1992 HIGHER SCHOOL CERTIFICATE SOLUTIONS

4 UNIT MATHEMATICS

QUESTION ONE

(a)(i) $\displaystyle\int \tan\theta\sec^2\theta\,d\theta$

$$= \int \tan\theta\, d(\tan\theta)$$

$$= \frac{1}{2}\tan^2\theta + C$$

(ii) $\displaystyle\int \frac{2x+6}{x^2+6x+1}\,dx$

$$= \log\left|x^2+6x+1\right| + C$$

(b) $\displaystyle\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{dx}{\sqrt{(x-1)(3-x)}}$ Let $u = x-2$

$x = \frac{3}{2}, \quad u = -\frac{1}{2}$

$x = \frac{5}{2}, \quad u = \frac{1}{2}$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{\sqrt{(u+1)(1-u)}}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$$

$$= \left[\sin^{-1} u\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}.$$

(c) Let $\dfrac{5(1-t)}{(t+1)(3-2t)} = \dfrac{a}{t+1} + \dfrac{b}{3-2t}$

$\therefore \quad 5(1-t) = a(3-2t) + b(1+t)$

Let $t = -1$: $\quad 10 = 5a$

$a = 2$

Let $t = \frac{3}{2}$: $\quad -\frac{5}{2} = 5\frac{b}{2}$

$b = -1$

$$\therefore \int_0^1 \frac{5(1-t)}{(t+1)(3-2t)}\,dt$$

$$= \int_0^1 \left\{\frac{2}{t+1} - \frac{1}{3-2t}\right\} dt$$

$$= \left[2\log(t+1) + \frac{1}{2}\log(3-2t)\right]_0^1$$

$$= 2\log 2 + \frac{1}{2}\log 1 - 2\log 1 - \frac{1}{2}\log 3$$

$$= 2\log 2 - \frac{1}{2}\log 3$$

$$= \log\frac{4}{\sqrt{3}}.$$

(d)(i) $\displaystyle\int x e^{x^2}\,dx$ Let $u = x^2$

$du = 2x\,dx$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2} + C$$

(ii) $\displaystyle\int_0^1 2x^3 e^{x^2}\,dx$

$$= \int_0^1 x^2 \frac{d}{dx}\left(e^{x^2}\right) dx$$

$$= \left[x^2 e^{x^2}\right]_0^1 - \int_0^1 2x.e^{x^2}\,dx$$

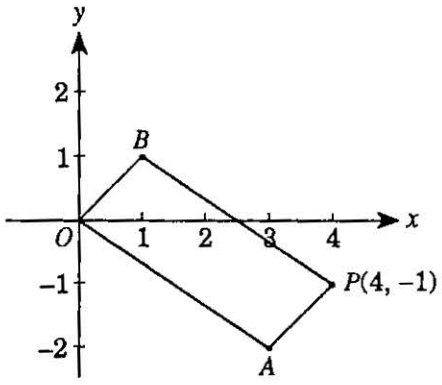
$$= \left[1.e^1 - 0.e^0\right] - 2\left[\frac{1}{2}e^{x^2}\right]_0^1 \quad \text{* from (i) above}$$

$$= e - \left(e^1 - e^0\right)$$

$$= 1.$$

QUESTION TWO

(a)(i)



(ii) P represents the complex number $4 - i$.

(b)(i) $z = a + ib$

$$\frac{1}{z} = \frac{1}{a+ib}$$

$$= \frac{a-ib}{(a+ib)(a-ib)}$$

$$= \frac{a-ib}{a^2+b^2}$$

$$\therefore \quad \text{Im}\left(\frac{1}{z}\right) = \frac{-b}{a^2+b^2}$$

If $\text{Im}(z) = b > 0$

then $\text{Im}\left(\frac{1}{z}\right) < 0$ since $a^2 + b^2 > 0$.

(ii) $$\left|\frac{1}{z}\right| = \left|\frac{a-ib}{a^2+b^2}\right|$$

$$= \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2}$$

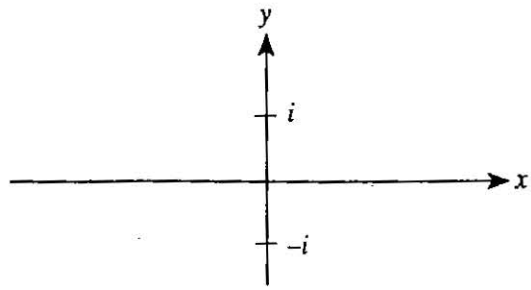
$$= \sqrt{\frac{a^2+b^2}{\left(a^2+b^2\right)^2}}$$

$$= \sqrt{\frac{1}{a^2+b^2}}$$

$$= \frac{1}{|z|}.$$

(c)(i) $|z - i| = |z + i|$

represents the locus of a point that is equidistant from the points i and $-i$, i.e. it represents the perpendicular bisector of the interval joining i and $-i$
The locus is the real axis.



OR $$|x + iy - i| = |x + iy + i|$$

$$|x + i(y-1)| = |x + i(y+1)|$$

$$\therefore \quad x^2 + (y-1)^2 = x^2 + (y+1)^2$$

$$x^2 + y^2 - 2y + 1 = x^2 + y^2 + 2y + 1$$

$$y = 0.$$

(ii) Let $z = x + iy$

$$|z - i| = \sqrt{2}|z + i|$$

$$|x + iy - i| = \sqrt{2}|x + iy + i|$$

$$x^2 + (y-1)^2 = 2\left\{x^2 + (y+1)^2\right\}$$

$$x^2 + y^2 - 2y + 1 = 2x^2 + 2y^2 + 4y + 2$$

$$x^2 + y^2 + 6y + 1 = 0$$

$$x^2 + (y+3)^2 = 8$$

which is a circle, centre $(0, -3)$, radius $2\sqrt{2}$.

(d)(i) If z_1 is a root of the polynomial $P(z) = 0$, then $\overline{z_1}$ is also a root.
$(1 + i)$ is a root,
$\therefore$ $\overline{1+i} = 1 - i$ is also a root.

(ii) $P(z) = 2z^2 - 3z^2 + rz + s$

$= \big(z-(1+i)\big)\big(z-(1-i)\big)(az+b)$

$= \big(z^2 - 2z + 2\big)(az+b)$

$= az^3 - (2a-b)z^2 + (2a-2b)z + 2b$

$\equiv 2z^2 - 3z^2 + rz + s$

$\therefore \quad a = 2, \quad 2a - b = 3$

$\therefore \quad b = 1$

$\therefore \quad r = 2a - 2b = 4 - 2 = 2$

and $\quad s = 2b = 2$

$\therefore P(z) = \big(z^2 - 2z + 2\big)(2z+1)$ over the reals.

Alternatively:

Roots of $P(z) = 0$ are $(1+i)$, $(1-i)$, α.

Sum roots $= (1+i) + (1-i) + \alpha$

$= 2 + \alpha \quad = -\dfrac{-3}{2}$

$\therefore \quad \alpha = -\dfrac{1}{2}.$

$\therefore \quad P(z) = k\big(z-(1+i)\big)\big(z-(1-i)\big)\left(z-\dfrac{-1}{2}\right)$

$= k'\big(z^2 - 2z + 2\big)(2z+1)$

$= \big(z^2 - 2z + z\big)(2z+1)$

as $k' = 1$ by inspection.

QUESTION THREE

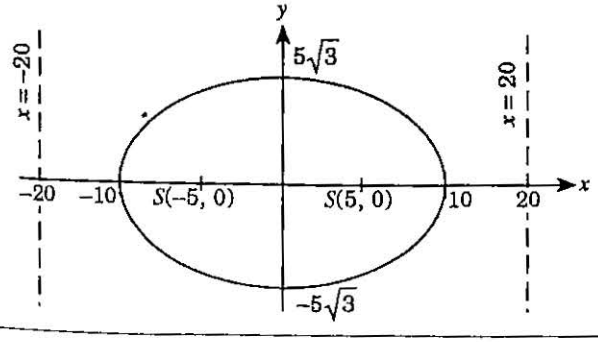
(a)(i) $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 1$

$\therefore \quad a = 10, \quad b = \sqrt{75} = 5\sqrt{3}$

and $\quad b^2 = a^2\big(1-e^2\big)$

$75 = 100\big(1-e^2\big)$

$e = \dfrac{1}{2}$



$\therefore$ Coordinates of foci are

$(\pm ae,\ 0) = (\pm 5,\ 0).$

Equations of directrices are

$x = \pm a / e = \pm 20.$

(ii) Differentiating w.r.t. x:

$\dfrac{2x}{100} + \dfrac{2y}{75}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x}{50}\cdot\dfrac{75}{2y} = -\dfrac{3x}{4y}$

At $P\big(5,\ 7{\cdot}5\big)$, $\dfrac{dy}{dx} = -\dfrac{15}{30} = -\dfrac{1}{2}$

$\therefore$ The gradient of the normal is 2.

Equation of normal at P is

$y - 7{\cdot}5 = 2(x-5)$

$y = 2x - 2{\cdot}5$

or $\quad 4x - 2y - 5 = 0 \qquad \ldots (1)$

(iii) By symmetry, the intersection of the normals at P and Q will lie on the x-axis. The centre of the required circle is the point of intersection of these normals.

Putting $y = 0$ in (1) gives $x = \dfrac{5}{4}$.

Let C be the point $\left(\dfrac{5}{4},\ 0\right)$.

$CP^2 = \left(5 - \dfrac{5}{4}\right)^2 + \big(7{\cdot}5 - 0\big)^2$

$= \left(\dfrac{15}{4}\right)^2 + \left(\dfrac{15}{2}\right)^2$

$= \dfrac{225}{16} + \dfrac{225}{4}$

$= \dfrac{1125}{16}$

$\therefore$ Equation of circle is

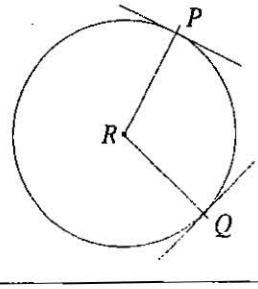
$\left(x - \dfrac{5}{4}\right)^2 + y^2 = \dfrac{1125}{16}$

Alternative solution to (iii):

$P(5,\ 7{\cdot}5)$; $Q(5,\ -7{\cdot}5)$

are on the ellipse ξ.

The normals meet at R, where R is the centre of the circle.



Normal at Q: $(y - -7{\cdot}5) = -2(x - 5)$ *

$y + 7{\cdot}5 = -2x + 10$

$y = -2x + 2{\cdot}5$

Normal at P: $y = 2x - 2{\cdot}5$

Solving: $R(1{\cdot}25,\ 0)$

* Normal at P has grad. 2.
$\therefore$ By symmetry normal at Q has grad. -2.

$$\text{Radius} = PR = \sqrt{(5 - 1{\cdot}25)^2 + (7{\cdot}5 - 0)^2}$$

$$= \sqrt{\left\{\left(3\frac{3}{4}\right)^2 + \left(7\frac{1}{2}\right)^2\right\}}$$

$$= \sqrt{\frac{225}{16} + \frac{225}{4}} = \frac{15\sqrt{5}}{4}.$$

Circle: $\left(x - 1\frac{1}{4}\right)^2 + y^2 = \left(\frac{15\sqrt{5}}{4}\right)^2$

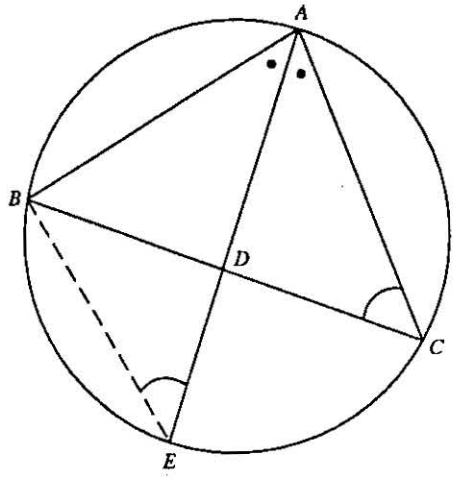
$$\frac{1}{16}(4x - 5)^2 + y^2 = \frac{1}{16} \times 1125$$

$$(4x - 5)^2 + 16y^2 = 1125.$$

$$16x^2 + 16y^2 - 40x = 1100$$

$$4x^2 + 4y^2 - 10x = 275.$$

(b)



(i) In Δs ABE, ADC

$\angle BAE = \angle DAC$ (given)

$\angle BEA = \angle ACD$ (angles at circumference standing on same arc)

$\therefore\ \Delta ABE \parallel\!\parallel \Delta ADC$ (two angles of one triangle respectively equal to two sides of the other)

(ii) Since $\Delta ABE \parallel\!\parallel \Delta ADC$

$$\frac{AB}{AD} = \frac{AE}{AC}$$ (corresponding sides of similar triangles)

$\therefore\ AB.AC = AD.AE$

(iii) $AD.DE = BD.DC$ (product of intercepts of intersecting chords)

$\therefore\ AB.AC - BD.DC$

$= AD.AE - BD.DC$ [from part (ii)]

$= AD.AE - AD.DE$

$= AD(AD + DE) - AD.DE$

$= AD^2$

QUESTION FOUR

(a)(i) TRUE

$\sin\theta$ is an odd function, therefore $\sin^7\theta$ is odd. The integral of an odd function between symmetrical limits is zero.

(ii) FALSE

$\sin^7\theta > 0$ for $0 < \theta < \pi$, $\therefore\ \int_0^{\pi} \sin^7\theta > 0$

(iii) FALSE

$e^{-x^2} > 0$ for all x, $\therefore\ \int_{-1}^{1} e^{-x^2}\,dx > 0$

(iv) TRUE $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$$\therefore \int_0^{\frac{\pi}{2}} \left(\sin^8\theta - \cos^8\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta - \int_0^{\frac{\pi}{2}} \sin^8\left(\frac{\pi}{2} - \theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta + \int_{\frac{\pi}{2}}^{0} \sin^8\phi\,d\phi, \quad \phi = \frac{\pi}{2} - \theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^8\theta\,d\theta - \int_0^{\frac{\pi}{2}} \sin^8\phi\,d\phi$$

$$= 0$$

(v) TRUE For $0 \le t \le 1$, $n = 1, 2, 3 \ldots$

$t^n \ge t^{n+1}$

$$\therefore\ \frac{1}{1 + t^n} \le \frac{1}{1 + t^{n+1}}$$

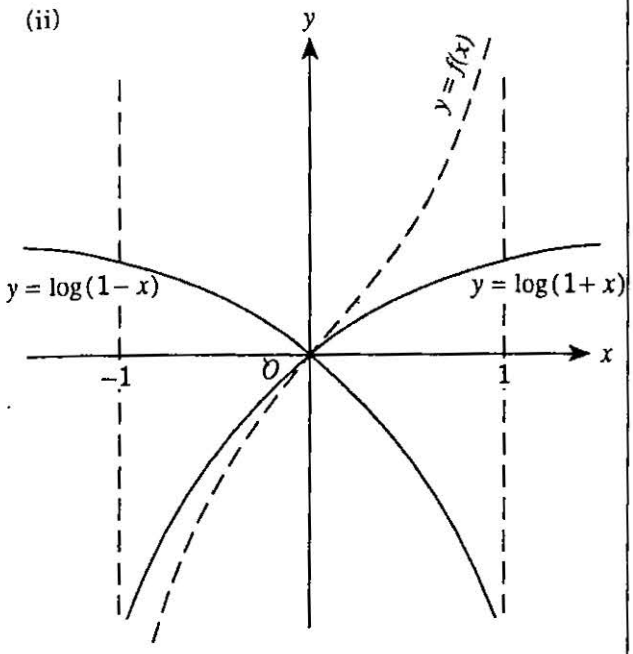
$$\therefore \int_0^1 \frac{dt}{1 + t^n} \le \int_0^1 \frac{dt}{1 + t^{n+1}}$$

(b)(i) $f(x) = \log(1+x) - \log(1-x), \quad -1 < x < 1$

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$$
$$= \frac{2}{1-x^2}$$
$$> 0$$

since $1-x^2 > 0$ for $-1 < x < 1$.

(ii)



N.B. $f'(x) > 0$ for $-1 < x < 1$

$f'(0) = 2$

(iii) $y = \log\dfrac{1+x}{1-x}$

The inverse function is $x = \log\dfrac{1+y}{1-y}$.

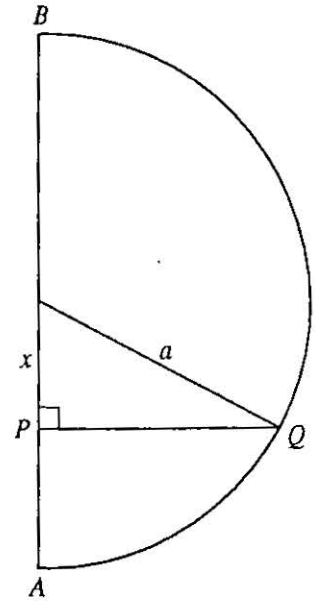
$$\frac{1+y}{1-y} = e^x$$
$$1+y = (1-y)e^x$$
$$= e^x - ye^x$$
$$y(e^x+1) = e^x - 1$$
$$\therefore \quad y = \frac{e^x-1}{e^x+1}$$

QUESTION FIVE

(a)(i) Consider the base of the solid T. Let PQ be the line of intersection of the required cross-section with this base.

By Pythagoras's Theorem

$$PQ^2 = a^2 - x^2$$



The cross-section is a right-angled isosceles triangle, and its area is

$$A_T(x) = \frac{1}{2}PQ.PQ = \frac{1}{2}\left(a^2 - x^2\right).$$

(ii)

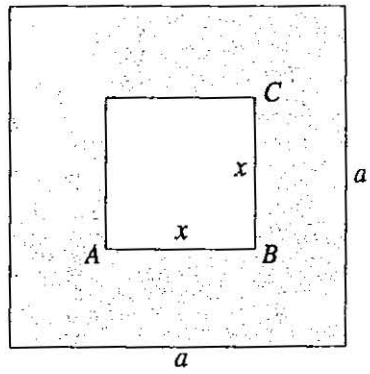


Fig. 1 Cross-section of S

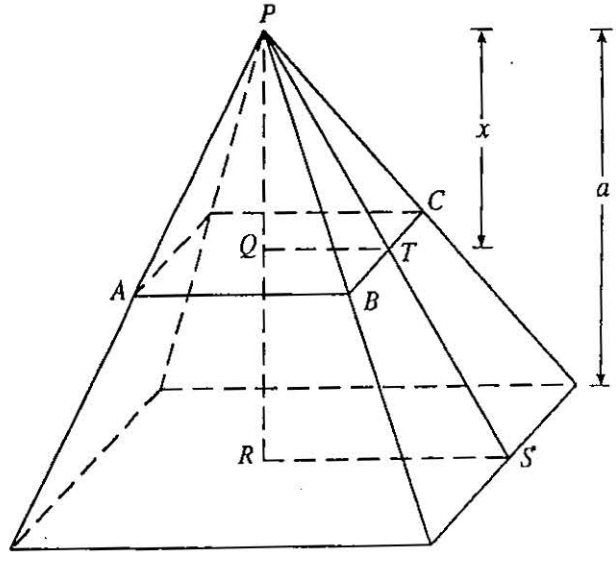


Fig. 2

From Fig. 2: $\frac{QT}{RS} = \frac{PQ}{PR} = \frac{x}{a}$

$$QT = RS.\frac{x}{a} = \frac{x}{2} \qquad \left(\text{since } RS = \frac{a}{2}\right)$$

$\therefore \quad AB = BC = x$

$\therefore$ Area of cross-section is

$$A_S(x) = a^2 - x^2.$$

(iii) $$V_S = 2\int_0^a A_S(x)\,dx$$

$$= 2\int_0^a \left(a^2 - x^2\right) dx$$

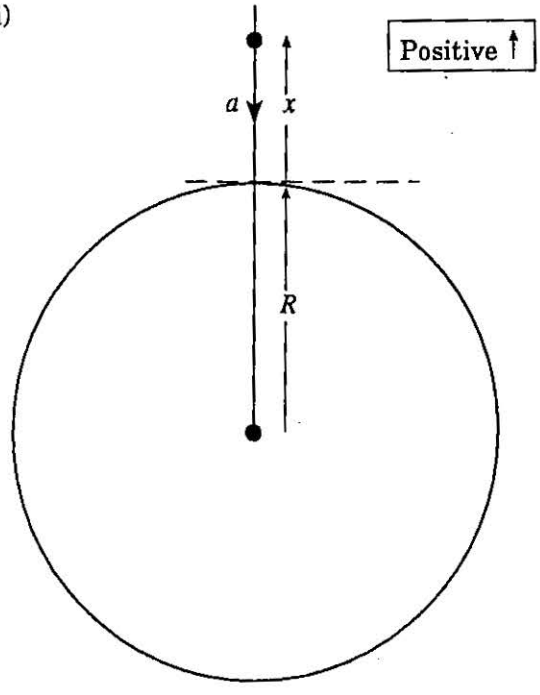
$$V_T = 2\int_0^a A_T(x)\,dx$$

$$= 2\int_0^a \frac{1}{2}\left(a^2 - x^2\right) dx$$

$$= \int_0^a \left(a^2 - x^2\right) dx$$

(iv) $V_S = 2V_T$

(b)(i)



Take the positive direction vertically upwards.

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx} \quad = \frac{dx}{dt}\frac{dv}{dx}$$

$$= \frac{dv}{dt} \quad = \frac{d^2x}{dt^2}$$

Since the acceleration (a) is directed towards the centre of the Earth,

$$a = -\frac{10}{\left(1+\frac{x}{R}\right)^2}$$

i.e. $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{-10}{\left(1+\frac{x}{R}\right)^2} \qquad \ldots (1)$$

(ii) Let h be the maximum height attained.

Then at $x = 0$, $v = 400$,

and at $x = h$, $v = 0$

$\therefore$ Integrating (1) w.r.t. x:

$$\int_{400}^{0} d\left(\frac{1}{2}v^2\right) = \int_0^h \frac{-10}{\left(1+\frac{x}{R}\right)^2} dx$$

$$\left[\frac{1}{2}v^2\right]_{400}^{0} = \int_0^h \frac{-10R^2}{(R+x)^2} dx$$

$$-\frac{1}{2}\times 400^2 = \left[\frac{10R^2}{R+x}\right]_0^h$$

$$-80000 = \frac{10R^2}{R+h} - \frac{10R^2}{R}$$

$$= \frac{10R^2}{R+h} - 10R$$

$$\therefore \quad \frac{10R^2}{R+h} = 10R - 80000$$

$$10R^2 = 10R^2 + 10Rh - 80000R - 80000h$$

$$h(10R - 80000) = 80000R$$

$$h = \frac{80000R}{10R - 80000}$$

$$\approx 8010$$

$\therefore$ The maximum height reached by the object is 8010 metres.

QUESTION SIX

(a)(i) Let s metres be the distance travelled by T in time t secs.

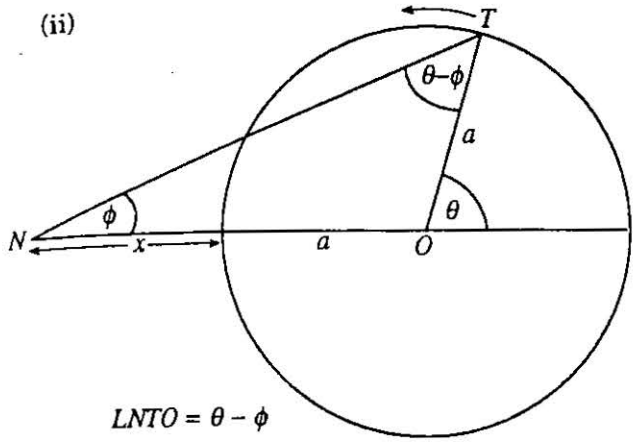
$\therefore \quad s = a\theta$

$$\frac{ds}{dt} = a.\frac{d\theta}{dt}$$

$$u = a\frac{d\theta}{dt}$$

$\therefore \quad \frac{d\theta}{dt} = \frac{u}{a}$

(ii)



$LNTO = \theta - \phi$

Using the Sine Rule:

$$\frac{a}{\sin\phi} = \frac{x+a}{\sin(\theta-\phi)}$$

$$\therefore\ a\sin(\theta-\phi) - (x+a)\sin\phi = 0$$

Differentiating w.r.t. t:

$$a\cos(\theta-\phi).\left(\dot\theta - \dot\phi\right) - (x+a)\cos\phi.\dot\phi = 0$$

$$a\cos(\theta-\phi).\left(\frac{u}{a} - \dot\phi\right) - (x+a)\cos\phi.\dot\phi = 0$$

$$-\dot\phi\left\{a\cos(\theta-\phi) + (x+a)\cos\phi\right\}$$

$$= -u\cos(\theta-\phi)$$

$$\dot\phi = \frac{u\cos(\theta-\phi)}{(x+a)\cos\phi + a\cos(\theta-\phi)}$$

(iii) When NT is tangential to the track

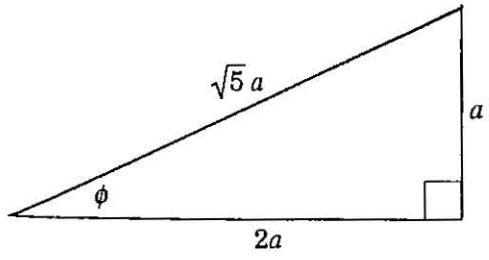
$$\theta - \phi = \frac{\pi}{2}$$

$$\frac{d\phi}{dt} = \frac{u\cos\left(\frac{\pi}{2}\right)}{(x+a)\cos\phi + a\cos\left(\frac{\pi}{2}\right)}$$

$$= 0.$$

(iv) If $x = a$ and $\theta = \frac{\pi}{2}$

$$\cos\phi = \frac{2}{\sqrt{5}},\quad \sin\phi = \frac{1}{\sqrt{5}}$$



Angular velocity about $N = \frac{d\phi}{dt}$

$$\therefore\ \omega_1 = \frac{d\phi}{dt} = \frac{u\cos\left(\frac{\pi}{2} - \phi\right)}{2a\cos\phi + a\cos\left(\frac{\pi}{2} - \phi\right)}$$

$$= \frac{u\sin\phi}{2a\cos\phi + a\sin\phi}$$

$$= \frac{\frac{u}{\sqrt{5}}}{\frac{4a}{\sqrt{5}} + \frac{a}{\sqrt{5}}}$$

$$= \frac{u}{5a}$$

When $\theta = 0,\quad \phi = 0$

$$\therefore\ \omega_2 = \frac{d\phi}{dt} = \frac{u\cos 0}{2a\cos 0 + a\cos 0}$$

$$= \frac{u}{3a}$$

$$\therefore\ \frac{\omega_1}{\omega_2} = \frac{u}{5a} \times \frac{3a}{u} = \frac{3}{5}$$

Alternative solution to (iv):

When $x = a$, $\frac{d\phi}{dt} = \frac{u\cos(\theta-\phi)}{2a\cos\phi + a\cos(\theta-\phi)}$

When $\theta = \frac{\pi}{2}$:

$$\frac{d\phi}{dt} = \frac{u}{a}.\frac{\cos\left(\frac{\pi}{2} - \phi\right)}{2\cos\phi + \cos\left(\frac{\pi}{2} - \phi\right)}$$

$$= \frac{u}{a}.\frac{\sin\phi}{2\cos\phi + \sin\phi}$$

$$= \frac{u}{a}.\frac{\sin\phi}{2(2\sin\phi) + \sin\phi} \qquad (*)$$

$$= \frac{u}{5a}$$

When $\theta = 0$:

$$\frac{d\phi}{dt} = \frac{u}{a}.\frac{\cos(-\phi)}{2\cos\phi + \cos(-\phi)}$$

$$= \frac{u}{a}.\frac{\cos\phi}{3\cos\phi}$$

$$= \frac{u}{3a}$$

(*) From (ii):

$$a\sin(\theta-\phi) - (x+a)\sin\phi = 0$$

When $x = a$, $\theta = \frac{\pi}{2}$:

$$a\sin\left(\frac{\pi}{2} - \phi\right) - 2a\sin\phi = 0$$

$$\cos\phi = 2\sin\phi$$

$$\therefore \quad \frac{\left.\frac{d\phi}{dt}\right|_{\phi=\frac{\pi}{2}}}{\left.\frac{d\phi}{dt}\right|_{\phi=0}} = \frac{\frac{u}{5a}}{\frac{u}{3a}}$$

$$= \frac{u}{5a} \times \frac{3a}{u} = \frac{3}{5}$$

(b)(i) Let $n = 2$

$$\sin(x_1 + x_2)$$
$$= \sin x_1 \cos x_2 + \cos x_1 \sin x_2$$
$$= a_1 \sin x_1 + a_2 \sin x_2$$

where $|a_1| = |\cos x_2| \le 1$

and $|a_2| = |\cos x_1| \le 1$.

So the result is true for $n = 2$.

Assume the result is true for $n = k$.

Then $\sin(x_1 + \ldots + x_k + x_{k+1})$

$$= \sin(X + x_{k+1}), \quad X = x_1 + \ldots + x_k$$

$$= \sin X \cos x_{k+1} + \cos X \sin x_{k+1}$$

$$= a\sin X + b_{k+1} \sin x_{k+1}$$

where $|a| = |\cos x_{k+1}| \le 1$

and $|b_{k+1}| = |\cos X| \le 1$

$$= a(a_1 \sin x_1 + \ldots + a_k \sin x_k) + b_{k+1} \sin x_{k+1}$$
$$= b_1 \sin x_1 + \ldots + b_k \sin x_k + b_{k+1} \sin x_{k+1}$$

where $|b_i| = |a a_i| \le 1, \quad i = 1, \ldots, k$

$$|b_{k+1}| \le 1.$$

Hence if the result is true for $n = k$, it is true for $n = k + 1$. Now the result is true for $n = 2$, therefore it is true for $n = 3$, $n = 4$, and so on. Hence the result is true for all positive integral n.

(ii) Let $x_1 = x_2 = \cdots = x_n = x$ and $0 < x < \pi$.

$$\therefore \quad \sin(nx) = a_1 \sin x + \cdots + a_n \sin x$$
$$\le \sin x + \cdots + \sin x$$

i.e. $\sin(nx) \le n \sin x$.

QUESTION SEVEN

(a)(i) To make the journey, Ayrton must choose 6 blocks heading south.

There are ${}^{10}C_6 = 210$ such choices;

OR there are ${}^{10}C_4$ choices of blocks heading east.

(ii) No. of routes via M

$$= \text{No. of routes from } A \text{ to } M \times \text{No. of routes from } M \text{ to } B$$

$$= {}^4C_2 \times {}^6C_4$$

$$= 6 \times 15$$

$$= 90$$

(iii) Ayrton must drive south from A to X. Thence he must drive 9 blocks to B.

$\therefore$ Number of routes $= {}^9C_5 = 126$.

(iv) He must start the journey with one of SS, ESS, EESS, EEESS, EEEESS. Thence the numbers of routes to 8 are, respectively,

$${}^8C_4, \ {}^7C_4, \ {}^6C_4, \ {}^5C_4, \ {}^4C_4$$

$\therefore$ Number of routes

$$= 70 + 35 + 15 + 5 + 1 = 126.$$

(b)(i) $z^7 = 1; \quad z \ne 1$

$$\therefore \quad z^7 - 1 = 0$$

$$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$$

$\because z \ne 1$: $\quad (z-1) \ne 0$

$$\therefore \quad z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$$

$(\div z^3 \ne 0)$: $z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0.$

(ii) $x = z + \frac{1}{z}$

$$x^2 = z^2 + 2 + \frac{1}{z^2}$$

$$\therefore \quad z^2 + \frac{1}{z^2} = x^2 - 2$$

$$x^3 = \left(z + \frac{1}{z}\right)^3$$
$$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$
$$= z^3 + \frac{1}{z^3} + 3x$$

$$\therefore \quad z^3 + \frac{1}{z^3} = x^3 - 3x.$$

Hence $z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$

$$= \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1$$

$$= \left(x^3 - 3x\right) + \left(x^2 + 2\right) + x + 1$$

$$= x^3 + x^2 - 2x - 1$$

i.e. the cubic equation is

$$x^3 + x^2 - 2x - 1 = 0.$$

(iii) $z^7 = 1$

$\therefore \quad |z^7| = |z|^7 = 1$

$\therefore \quad |z| = 1$

Let $z = \cos\theta + i\sin\theta$

Then $z^7 = (\cos\theta + i\sin\theta)^7$

$= \cos 7\theta + i\sin 7\theta$

$= 1.$

Equating real and imaginary parts:

$\cos 7\theta = 1, \quad \sin 7\theta = 0$

$\therefore \quad 7\theta = 2\pi k, \quad k = 0, 1, \ldots, 6$

$$\theta = \frac{2\pi k}{7}, \quad k = 0, 1, \ldots, 6$$

Since $z \neq 1$,

$$\therefore \quad z = \cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}, \quad k = 1, 2, \ldots, 6$$

Also $\dfrac{1}{z} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

$\therefore \quad x = z + \dfrac{1}{z}$

$= 2\cos\theta$

$$= 2\cos\frac{2\pi k}{7}, \quad k = 1, 2, \ldots, 6$$

Only three of these six values of x are distinct since

$$\cos\frac{12\pi}{7} = \cos\frac{2\pi}{7}$$

$$\cos\frac{10\pi}{7} = \cos\frac{4\pi}{7} = \cos\left(\pi - \frac{3\pi}{7}\right) = -\cos\frac{3\pi}{7}$$

$$\cos\frac{8\pi}{7} = \cos\frac{6\pi}{7} = \cos\left(\pi - \frac{6\pi}{7}\right) = -\cos\frac{\pi}{7}$$

Hence, forming the product of the roots of the cubic equation in x in part (ii),

$$\left(2\cos\frac{2\pi}{7}\right)\left(-2\cos\frac{3\pi}{7}\right)\left(-2\cos\frac{\pi}{7}\right) = 1$$

i.e. $$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$$

QUESTION EIGHT

(a)(i) $f(x) = e^x\left(1 - \dfrac{x}{10}\right)^{10}$

$$f'(x) = e^x.10\left(1 - \frac{x}{10}\right)^9\left(-\frac{1}{10}\right) + \left(1 - \frac{x}{10}\right)^{10}.e^x$$

$$= -e^x\left(1 - \frac{x}{10}\right)^9 + e^x\left(1 - \frac{x}{10}\right)^{10}$$

$$= e^x\left(1 - \frac{x}{10}\right)^9\left(1 - \frac{x}{10} - 1\right)$$

$$= -\frac{xe^x}{10}\left(1 - \frac{x}{10}\right)^9$$

$\therefore \quad f'(x) = 0 \quad \Rightarrow \quad x = 0, \ 10$

$f(0) = 1 \quad$ and $\quad f(10) = 0$

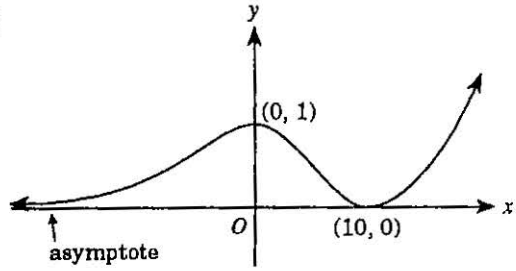
For $\quad x < 0, \quad f'(x) > 0$

$0 < x < 10, \quad f'(x) < 0$

$x > 10, \quad f'(x) > 0$

$\therefore$ (0, 1) is a maximum turning point and (10, 0) is a minimum turning point.

(ii)



$f(x) \to \infty$ as $x \to \infty$

Since $f(x) \geq 0$ for all x and (0, 1) is a maximum turning point

$\therefore \quad f(x) \to 0$ as $x \to -\infty$

(iii) $f(x) \leq 1$ for $x < 10$

i.e. $e^x\left(1 - \dfrac{x}{10}\right)^{10} \leq 1$

$e^x \leq \left(1 - \dfrac{x}{10}\right)^{-10}$ because $\left(1 - \dfrac{x}{10}\right)^{10} > 0$

for $x < 10$, and division by a positive quantity does not alter the inequality.

(iv) Let $x = 1, \quad \therefore \quad e \leq \left(\dfrac{9}{10}\right)^{-10} = \left(\dfrac{10}{9}\right)^{10}$

Let $x = -1, \ \therefore \ e^{-1} \leq \left(\dfrac{11}{10}\right)^{-10}$

$$e \geq \left(\frac{11}{10}\right)^{10}$$

Hence $$\left(\frac{11}{10}\right)^{10} \leq e \leq \left(\frac{10}{9}\right)^{10}.$$

(b) $x \approx \sqrt{n}$ where $x, n > 0$

$xy = n$ $\therefore\ y > 0$

(i) $0 \le \left(\sqrt{x} - \sqrt{y}\right)^2 = x + y - 2\sqrt{xy}$

$= x + y - 2\sqrt{n}$

$\therefore\quad 2\sqrt{n} \le x + y$

$\therefore\quad \dfrac{x+y}{2} \ge \sqrt{n}$

N.B. Equality occurs for $x = y$ only.

(ii) $x > \sqrt{n}$

$\therefore\quad \dfrac{n}{y} > \sqrt{n}.$

$\therefore\quad \dfrac{y}{n} < \dfrac{1}{\sqrt{n}}$ taking reciprocals

$y < \dfrac{n}{\sqrt{n}}$

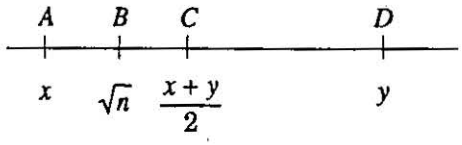
$\therefore\quad y < \sqrt{n}.$

$\therefore\quad y < \sqrt{n} < x$

Then $\sqrt{n} \le \dfrac{x+y}{2} \le \dfrac{x+x}{2} = x.$

So $\dfrac{x+y}{2}$ is a closer approximation to $\sqrt{n}$ than x is.

(iii)



$x < \sqrt{n}$

$\dfrac{n}{y} < \sqrt{n}$ $\because\ xy = n$

$\therefore\quad \dfrac{y}{n} > \dfrac{1}{\sqrt{n}}$ (reciprocals)

$y > \sqrt{n}.$

Also from (i):

$\dfrac{x+y}{2} > \sqrt{n}$ $(x \ne y$ here)

$\therefore\quad x < \sqrt{n} < \dfrac{x+y}{2} < y.$

For $\dfrac{x+y}{2}$ to be a closer approximation to $\sqrt{n}$ than x is, then $BC < AB$.

$\dfrac{x+y}{2} - \sqrt{n} < \sqrt{n} - x$

$\therefore\quad \dfrac{x+y}{2} + x < 2\sqrt{n}$

$3x + y < 4\sqrt{n}$

$3x + \dfrac{n}{x} < 4\sqrt{n}$

$3x^2 - 4\sqrt{n}.x + n < 0$

$\left(3x - \sqrt{n}\right)\left(x - \sqrt{n}\right) < 0$

$\dfrac{\sqrt{n}}{3} < x < \sqrt{n}.$

i.e. $\dfrac{x+y}{2}$ will be a better approximation to $\sqrt{n}$ than x is, for $\dfrac{\sqrt{n}}{3} < x < \sqrt{n}.$