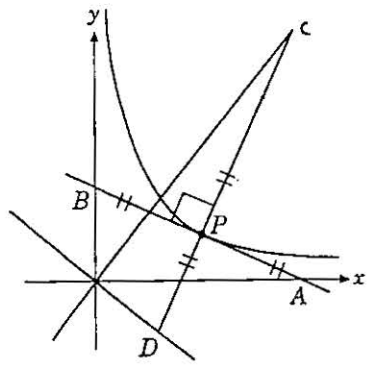
$$= \left[\frac{ct^2 - ct^2 + c)}{t}\right]^2 + \left[\frac{c - ct^2 + c)}{t}\right]^2$$

$$= \frac{c^2}{t^2} + c^2t^2,$$

$$\therefore \quad PA^2 = c^2t^2 + \frac{c^2}{t^2},$$

$$\therefore \quad PC = PD = PA.$$

(iv)



(v) The diagonals AB, CD are equal and bisect each other at right angles.
Hence $ABCD$ is a square.

(b) (i) $d^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2$.

Since $R(x_0, y_0)$ and $P(x_1, y_1)$ both lie on the circle,

$$x_0^2 + y_0^2 + 2gx_0 + 2fy_0 + c = 0 \qquad \text{—①}$$

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \qquad \text{—②}$$

Adding and rearranging ① and ②:

$$x_0^2 + x_1^2 + y_0^2 + y_1^2$$
$$= -2g(x_0 + x_1) - 2f(y_0 + y_1) - 2c,$$

$$\therefore \quad (x_0 - x_1)^2 + (y_0 - y_1)^2$$
$$= -2g(x_0 + x_1) - 2f(y_0 + y_1) - 2c$$
$$- 2x_0x_1 - 2y_0y_1,$$

i.e. $\quad d^2 = -2g(x_0 + x_1) - 2f(y_0 + y_1)$
$$- 2c - 2x_0x_1 - 2y_0y_1$$

$$\therefore \quad \frac{-d^2}{2} = x_0x_1 + y_0y_1 + g(x_0 + x_1)$$
$$+ f(y_0 + y_1) + c.$$

(ii) It follows from (i) that $P(x_1, y_1)$ lies on the line

$$\frac{-d^2}{2} = x_0x + y_0y + g(x_0 + x) + f(y_0 + y) + c. \qquad —③$$

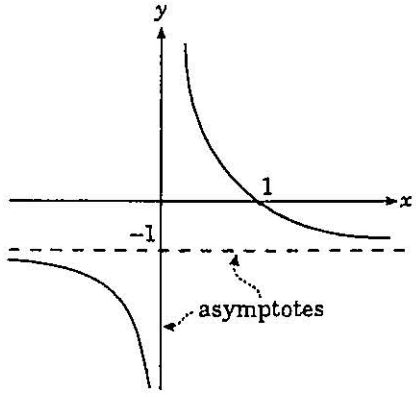
If Q is also distant d from R, then

$$\frac{-d^2}{2} = x_0x_2 + y_0y_2 + g(x_0 + x_2) + f(y_0 + y_2) + c,$$

and hence $Q(x_2, y_2)$ also lies on the line ③,
$\therefore$ ③ is the eqn. of the chord PQ.

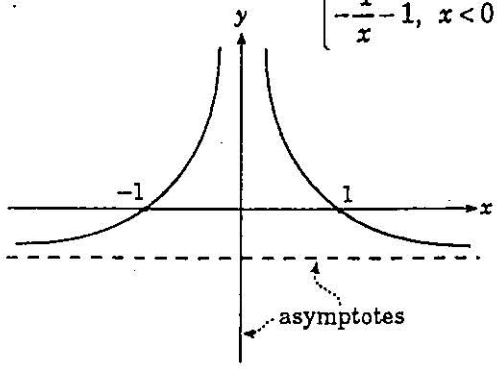
QUESTION FOUR

(a) (i) $f(x) = \dfrac{1-x}{x} = \dfrac{1}{x} - 1.$



(ii) $$y = \frac{1-|x|}{|x|} = \begin{cases} \dfrac{1-x}{x}, & x > 0 \\ \dfrac{1+x}{-x}, & x < 0 \end{cases}$$

$$= \begin{cases} \dfrac{1}{x} - 1, & x > 0 \\ -\dfrac{1}{x} - 1, & x < 0 \end{cases}$$



(iii) From the graph in (i).

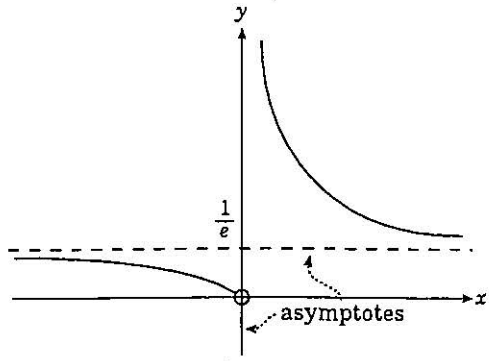
As $x \to 0^+$, $y \to e^{\infty} = \infty$

$x \to 0^-$, $y \to e^{-\infty} = 0$

$x \to \infty$, $y \to e^{-1} = \dfrac{1}{e}$

$x \to -\infty$, $y \to e^{-1} = \dfrac{1}{e}$

[OR $y = e^{\frac{1}{x}-1} = \dfrac{1}{e}e^{\frac{1}{x}}$ from which limits can be determined.]



(iv) $y^2 = f(x)$

$y = \pm\sqrt{f(x)}$. The function is symmetrical about the x axis.
The function is defined only for $f(x) \geq 0$, i.e. $0 < x \leq 1$.

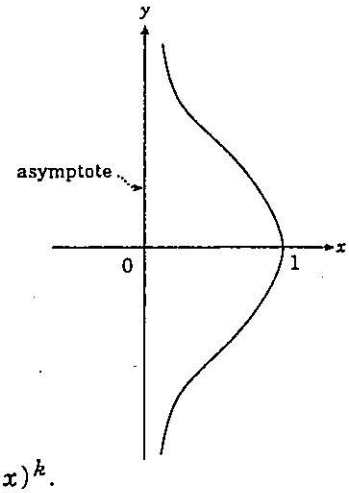
Now $y^2 = \dfrac{1}{x} - 1.$

Differentiating: $2y\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

$$\frac{dy}{dx} = -\frac{1}{2x^2y}$$

At $(1, 0)$, $\dfrac{dy}{dx}$ is undefined.

There is a vertical tangent at $(1, 0)$.



(b) (i) Let $u = (\tan x)^k$.

Then $\left\{\tan\left(\dfrac{\pi}{2} - x\right)\right\}^k$ $\qquad 0 \leq x \leq \dfrac{\pi}{2}$

$= \{\cot x\}^k = \dfrac{1}{u}.$

Then $f(x) + f\left(\dfrac{\pi}{2} - x\right) = \dfrac{1}{1+u} + \dfrac{1}{1+\frac{1}{u}}$

$= \dfrac{1}{1+u} + \dfrac{1}{\frac{u+1}{u}}$

$= \dfrac{1}{1+u} + \dfrac{u}{1+u}$

$= 1$ for $0 \leq x \leq \dfrac{\pi}{2}$.

(ii) $f'(x)-f'\left(\frac{\pi}{2}-x\right)=0$

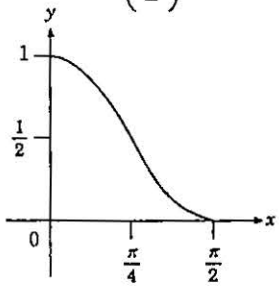
$\therefore\ f'(x)=f'\left(\frac{\pi}{2}-x\right).$

Assume $f'(0)=0$ $\quad\therefore\ f'\left(\frac{\pi}{2}\right)=0.$

Also $f(0)=1,$

and $f'\left(\frac{\pi}{2}\right)=0$ [given],

and $f\left(\frac{\pi}{4}\right)=\frac{1}{2}.$



(iii) Let $I=\int_0^{\frac{\pi}{2}} f(x)\,dx.$

If $u=\frac{\pi}{2}-x,$ $\quad\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)dx$

$=-\int_{\frac{\pi}{2}}^{0} f(u)\,du$

$=\int_0^{\frac{\pi}{2}} f(u)\,du$

$=I.$

Then, integrating the result in (i):

$$\int_0^{\frac{\pi}{2}} f(x)\,dx+\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)dx=\int_0^{\frac{\pi}{2}} 1\,dx,$$

$$I \quad + \quad I \quad =\frac{\pi}{2},$$

$$\therefore \quad I=\frac{\pi}{4}.$$

Alternative solution:

$I=\int_0^{\frac{\pi}{2}} f(x)\,dx$

$=\int_0^{\frac{\pi}{4}} f(x)\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)\,dx$

$=I_1+I_2.$

For I_2, let $x=\frac{\pi}{2}-u$ $\quad x=\frac{\pi}{4}\ \ u=\frac{\pi}{4}$

$dx=-du$ $\quad x=\frac{\pi}{2}\ \ u=0$

$\therefore\quad I_2=-\int_{\frac{\pi}{4}}^{0} f\left(\frac{\pi}{2}-u\right)du$

$=\int_0^{\frac{\pi}{4}} f\left(\frac{\pi}{2}-u\right)du$

$=\int_0^{\frac{\pi}{4}} f\left(\frac{\pi}{2}-x\right)dx.$

$\therefore\ I=I_1+I_2$

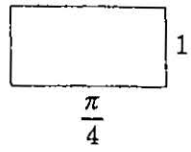
$=\int_0^{\frac{\pi}{4}} f(x)\,dx+\int_0^{\frac{\pi}{4}} f\left(\frac{\pi}{2}-x\right)dx$

$=\int_0^{\frac{\pi}{4}} \left\{f(x)+f\left(\frac{\pi}{2}-x\right)\right\}dx$

$=\int_0^{\frac{\pi}{4}} 1\,.\,dx$

$=\frac{\pi}{4}.$

Note This is the area of a rectangle of length $\frac{\pi}{4}$ and width 1.



That is, $\frac{1}{2}$ of the interval $\left[0,\ \frac{\pi}{2}\right]$ and the sum $f(x)+f\left(\frac{\pi}{2}-x\right).$

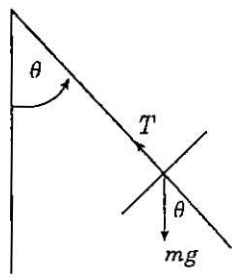
QUESTION FIVE

(a) (i) $s=l\theta$

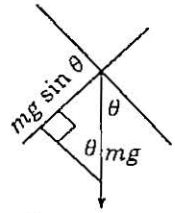
$\dot{s}=l\dot{\theta}$

$\ddot{s}=l\ddot{\theta}=l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right).$

(ii)



The only forces acting are the tension T and the weight mg of the particle.
Resolving forces tangentially:



$$ml\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)=-mg\sin\theta,$$

i.e. $\quad l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)=-g\sin\theta.$ $\quad$ —①

(iii) Integrating ① from $\theta = 0$, $\dot{\theta} = \sqrt{\frac{g}{l}}$ to θ, $\dot{\theta}$.

$$\int_{\sqrt{\frac{g}{l}}}^{\dot{\theta}} l\, d\left(\frac{1}{2}\dot{\theta}^2\right) = -\int_0^{\theta} g\sin\theta\, d\theta$$

$$l\left(\frac{1}{2}\dot{\theta}^2 - \frac{1}{2}\cdot\frac{g}{l}\right) = \left[g\cos\theta\right]_0^{\theta}$$

$$\frac{1}{2}l\dot{\theta}^2 - \frac{g}{2} = g\cos\theta - g$$

$$\frac{1}{2}l\dot{\theta}^2 = g\left(\cos\theta - \frac{1}{2}\right).$$

OR

$$l\left[\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)\right] = -g\sin\theta$$

$$\frac{1}{2}l\dot{\theta}^2 = g\cos\theta + k.$$

At $\theta = 0$, $\dot{\theta}^2 = \frac{g}{l}$

$$\therefore \quad \frac{1}{2}g = g + k$$

$$k = -\frac{1}{2}g$$

$$\frac{1}{2}l\dot{\theta}^2 = g\cos\theta - \frac{1}{2}g$$

$$= g\left(\cos\theta - \frac{1}{2}\right).$$

Pendulum reaches max. θ when $\dot{\theta} = 0$,

i.e. $g\left(\cos\theta - \frac{1}{2}\right) = 0$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}.$$

(iv) Let $f(\theta)$:

$$\frac{1}{2}l\dot{\theta}^2 = g\left(\cos\theta - \frac{1}{2}\right) - \frac{g}{10}\left(2\sin\theta - \theta\right)$$

$\theta = 0$, $\quad f(\theta) = \frac{1}{2}g$

$\theta = \frac{\pi}{6}$, $\quad f(\theta) = g\left(\frac{\sqrt{3}-1}{2}\right) - \frac{g}{10}\left(1 - \frac{\pi}{6}\right)$

$$= g\left(\frac{\sqrt{3}-1}{2}\right) - \frac{g}{10}\left(\frac{6-\pi}{6}\right)$$

$$\approx g.(0{\cdot}318).$$

$\theta = \frac{\pi}{4}$, $\quad f(\theta) = g\left\{\left(\frac{\sqrt{2}-1}{2}\right) - \frac{1}{10}\left(\frac{2}{\sqrt{2}} - \frac{\pi}{4}\right)\right\}$

$$= g\left\{\left(\frac{\sqrt{2}-1}{2}\right) - \frac{1}{10}\left(\frac{8-\pi\sqrt{2}}{4\sqrt{2}}\right)\right\}$$

$$= g\left\{\left(\frac{\sqrt{2}-1}{2}\right) - \frac{1}{10}\left(\frac{8\sqrt{2}-2\pi}{4}\right)\right\}$$

$$= g\left(\frac{\sqrt{2}-1}{2} - \frac{4\sqrt{2}-\pi}{20}\right)$$

$$\approx g.(0{\cdot}318).$$

$\dot{\theta}$ a max.

$\theta = \frac{\pi}{3}$, $\quad f(\theta) = -\frac{g}{10}\left(\sqrt{3} - \frac{\pi}{3}\right)$

$$\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$$

$$= \frac{\pi}{3} - \frac{\frac{-g}{10}\left(\sqrt{3} - \frac{\pi}{3}\right)}{-\frac{\sqrt{3}g}{2}}$$

$$= \frac{\pi}{3} - \frac{g}{10}\left(\sqrt{3} - \frac{\pi}{3}\right)\cdot\frac{2}{\sqrt{3}g}$$

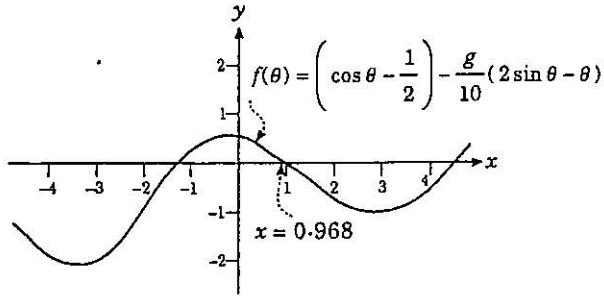
$$= \frac{\pi}{3} - \frac{1}{5\sqrt{3}}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{15}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} - \frac{1}{5} + \frac{\pi\sqrt{3}}{45}$$

$$= \frac{15\pi - 9 + \pi\sqrt{3}}{45}$$

$$\approx 0{\cdot}968.$$



Above is the graph of

$$f(\theta) = \left(\cos\theta - \frac{1}{2}\right) - \frac{g}{10}(2\sin\theta - \theta).$$

We see our final result of approximately 0·968 cutting the x axis near $x = 1$.

Observe though that $\frac{\pi}{3} > 1$ and that if

$$f\left(\frac{\pi}{3}\right) < 0,$$

$f(\theta)$ represents $\frac{1}{2}l\dot{\theta}^2$, which must be positive.

By choosing a physically meaningless first approximation, we solve the problem.

The approximation $x=\frac{\pi}{3}$ has meaning only in an abstract mathematical sense.

(b) (i) If a is a multiple root of the polynomial equation $P(x)=0$, then

$$P(x)=(x-a)^m Q(x)$$
$$P'(x)=m(x-a)^{m-1}Q(x)+(x-a)^m Q'(x)$$
$$\therefore \quad P'(a)=0.$$

(ii) Let $P(x)=18x^3+3x^2-28x+12$,

$$\therefore \quad P'(x)=54x^2+6x-28$$
$$=2(27x^2+3x-14)$$
$$=2(9x+7)(3x-2).$$

The roots of $P'(x)=0$ are

$$x=\frac{-7}{9},\ \frac{2}{3}.$$

Now $P\left(\frac{2}{3}\right)=0,\quad P\left(\frac{-7}{9}\right)\neq 0,$

$\therefore\ x=\frac{2}{3}$ is a double root of $P(x)=0$.

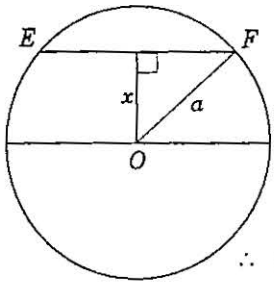
Let β be the third root.

$$\frac{2}{3}\times\frac{2}{3}\beta=-\frac{12}{18}$$
$$\beta=-\frac{2}{3}\cdot\frac{3}{2}\cdot\frac{3}{2}$$
$$=-\frac{3}{2},$$

$\therefore$ the three roots are $\frac{2}{3},\ \frac{2}{3},\ -\frac{3}{2}$.

QUESTION SIX

(a) (i)



$$EF=2\sqrt{a^2-x^2}$$
$$\therefore \text{ Area } EFG=\sqrt{a^2-x^2}\,.d$$

(ii) Volume of $S=\int_{-a}^{a}\sqrt{a^2-x^2}\ d\cdot dx$

$$=d\int_{-a}^{a}\sqrt{a^2-x^2}\ dx$$
$$=d\times\frac{1}{2}\pi a^2$$
$$=\frac{1}{2}\pi a^2 d.$$

since $\int_{-a}^{a}\sqrt{a^2-x^2}\ dx$ represents the area of a semicircle, radius a.

(b) (i) Diagram copied.

(ii) $\angle PBQ=\angle PBC-\angle QBC$

$$=\frac{1}{2}\angle ABC-\frac{1}{2}\angle DBC$$

[P, Q are incentres of Δs ABC, BCD]

$$=\frac{1}{2}\angle ABD.$$

(iii) $\angle PCQ=\angle BCQ-\angle BCP$

$$=\frac{1}{2}\angle BCD-\frac{1}{2}\angle BCA$$

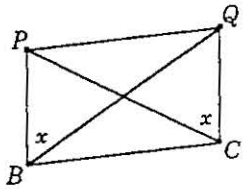
[P, Q are incentres of Δs ABC, BCD]

$$=\frac{1}{2}\angle ACD.$$

But $\angle$s ACD, ABD both stand on the same arc, AD,

$$\therefore \quad \angle ACD=\angle ABD$$
$$\therefore \quad \angle PBQ=\angle PCQ.$$

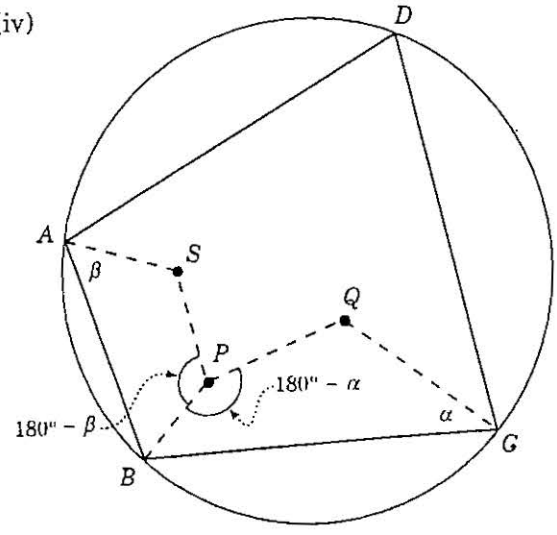
If an interval subtends equal angles at two points on the same side of it, then the end-points of the interval and the two points are concyclic. This is the converse of 'angles standing on the same arc of a circle are equal'. We have:



PQ as arc.
B and C must be on the circumference.

Now, the interval PQ subtends equal angles PBQ and PCQ, hence $PBCQ$ is a cyclic quadrilateral.

(iv)



Let $\angle BCQ=\alpha°$

$\therefore \quad \angle BPQ=180°-\alpha°$

[opp. $\angle$s of cyclic quad.].

Also, similarly to the proof in part (iii), $ASPB$ is a cyclic quadrilateral.

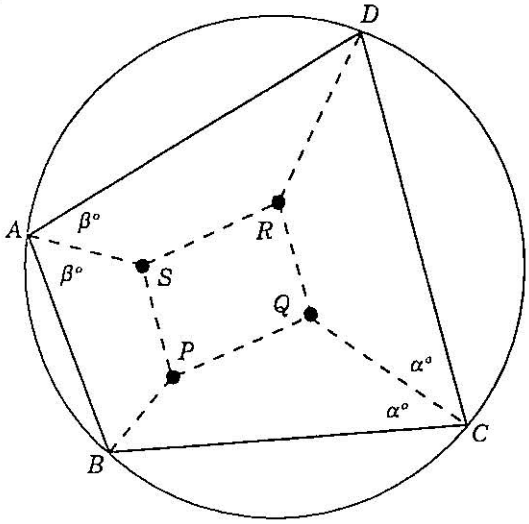
Let $\angle BAS = \beta°$

$\therefore \quad \angle BPS = 180° - \beta°$
[opp. $\angle$s of cyclic quad.].

$\therefore \angle SPQ + (180° - \beta°) + (180° - \alpha)$
$= 360°$ [$\angle$s at a point]

$\therefore \quad \angle SPQ = \alpha° + \beta°$
$= \angle BAS + \angle BCQ.$

(v)



$\angle QCD = \angle QCB = \alpha°$,
[Q is incentre of ΔBCD].

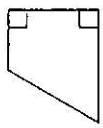
$\angle SAD = \angle BAS = \beta°$,
[S is incentre of ΔBAD].

$\therefore$ Similarly to (iv), $\angle SRQ = \alpha° + \beta°$.

Now $2\alpha° + 2\beta° = 180°$,
[opp. angles of cyclic quad. $ABCD$]

$\therefore \quad \alpha + \beta = 90°,$

$\therefore \angle SPQ = \angle SRQ = 90°.$



Note This is *not* sufficient.

Similarly $\angle PSR = \angle PQR = 90°$,

$\therefore$ $PQRS$ is a rectangle.

QUESTION SEVEN

(a) (i) There are 5 ways to choose three consecutive days: 123; 234; 345; 456; 567. For two independent choices there are 25 ways.

For day 1 to be common, both Esther and George must choose (123).

Hence probability is $\frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$.

(ii) $E(123)$ $G(234)$
or $E(234)$ $G(123)$
or $E(234)$ $G(234)$,

hence probability is $\frac{3}{25}$.

(iii) For *no* days off together:

	E	G
	(123)	(456)
or	(123)	(567)
	(234)	(567)
	(456)	(123)
	(567)	(123)
	(567)	(234)

$\therefore$ They have at least one day off together in 19 cases.

Probability is $\frac{19}{25}$.

(b) A proof that e is irrational.

(i) If $n = 1$, $\text{LHS} = e - s_1 = e - 2$

$$\text{RHS} = e\int_0^1 xe^{-x}\,dx$$

$$= e\int_0^1 x\,d(e^{-x})$$

$$= -e\left[xe^{-x}\right]_0^1 + e\int_0^1 xe^{-x}\,dx \quad \text{[by parts]}$$

$$= \frac{-e}{e} - e\left[e^{-x}\right]_0^1$$

$$= -1 - 1 + e$$

$$= e - 2$$

$$= \text{LHS},$$

$\therefore$ the result is true for $n = 1$.

Assume the result is true for $n = k$.

Then $e\int_0^1 \frac{x^{k+1}}{(k+1)!}e^{-x}\,dx$

$$= -\frac{e}{(k+1)!}\int_0^1 x^{k+1}\,d(e^{-x})$$

$$= -\frac{e}{(k+1)!}\left[x^{k+1}e^{-x}\right]_0^1 + \frac{e}{(k+1)!}\int_0^1 e^{-x}(k+1)x^k\,dx \quad \text{[by parts]}$$

$$= -\frac{e}{(k+1)!}\cdot\frac{1}{e} + \frac{e}{k!}\int_0^1 e^{-x}x^k\,dx$$

$$= -\frac{1}{(k+1)!} + e\int_0^1 \frac{x^k}{k!}e^{-x}\,dx$$

$$= -\frac{1}{(k+1)!} + e - s_k \quad \left[s_{k+1} = s_k + \frac{1}{(k+1)!}\right]$$

$$= e - s_{k+1}.$$

Hence if the result is true for $n = k$, it is also true for $n = k + 1$. Now the result is true for $n = 1$, that is, it is true for $n = 2$, 3, ...

Hence it is true for all positive integral n.

(ii) Since $x^n \geq 0$ for $x \geq 0$ and $e^{-x} > 0$,

$$e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx > 0.$$

Also $e\int_0^1 \frac{x^n}{n!}e^{-x}\,dx$

$$< 3\int_0^1 \frac{x^n}{n!}\,dx \quad \begin{matrix}[\text{since } e^{-x} \leq 1 \\ \text{and } e < 3]\end{matrix}$$

$$= \frac{3}{(n+1)!} \quad \text{since } \int_0^1 x^n\,dx = \frac{1}{n+1},$$

$$\therefore\ 0 < e - s_n < \frac{3}{(n+1)!}.$$

(iii) $(e-s_n)n! < \frac{3}{n+1} \leq 1$ for $n = 2, 3, \ldots,$

$\therefore (e-s_n)n!$ is not an integer for $n = 2, 3, \ldots.$

(iv) Assume $e = \frac{p}{q}$ where p, q are positive integers.

Choose $n \geq q$.

Then $(e-s_n)n! = \left(\frac{p}{q} - 1 - \sum_{r=1}^{n}\frac{1}{r!}\right)n!$

$$= \frac{pn!}{q} - n! - \left(\sum_{r=1}^{n}\frac{1}{r!}\right)n!,$$

which is an integer since q divides $n!$ ($\because n \geq q$) and $r!$ divides $n!$ for $r \leq n$. This is in contradiction to (ii),

$\therefore e \neq \frac{p}{q}$, where p and q are positive integers.

QUESTION EIGHT

(a) (i) If $w_1, \cdots, w_n$ are complex roots of the equation $z^n - 1 = 0$, then $w_1 + w_2 + \cdots + w_n = 0$ [sum of the roots and the coefficient of z^{n-1} is zero].

(ii) $|PA_i| = |z - w_i|$

$$PA_i^2 = (z-w_i)^2$$

$$= (z-w_i)\overline{(z-w_i)}, \text{ since } |z|^2 = z.\bar{z}$$

$$= (z-w_i)(\bar{z}-\bar{w}_i).$$

(iii) $\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n}(z-w_i)(\bar{z}-\bar{w}_i)$

$$= \sum_{i=1}^{n}(z\bar{z} - z\bar{w}_i - w_i\bar{z} + w_i\bar{w}_i)$$

$$= \sum_{i=1}^{n}(2 - z\bar{w}_i - w_i\bar{z}),$$

since $z\bar{z} = |z|^2 = 1;\ w\bar{w} = |w|^2 = 1$

$$= 2n - z\sum_{i=1}^{n}\bar{w}_i - \bar{z}\sum_{i=1}^{n}w_i$$

$$= 2n - z.0 - \bar{z}.0 \quad \text{[by (i)]}$$

$$= 2n.$$

(b) $f(x) = 1 + x^2,\ f'(x) = 2x$

$$x_{n+1} = x_n - \frac{1+x_n^2}{2x_n} \qquad \text{—①}$$

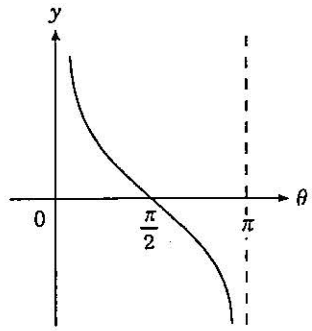
(i) Now $(1-x_n)^2 \geq 0$

$$\Rightarrow \quad 1 + x_n^2 \geq 2x_n$$

$$\frac{1+x_n^2}{2x_n} \geq 1$$

$$\therefore\ |x_{n+1} - x_n| = \left|\frac{1+x_n^2}{2x_n}\right|, \text{ from ①.}$$

(ii)



(iii) $\cot\theta$ takes all real values exactly once in the interval $0 < \theta < \pi$, hence there exists a real number θ_n for all real x_n.

(iv) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$\therefore \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$$

$$= \frac{\cot^2\theta - 1}{2\cot\theta} \quad \begin{matrix}[\text{dividing numerator} \\ \text{and denominator by } \tan^2\theta]\end{matrix}$$

Hence:

$$\cot\theta_{n+1} = x_{n+1}$$

$$= x_n - \frac{1+x_n^2}{2x_n} \quad \text{[by def'n. of } x_n\text{]}$$

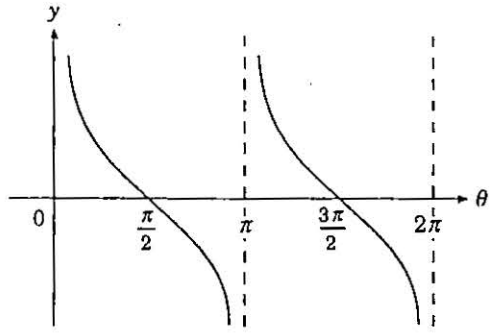
$$= \cot\theta_n - \frac{1+\cot^2\theta_n}{2\cot\theta_n}$$

$$= \frac{\cot^2\theta_n - 1}{2\cot\theta_n}$$

$$= \cot 2\theta_n.$$

(v) From (iv): $\cot\theta_2 = \cot 2\theta_1$

$\Rightarrow \theta_2 = 2\theta_1$ or $\theta_2 = 2\theta_1 - \pi$, since $0 < \theta < \pi$.



i.e. $\theta_2 = 2\theta_1 - m_1\pi \qquad [m_1 = 0, 1]$

$\therefore \quad \theta_3 = 2\theta_2$

$= 4\theta_1 - m_2\pi \qquad [m_2 = 0, 1, 2]$

...

$\therefore \quad \theta_{n+1} = 2^n\theta_1 - m\pi \quad [m = 0, 1, \ldots, n]$

If $x_1 = x_{n+1}$, then $\theta_1 = \theta_{n+1}$.

From (i): $\theta_1 = 2^n\theta_1 - m\pi$

$$\theta_1 = \frac{m\pi}{2^n - 1}.$$

Hence the required points are:

$$x_1 = \cot\theta_1 = \cot\frac{m\pi}{2^n - 1} \left[\begin{matrix} m,\ n \text{ integers,} \\ n \geq 1,\ 0 \leq m \leq n \end{matrix}\right]$$

Observe:

$m = 1,\ n = 2, \qquad \cot\theta_1 = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

$$x_1 = \frac{1}{\sqrt{3}}, \qquad x_{n+1} = x_n - \frac{1 + x_n^2}{2x_n}$$

$$x_2 = \frac{1}{\sqrt{3}} - \frac{1 + \frac{1}{3}}{2(\frac{1}{\sqrt{3}})}$$

$$= \frac{1}{\sqrt{3}} - \frac{\frac{4}{3}}{\frac{2}{\sqrt{3}}} = \frac{1}{\sqrt{3}} - \frac{4}{3} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

$$x_3 = -\frac{1}{\sqrt{3}} - \frac{1 + \frac{1}{3}}{2(-\frac{1}{\sqrt{3}})}$$

$$= -\frac{1}{\sqrt{3}} + \frac{\frac{4}{3}}{\frac{2}{\sqrt{3}}}$$

$$= -\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

For $n = 2$, $x_1 = x_3$.

COMMENTS ON THE 1993 4 UNIT PAPER

2. (d) Many students baulk at using the equation:

$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$ to prove

$\frac{\pi}{4} = \tan^{-1}2 - \tan^{-1}\frac{1}{3}$, and prefer to expand $\tan(\alpha - \beta)$.

It is a harder exercise to prove

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

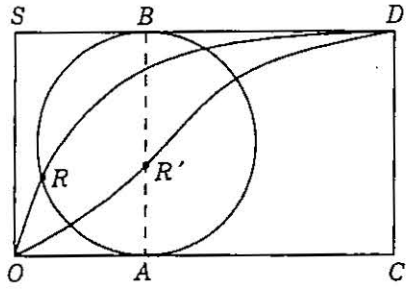
by two different methods.

3. (a) This result is due to Apollonius (circa 230 BC).

(b) This should be compared with the derivation of the chord of contact.

4. (a) The curve $y^2 = \frac{1-x}{x}$ is the celebrated 'Witch of Agnesi' (circa 1750). Contrary to what is stated in the stem of the question, it is not the graph of a function.

(b) Probably the first curve studied which exhibited this symmetry property is the 'companion of the cycloid'. This curve was defined by Roberval (circa 1634).



Let R be a reflector on the circumference of a wheel and suppose R is initially at the origin O. As the wheel rolls without slipping along the base line OC, R traces out a 'cycloid'. Now imagine the wheel after it has rolled from O to a general position with base point A so that the reflector has moved from O to R (where $OA = \text{arc } AR$). From R, draw a horizontal line to meet the vertical diameter AB at R'. As the wheel rolls along, completing half a revolution, R' traces out a curve called the companion of the cycloid. A little thought reveals that the companion curve bisects the rectangle $OCDS$. A little more thought reveals that the area between the cycloid and its companion is one half the area of the wheel. Remember Cavalieri's principle, which is particularly relevant when finding the volumes of solids by slicing.

An alternative way of generating examples is to let $f(x) = \dfrac{g(x)}{g(x)+g(a-x)}$.

Since we have $\displaystyle\int_0^a f(x)\,dx = \frac{a}{2}$, we can evaluate an infinite number of integrals in one swoop, for example, let $g(x) = x^2$ with $a = 2$, $g(x) = (\sin x)^k$ with $a = \dfrac{\pi}{2}$, and so on.

5. **(a)** Equation (iv) comes from assuming that there is a damping force proportional to $\left(\dfrac{ds}{dt}\right)^2$. If this is the case, equation (ii) becomes $\dfrac{d^2s}{dt^2} = -g\sin\theta - k\left(\dfrac{ds}{dt}\right)^2$ (*)

Rather than solve this equation directly, approximate $\left(\dfrac{ds}{dt}\right)^2$ by the v^2 obtained from the solution of (ii), that is,

$$v^2 = l^2\dot{\theta}^2 - 2lg\left(\cos\theta - \frac{1}{2}\right).$$

The equation (*) becomes

$$l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = -g\sin\theta - \frac{g}{5}\left(\cos\theta - \frac{1}{2}\right),$$

where $k = \dfrac{1}{10l}$. The solution of this equation is (iv). Of course students do not need to realize all this to answer (iv).

6. **(a)** The solid S is an example of a 'conoid'. New solids can be generated in a similar way, for example, replace l by an appropriate square.

(b) This result appeared in 1992 in *Mathematical Spectrum*, a British magazine for students and teachers of mathematics in schools, colleges, and universities. It is due to J.H. Littlewood, who proved more. Draw the incircles of triangles ABC and BCD. (They have P and Q as centres.) Then the other common tangent to these circles is parallel to AD.
(If you are keen: suppose that the common tangent intersects BC produced at E. Then E, P and Q are collinear. Now once more use the fact that $BCQP$ is cyclic.)

7. **(b)** This is essentially Euler's proof that e is irrational. In 1992, Question 8(a), candidates were essentially asked to prove that

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

and this year that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

The series expansion converges too quickly for e to be rational. (The former limit expression can be traced back to Bernoulli's problem (circa 1685) about continuous compound interest: if we compound interest over sufficiently short periods, can we earn 'pots of money' in interest from a given principal over a given time period?)

8. **(a)** The geometric interpretation of the result is this: Let A_1, A_2, ..., A_n be equally spaced points on the unit circle and let P be a variable point on the circle. Then $\displaystyle\sum_{k=1}^{n} PA_k^2$ is a constant. A similar argument reveals that $\displaystyle\sum_{k=1}^{n} PA_k^4$ is constant.

If we let P be a point on OA_1 such that $OP = x$, then we obtain

$$\prod_{k=1}^{n} PA_k = 1 - x^n,$$

that is, we have a geometric factorization of $1-x^n$ due to Cotes (circa 1720).

(b) Newton's method applied to finding a zero for $1+x^2$ also provides a classical example of chaotic behaviour. Suppose $x_1 = \cot\theta$ is our initial approximation. Then our $(n+1)$th iterative point is $x_{n+1} = \cot 2^n\theta$. Think what effect the multiplication 2^n has on θ. What happens after twelve iterations when we incorrectly read $x_1 = 0{\cdot}501$ as $x_1 = 0{\cdot}5$?