1991 HIGHER SCHOOL CERTIFICATE SOLUTIONS

4 UNIT MATHEMATICS

QUESTION ONE

(a) (i) $\displaystyle\int \frac{t-1}{t^3}\,dt = \int \left(t^{-2}-t^{-3}\right)\,dt$

$\displaystyle = -\frac{1}{t}+\frac{1}{2t^2}+C$

(ii) $\displaystyle\int \frac{e^x}{e^{2x}+9}\,dx, \quad u=e^x, \quad du=e^x\,dx$

$\displaystyle = \int \frac{du}{u^2+9}$

$\displaystyle = \frac{1}{3}\tan^{-1}\frac{u}{3}+C$

$\displaystyle = \frac{1}{3}\tan^{-1}\frac{e^x}{3}+C$

(b) (i) Let $\displaystyle\frac{5}{(2t+1)(2-t)} = \frac{a}{2t+1}+\frac{b}{2-t}$

$\therefore \quad 5 = a(2-t)+b(2t+1)$

Let $t=-\frac{1}{2}, \quad 5=\frac{5}{2}a,$

$\therefore \quad a=2$

Let $t=2, \quad 5=5b,$

$\therefore \quad b=1$

$\displaystyle\int_0^1 \frac{5}{(2t+1)(2-t)}\,dt$

$\displaystyle = \int_0^1 \left(\frac{2}{2t+1}+\frac{1}{2-t}\right)\,dt$

$= \left[\log|2t+1|-\log|2-t|\right]_0^1$

$\displaystyle = \left[\log\frac{|2t+1|}{|2-t|}\right]_0^1$

$= \log 43 - \log\frac{1}{2}$

$= \log 6$

(ii) Let $t=\tan\frac{\theta}{2}$

When $\theta=0, \quad t=0$

$\theta=\frac{\pi}{2}, \quad t=1$

$\displaystyle\sin\theta = \frac{2t}{1+t^2}$

$\displaystyle\cos\theta = \frac{1-t^2}{1+t^2}$

$\displaystyle d\theta = \frac{2dt}{1+t^2}$

$\therefore \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta+4\cos\theta}$

$\displaystyle = \int_0^1 \frac{\frac{2dt}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}$

$\displaystyle = \int_0^1 \frac{dt}{3t+2-2t^2}$

$\displaystyle = \int_0^1 \frac{dt}{(2t+1)(2-t)}$

$= \log 6$ from (i)

(c) (i) $\displaystyle I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx, \quad n\geq 2$

$\displaystyle = \int_0^{\frac{\pi}{2}} \sin^{n-1}x\;\; d(\cos x)$

$\displaystyle = -\left[\sin^{n-1}x\;\;\cos x\right]_0^{\frac{\pi}{2}}$

$\displaystyle + \int_0^{\frac{\pi}{2}} \cos x\;\; d\left(\sin^{n-1}x\right)$

$\displaystyle = 0+\int_0^{\frac{\pi}{2}} \cos x(n-1)\sin^{n-2}x\;\;\cos x\,dx$

$\displaystyle = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,\cos^2 x\;\;dx$

(ii) Hence

$\displaystyle I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\left(1-\sin^2 x\right)dx$

$$= (n-1)\int_0^{\frac{\pi}{2}} \left(\sin^{n-2} x - \sin^n x\right)\, dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore \quad nI_n = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2} \qquad (n \geq 2)$$

(iii) $I_4 = \frac{3}{4} I_2$

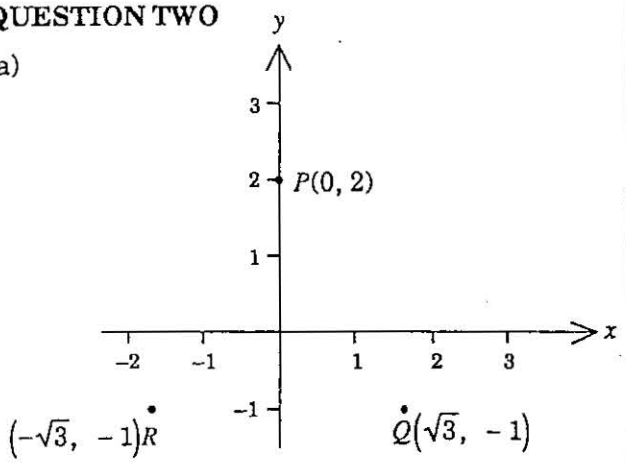
$$= \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$= \frac{3}{8} \int_0^{\frac{\pi}{2}} dx$$

$$= \frac{3\pi}{16}$$

QUESTION TWO

(a)



$$PQ^2 = \sqrt{3}^2 + (-1-2)^2 = 12$$

$$PR^2 = \left(0-\sqrt{3}\right)^2 + (2--1)^2 = 12$$

$$RQ^2 = \left(2\sqrt{3}\right)^2 = 12$$

$\therefore$ ΔPQR is equilateral

(b) (i) $\frac{1}{z_1} = \left(\cos\theta_1 + i\sin\theta_1\right)^{-1}$

$$= \cos\left(-\theta_1\right) + i\sin\left(-\theta_1\right),$$

by De Moivre's Theorem

$$= \cos\theta_1 - i\sin\theta_1$$

Alternatively,

$$\frac{1}{z_1} = \left(\frac{1}{\cos\theta_1 + i\sin\theta_2}\right)\left(\frac{\cos\theta_1 - i\sin\theta_1}{\cos\theta_1 - i\sin\theta_1}\right)$$

$$= \frac{\cos\theta_1 - i\sin\theta_1}{\cos^2\theta_1 + \sin^2\theta_1}$$

$$= \cos\theta_1 - i\sin\theta_1$$

(ii) $z_1 z_2$

$$= \left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\sin\theta_2\right)$$

$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$+ i\left(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)$$

$$= \cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)$$

(c) (i) $(x+iy)^2 = -3-4i$

$$x^2 - y^2 + 2ixy = -3-4i$$

Equating real and imaginery parts:

$$x^2 - y^2 = -4 \qquad \ldots(1)$$

$$2xy = -4 \qquad \ldots(2)$$

From (2), $y = -\frac{2}{x}$ $\qquad \ldots(3)$

Substituting into (1)

$$x^2 - \frac{4}{x^2} = -3$$

$$x^4 + 3x^2 - 4 = 0$$

$$\left(x^2+4\right)\left(x^2-1\right) = 0$$

$\therefore$ $x^2 = 1, \quad x = \pm 1$

From (3), when $x = 1, \quad y = -2$

and when $x = -1, \quad y = 2$

i.e. $\sqrt{-3-4i} = \pm(1-2i)$

(ii) $z^2 - 3z + (3+i) = 0$

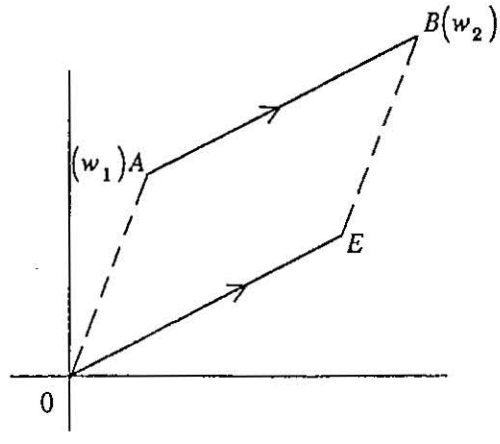
$$z = \frac{3 \pm \sqrt{9-4(3+i)}}{2}$$

$$= \frac{3 \pm \sqrt{-3-4i}}{2}$$

$$= \frac{3 \pm (1-2i)}{2}$$

$$= 2-i, \quad 1+\frac{5}{2}i$$

(d)



(i) $OE \parallel AB$ and $OE = AB$

$\therefore$ $OABE$ is a parallelogram.

Let E correspond to the complex number z, then, since addition of complex numbers corresponds to a parallelogram construction in the complex plane,

$$z + w_1 = w_2$$
$$z = w_2 - w$$

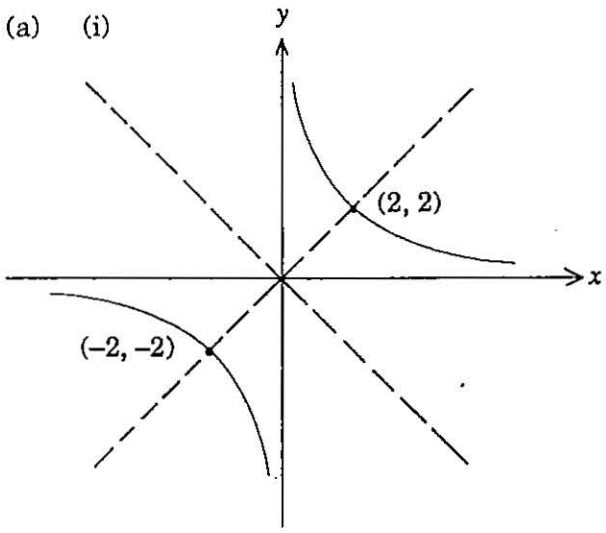
i.e. E corresponds to the complex number $w_2 - w_1$.

(ii) $OF \perp OE$ and $OF = OE$
Since multiplying a complex number by i corresponds to an anticlockwise rotation about the origin through 90°, F corresponds to the complex number $i(w_2 - w_1)$.

(iii) Since $AD \parallel OF$ and $AD = OF$, D corresponds to the complex number
$$w_1 + i(w_2 - w_1) = w_1(1 - i) + iw_2.$$

QUESTION THREE

(a) (i)



(ii) $y = \frac{4}{x}$ $(x \neq 0)$

$$\frac{dy}{dx} = \frac{-4}{x^2}$$

At P, $x = 2t$, $\frac{dy}{dx} = \frac{-4}{4t^2} = -\frac{1}{t^2}$ $(t \neq 0)$

$\therefore$ equation of tangent at P is

$$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$$
$$= -\frac{x}{t^2} + \frac{2}{t}$$

$\therefore$ $t^2 y - 2t = -x + 2t$
$x + t^2 y = 4t$

(iii) Tangent at P is $x + t^2 y = 4t$ ...(1)
Tangent at Q is $x + s^2 y = 4s$ ...(2)
(1) – (2): $(t^2 - s^2)y = 4(t - s)$

$\therefore$ $y = \frac{4}{t + s}$ since $s^2 \neq t^2$

Substituting into (1):

$$x + \frac{4t^2}{t + s} = 4t$$
$$x = 4t - \frac{4t^2}{t + s}$$
$$= \frac{4ts}{s + t}$$

$\therefore$ point of intersection M is
$$\left(\frac{4ts}{s + t}, \frac{4}{t + s}\right)$$

(iv) Putting $s = -\frac{1}{t}$ in M, we get

$$x = \frac{-4}{-\frac{1}{t} + t}$$
$$= \frac{4t}{1 - t^2}$$
$$y = \frac{4}{-\frac{1}{t} + t} = \frac{-4t}{1 - t^2}$$

Since $t \neq 0$, $x \neq 0$ and $y \neq 0$
$\therefore$ the locus is $y = -x$ $(x \neq 0)$, which is a straight line through, but excluding, the origin.

(b) (i) The points of intersection of the ellipses satisfy both $C_1 = 0$ and $C_2 = 0$.
$\therefore$ at these points $C_1 + \lambda C_2 = 0$

(ii) $C_1 + \lambda C_2$
$= x^2 + 3y^2 - 1 + \lambda(4x^2 + y^2 - 1)$
$= (1 + 4\lambda)x^2 + (3 + \lambda)y^2 - (1 + \lambda)$

i.e. $\frac{(1 + 4\lambda)x^2}{1 + \lambda} + \frac{(3 + \lambda)y^2}{1 + \lambda} = 1$

For an ellipse we require
$1 + 4\lambda > 0$, $3 + \lambda > 0$, and $1 + \lambda > 0$
$\lambda > -\frac{1}{4}$, $\lambda > -3$, and $\lambda > -1$

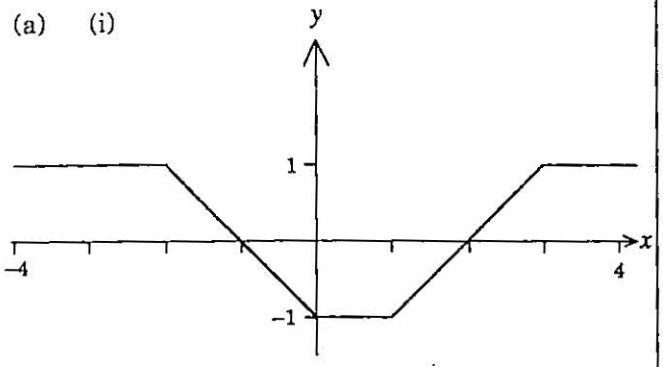
i.e. $\lambda > -\frac{1}{4}$, or

$1 + 4\lambda < 0$, $3 + \lambda < 0$, and $1 + \lambda < 0$
$\lambda < -\frac{1}{4}$, $\lambda < -3$, and $\lambda < -1$

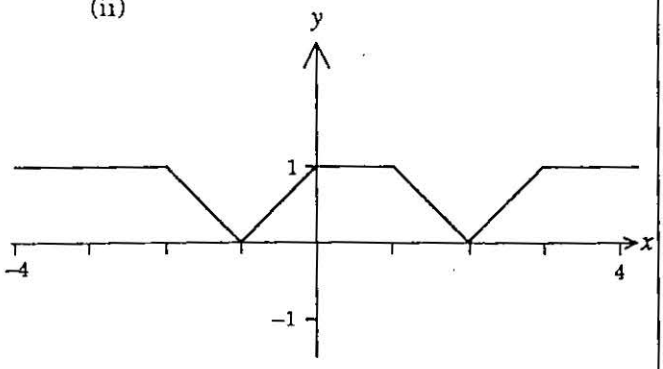
i.e. $\lambda < -3$.
$\therefore$ The curve is an ellipse for $\lambda < -3$ or $\lambda > -\frac{1}{4}$.

QUESTION FOUR

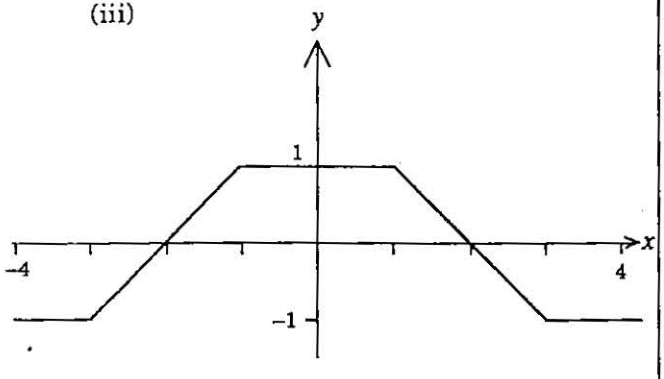
(a) (i)



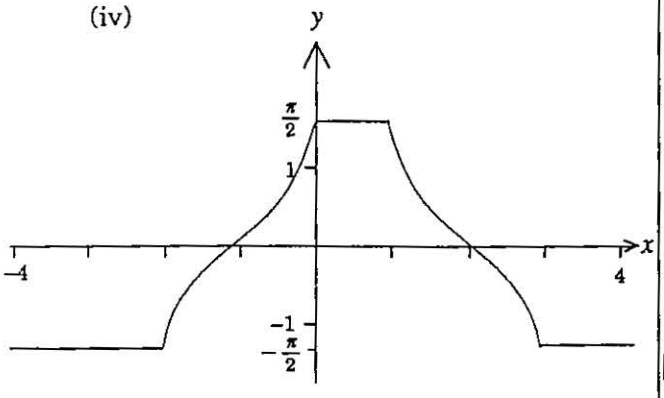
(ii)



(iii)



(iv)



(b) (i) $g(x) = 4e^{-x} - 6e^{-2x} \qquad x \geq 0$

$g'(x) = -4e^{-x} + 12e^{-2x}$

$g''(x) = 4e^{-x} - 24e^{-2x}$

$g'(x) = 0 \Rightarrow 12e^{-2x} - 4e^{-x} = 0$

$4e^{-x}(3e^{-x} - 1) = 0$

$3e^{-x} = 1$

$e^{-x} = \frac{1}{3}$

$-x = \ln\left(\frac{1}{3}\right)$

$x = \ln 3$

$g(\ln 3) = 4 \times \frac{1}{3} - 6 \times \frac{1}{9} = \frac{2}{3}$

$g''(\ln 3) = 4 \times \frac{1}{3} - 24 \times \frac{1}{9} = -\frac{4}{3}$

$\therefore \left(\ln 3, \frac{2}{3}\right)$ is a maximum turning point.

(ii) For $x \geq 0$, the only turning point is $\left(\ln 3, \frac{2}{3}\right)$. For inflexions $g''(x) = 0$.

$\Rightarrow \quad 4e^{-x} - 24e^{-2x} = 0$

$4e^{-x}(1 - 6e^{-x}) = 0$

i.e. $6e^{-x} = 1$

$x = \ln 6$

and $g(\ln 6) = \frac{1}{2}$

For $0 < x < \ln 6$, $g''(x) < 0$

and $x > \ln 6$, $g''(x) > 0$

$\therefore \left(\ln 6, \frac{1}{2}\right)$ is a point of inflexion.

The graph cuts the x-axis when $g(x) = 0$

$\Rightarrow \quad 4e^{-x} - 6e^{-2x} = 0$

$2e^{-x}(2 - 3e^{-2}) = 0$

$3e^{-x} = 2$

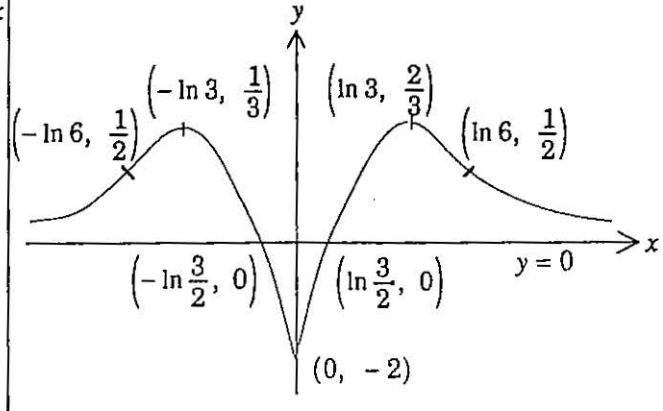
$x = \ln\frac{3}{2}$

As $x \to \infty$, $g(x) \to 0$,

$\therefore$ the x-axis is an asymptote.

$g(0) = -2$

$g(x) = g(-x)$ so the function is odd.



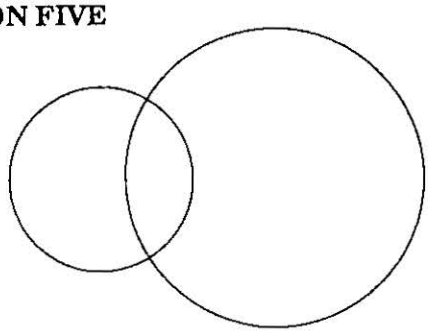
(iii) At $x = 0$, $g(x) = -2$,

$g'(x^+) = 8$, $g'(x^-) = -8$

The curve has a cusp at $(0, -2)$.

QUESTION FIVE

(a)



(i) In ΔTPR, $\angle STQ = \angle YRQ + \angle YPS$ because the ext. angle is the sum of the interior opposite angles.

(ii) $\angle YPS = \angle YXS$ ($\angle$s standing on the same arc)

$\angle YRQ = \pi = \angle YXQ$ (opp. $\angle$s of a cyclic quadrilateral)

$= \angle PXY$ (PXQ is a st. line)

$\therefore \quad \angle YRQ + \angle YPS + \angle SXQ$

$= \angle PXY + \angle YPS + \angle SXQ$

$= \pi$ (adj. $\angle$s, PXQ is a st. line)

(iii) $\angle STQ + \angle SXQ$

$= \angle YRQ + \angle YPS + \angle SXQ$, by (i)

$= \pi$, by (ii)

$\therefore$ $STQX$ is a cyclic quadrilateral as the opposite angles are supplementary.

(iv) $\angle XYS = \angle XPS$ (angles standing on the same arc)

$\angle SXY = \angle SPY$ (angles standing on the same arc)

$\therefore \quad d = \angle QPY = \angle XPS + \angle SPY$

$= \angle XYS + \angle SXY$

$= \angle XSQ$ (ext. $\angle$ of ΔSXY)

also $\angle STQ = \pi - \angle SXQ$ (opp. $\angle$s of cyclic quad. $STQX$)

$= \angle PXS$ (PXQ is a st. line)

$= \angle XQS + \angle XSQ$ (ext. $\angle$ of ΔQXS)

$= \angle PQY + \angle XQS$

$= \beta + \alpha$

(b) (i) Triangles COD, FHD are similar,

$$\therefore \quad \frac{FH}{CO} = \frac{HD}{OD}$$

$$\frac{FH}{a} = \frac{h-x}{h}$$

$$FH = \frac{a}{h}(h-x)$$

(ii) $C'F = a - FH$

$$= a - \frac{a}{h}(h-x)$$

$$= \frac{ax}{h}$$

$$\cos \angle HC'G = \frac{C'F}{a}$$

$$= \frac{x}{h}$$

$$\therefore \quad \angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$$

(iii) Shaded area

= area sector $C'EHG - 2 \times$ area $\Delta C'FG$

$$= \frac{1}{2}a^2 \cdot 2\cos^{-1}\left(\frac{x}{h}\right) - 2 \cdot \frac{1}{2} C'F \cdot FG$$

$$= a^2 \cos^{-1}\left(\frac{x}{h}\right) - \frac{ax}{h}\sqrt{a^2 - \frac{a^2x^2}{h^2}}$$

$$= a^2 \cos^{-1}\left(\frac{x}{h}\right) - \frac{a^2 x}{h}\sqrt{1-\left(\frac{x}{h}\right)^2}$$

$$= a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1-\left(\frac{x}{h}\right)^2}\right]$$

(iv) Volume

$$= \int_0^h a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1-\left(\frac{x}{h}\right)^2}\right] dx$$

$$= a^2 h \int_0^1 \left(\cos^{-1}\theta - \theta\sqrt{1-\theta^2}\right) d\theta$$

$$= a^2 h \int_0^1 \cos^{-1}\theta \, d\theta + \frac{a^2 h}{2}\int_0^1 \sqrt{1-\theta^2}\; d(1-\theta^2)$$

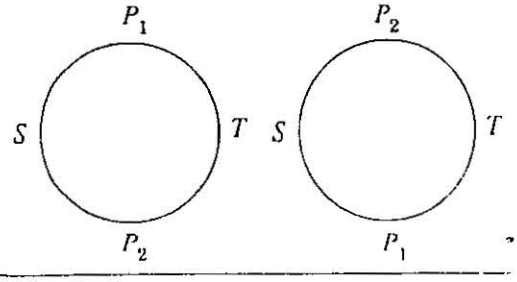
$$= a^2 h \left[\theta\cos^{-1}\theta - \sqrt{1-\theta^2}\right]_0^1 + \frac{a^2 h}{3}\left[(1-\theta^2)^{\frac{3}{2}}\right]_0^1$$

$$= a^2 h - \frac{a^2 h}{3}$$

$$= \frac{2a^2 h}{3}$$

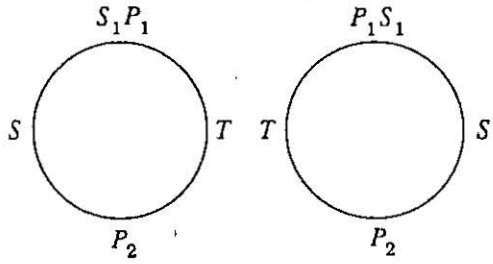
QUESTION SIX

(a) (i) If S and T denote the sets 'students' and 'teachers' respectively, and P_1, P_2 are the parents, then all the possible arrangements are shown below.



There are 4! ways of arranging the students and 3! ways of arranging the teachers. Thus the number of arrangements of the 9 members is $P_2 P_1 2 \times 3! \times 4! = 288$.

(ii) Let S_1, P_1 be the related student and parent. Let S' and P_2 denote the remaining set of students and the other parent, and let T be the set of teachers. Then the possible arrangements are shown below.

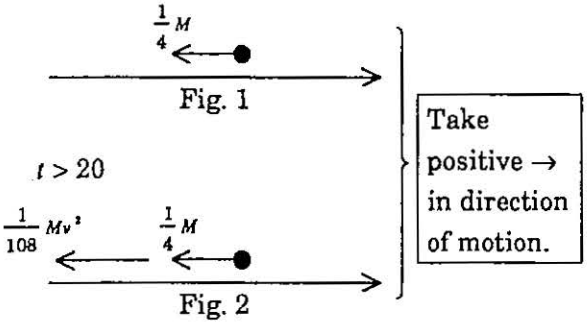


There are 3! ways of arranging the students S and 3! ways of arranging the teachers T. Thus the number of arrangements of the 9 members is $2 \times 3! \times 3! = 72$.

Hence the probability that this pair of people sit together is

$$\frac{72}{288} = \frac{1}{4}.$$

(b) $0 \le t \le 20$



(i) From Fig. 1, for $0 \le t \le 20$

$$M\ddot{x} = -\frac{1}{4}M$$

$$\ddot{x} = -\frac{1}{4}$$

From Fig. 2, for $t > 20$

$$M\ddot{x} = -\frac{1}{4}M - \frac{1}{108}Mv^2$$

$$\ddot{x} = -\frac{1}{108}(27 + v^2)$$

(ii) Since $v = \frac{dx}{dt}$,

$$\therefore \quad v\frac{dv}{dx} = \frac{dx}{dt}\frac{dv}{dx}$$

$$= \frac{dv}{dt}$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$= \frac{d^2x}{dt^2}.$$

(iii) $\ddot{x} = -\frac{1}{4}, \quad 0 \le t \le 20$

i.e. $\frac{dv}{dt} = -\frac{1}{4}$

$$v = -\frac{1}{4}t + C_1$$

at $t = 0, \quad v = 60$

$\therefore \quad 60 = C_1$

$$v = -\frac{1}{4}t + 60 \qquad \ldots(1)$$

Integrating again w.r.t. t

$$x = -\frac{1}{8}t^2 + 60t + C_2$$

at $t = 0, \quad x = 0$

$\therefore \quad C_2 = 0$

$$x = -\frac{1}{8}t^2 + 60t \qquad \ldots(2)$$

at $t = 20$, (1) and (2) give

$$v = -\frac{1}{4} \times 20 + 60 = 55$$

$$x = -\frac{1}{8} \times 400 + 60 \times 20 = 1150$$

(iv) When $t > 20$

$$\ddot{x} = -\frac{1}{108}(27 + v^2)$$

$$v\frac{dv}{dx} = -\frac{1}{108}(27 + v^2)$$

$$\frac{v\,dv}{27 + v^2} = -\frac{1}{108}dx$$

From (iii), at $x = 1150, \quad v = 55$

$$\therefore \quad \int_{55}^{v} \frac{v\,dv}{27 + v^2} = -\frac{1}{108}\int_{1150}^{x} dx$$

$$\left[\frac{1}{2}\log(27 + v^2)\right]_{55}^{v} = -\frac{1}{108}[x]_{1150}^{x}$$

$$\frac{1}{2}\left\{\log(27 + v^2) - \log(27 + 55^2)\right\}$$

$$= -\frac{1}{108}(x - 1150)$$

$$\therefore \quad x - 1150$$

$$= 54\left\{\log(27 + 55^2) - \log(27 + v^2)\right\}$$

$$x = 1150 + 54\{\log(27 + 55^2) - \log(27 + v^2)\}$$

(v) The jet comes to rest when $v = 0$, i.e.

$$x = 1150 + 54\{\log(27 + 55^2) - \log 27\}$$
$$= 1405{\cdot}296\,638$$
$$\approx 1405$$

The jet comes to rest 1405 metres from the touchdown point.

QUESTION SEVEN

(a) (i) $\cos 3A = (\cos 2A + A)$

$$= \cos 2A \cos A - \sin 2A \sin A$$
$$= (2\cos^2 A - 1)\cos A - 2\cos A \sin A \sin A$$
$$= 2\cos^3 A - \cos A - 2\cos A \sin^2 A$$
$$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$$
$$= 4\cos^3 A - 3\cos A$$

$$\therefore \quad \cos^3 A - \frac{3}{4}\cos A = \frac{1}{4}\cos 3A$$

(ii) Substituting $x = 2\sqrt{2}\cos A$ into $x^3 - 6x = -2$ gives

$$16\sqrt{2}\cos^3 A - 12\sqrt{2}\cos A = -2$$
$$\cos^3 A - \frac{3}{4}\cos A = -\frac{2}{16\sqrt{2}}$$
$$= -\frac{1}{8\sqrt{2}}$$

i.e. $\frac{1}{4}\cos 3A = -\frac{1}{8\sqrt{2}}$ by (i)

$$\therefore \quad \cos 3A = -\frac{1}{2\sqrt{2}}$$

(iii) $\cos 3A = -\frac{1}{2\sqrt{2}}$

$$\Rightarrow 3a = \pm\cos^{-1}\left(-\frac{1}{2\sqrt{2}}\right) + 2n\pi,$$

where n is an integer.

Taking the positive branch* of the inverse cosine and $n = 0, 1, 2$ gives

$3A = 1{\cdot}93\,216, \quad 8{\cdot}215\,35, \quad 14{\cdot}49\,853$

$A = 0{\cdot}64\,405, \quad 2{\cdot}73\,845, \quad 4{\cdot}83\,284$

$\therefore \quad x = 2\sqrt{2}\cos A$

$= 2{\cdot}2618, \quad -2{\cdot}6017, \quad 0{\cdot}3399$

to 4 dec. places.

* Note that taking the negative branch leads to the same values for x since $\cos A = \cos(-A)$.

(b) (i) $P(0) = 1 \neq 0$

$P(1) = 2A + B + 2 \neq 0$

since $2 + B \neq -2A$

$P(-1) = -2A + B + 2 \neq 0$

since $2 + B \neq 2A$

(ii) $P\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^4} + \frac{A}{\alpha^3} + \frac{B}{\alpha^2} + \frac{A}{\alpha} + 1$

$$= \frac{1}{\alpha^4}(1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4)$$
$$= \frac{1}{\alpha^4}P(\alpha)$$
$$= 0 \text{ since } P(\alpha) = 0$$

$\therefore \quad \frac{1}{\alpha}$ is a root of $P(x) = 0$

Since $\alpha \neq 0,\ 1,\ -1$

$\therefore \quad \alpha$ and $\frac{1}{\alpha}$ are distinct.

Since α is a multiple root and $P(x)$ has degree 4 we can write

$$P(x) = (x - \alpha)^2\left(x - \frac{1}{\alpha}\right)(ax + b)$$

(iii) $P(x) = (x - \alpha)^2\left(x - \frac{1}{\alpha}\right)(ax + b)$

Clearly $a = 1$ and, equating constants,

$$1 = \alpha b \Rightarrow b = -\frac{1}{\alpha}$$

$$\therefore \quad P(x) = (x - \alpha)^2\left(x - \frac{1}{\alpha}\right)^2$$

i.e. α is a root of multiplicity 2.

$$x^4 + Ax^3 + Bx^2 + Ax + 1$$
$$= (x - \alpha)^2\left(x - \frac{1}{\alpha}\right)^2$$
$$= \left\{(x - \alpha)\left(x - \frac{1}{\alpha}\right)\right\}^2$$
$$= \left\{x^2 - \left(\alpha + \frac{1}{\alpha}\right)x + 1\right\}^2$$

Equating coefficients of x^3:

$$A = -2\left(\alpha + \frac{1}{\alpha}\right)$$

Equating coefficients of x^2:

$$B = \left\{-\left(\alpha + \frac{1}{\alpha}\right)\right\}^2 + 2$$
$$= 2 + \left(\alpha + \frac{1}{\alpha}\right)^2$$

$$\therefore \quad 4B = 8 + 4\left(\alpha + \frac{1}{\alpha}\right)^2$$
$$= 8 + A^2$$

QUESTION EIGHT

(a) (i) $f(x) = \dfrac{\sin x}{x}$ for $0 < x < \dfrac{\pi}{2}$

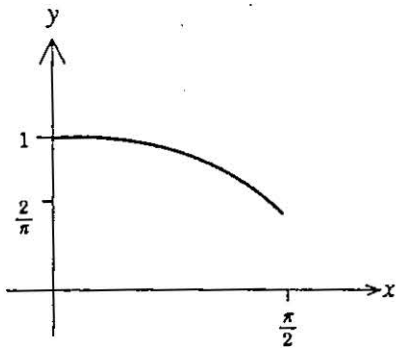
$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$= \frac{\cos x(x - \tan x)}{x^2}$$

$$< 0$$

since $\tan x > x$ for $0 < x < \dfrac{\pi}{2}$

(ii)



$f(0) = 1$

$f\left(\dfrac{\pi}{2}\right) = \dfrac{2}{\pi}$

$f(x) \to 1$ as $x \to 0$

From the graph $f(x) \geq \dfrac{2}{\pi}$ for $0 \leq x \leq \dfrac{\pi}{2}$

i.e. $\dfrac{\sin x}{x} \geq \dfrac{2}{\pi}$

$$\sin x \geq \frac{2x}{\pi}$$

(b) (i) Let $f(x) = \dfrac{1}{x} + \dfrac{1}{s-x}$

For $0 < x < s$, $f'(x) = -\dfrac{1}{x^2} + \dfrac{1}{(s-x)^2}$

$$f'(x) = 0 \Rightarrow \frac{1}{(s-x)^2} = \frac{1}{x^2}$$

$$(s-x)^2 = x^2$$

$$x = \frac{s}{2}$$

$$f\left(\frac{s}{2}\right) = \frac{4}{s}$$

When $0 < x < \dfrac{s}{2}$, $f'(x) < 0$.

When $\dfrac{s}{2} < x < s$, $f'(x) > 0$.

Hence $f(x)$ has a minimum turning point at $\left(\dfrac{s}{2}, \dfrac{4}{s}\right)$, that is, since $y = s - x$,

$\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{s}$ $(x > 0,\ y > 0,\ x + y = s)$.

(b) (ii) Now let $g(x) = \dfrac{n^2}{x} + \dfrac{1}{t-x}$, $0 < x < t$

$$g'(x) = -\frac{n^2}{x^2} + \frac{1}{(t-x)^2}$$

$$g'(x) = 0 \Rightarrow \frac{n^2}{x^2} = \frac{1}{(t-x)^2}$$

i.e. $x = \dfrac{nt}{1+n}$

$$g\left(\frac{nt}{1+n}\right) = \frac{n^2}{\frac{nt}{1+n}} + \frac{1}{t - \frac{nt}{1+n}}$$

$$= \frac{(1+n)^2}{t}$$

When $0 < x < \dfrac{nt}{1+n}$, $g'(x) > 0$

When $\dfrac{nt}{1+n} < x < t$, $g'(x) < 0$

Hence $g(x)$ has a minimum turning point at $\left(\dfrac{nt}{1+n}, \dfrac{(1+n)^2}{t}\right)$ ...(1)

If $x_i > 0$ for $i = 1, 2, \ldots, n$ $(n \geq 2)$ and $s = x_1 + x_2 + \ldots + x_n$, then the result $\dfrac{1}{x_1} + \dfrac{1}{x_2} + \ldots + \dfrac{1}{x_n} \geq \dfrac{n^2}{s}$ is true for $n = 2$ by part (i).

Assume it is time for $n = k$, then, putting $t = x_1 + x_2 + \ldots + x_{k+1}$,

$$\frac{1}{x_1} + \ldots + \frac{1}{x_k} + \frac{1}{x_{k+1}} \geq \frac{n^2}{s} + \frac{1}{t-s}$$

$$\geq \frac{(n+1)^2}{t} \text{ by (i).}$$

Thus if the result is true for $n = k$, it is also true for $n = k + 1$. Now it is true for $n = 2$, therefore it is true for $n = 3$, 4, and so on.

Hence it is true for all positive integral n.