NSW HSC 4 Unit Mathematics Examination 1994

1. **(a)** Find **(i)** $\int \frac{4x-12}{x^2-6x+13}\,dx$ **(ii)** $\int \frac{1}{x^2-6x+13}\,dx$.

(b) Evaluate $\int_{\frac{3}{2}}^{3} \sqrt{9-u^2}\,du$. **(c)** Evaluate $\int_1^2 \frac{11-2t}{(2t-1)(3-t)}\,dt$.

(d) Evaluate $\int_0^{\pi} e^{2x}\sin x\,dx$.

2. **(a)** Let $z = a + ib$, where a and b are real. Find:

(i) $\Im(4i - z)$

(ii) $\overline{(3iz)}$ in the form $x + iy$, where x and y are real

(iii) $\tan\theta$, where $\theta = \arg(z^2)$.

(b) Express in modulus-argument form:

(i) $-1 + i$ **(ii)** $(-1+i)^n$, where n is a positive integer.

(c) **(i)** On the same diagram, draw a neat sketch of the locus specified by each of the following:

(α) $|z - (3 + 2i)| = 2$ **(β)** $|z + 3| = |z - 5|$.

(ii) Hence write down all the values of z which satisfy simultaneously

$$|z - (3 + 2i)| = 2 \text{ and } |z + 3| = |z - 5|$$

(iii) Use your diagram in **(i)** to determine the values of k for which the simultaneous equations $|z - (3 + 2i)| = 2$ and $|z - 2i| = k$ have exactly one solution for z.

3. **(a)** The hyperbola $\mathcal{H} : 16x^2 - 9y^2 = 144$ has foci $S(5, 0)$ and $S'(-5, 0)$.

The directrices are $x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

(i) Find the equation of each asymptote of $\mathcal{H}$.

(ii) Sketch $\mathcal{H}$ and indicate on your diagram the foci, directrices, and asymptotes.

(iii) By differentiation, find the gradient of the tangent to $\mathcal{H}$ at $P(3\sec\theta, 4\tan\theta)$.

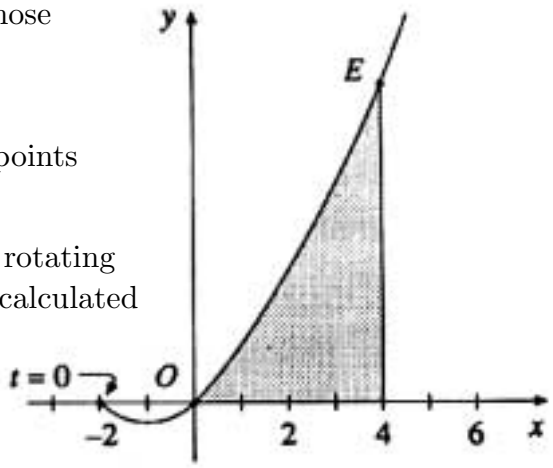
(iv) Show that the tangent to $\mathcal{H}$ at P has equation $4x = (3\sin\theta)y + 12\cos\theta$.

(v) Given that $0 < \theta < \frac{\pi}{2}$, show that Q, the point of intersection of the tangent to $\mathcal{H}$ at P and the nearer directrix, has y coordinate $\frac{12-20\cos\theta}{5\sin\theta}$.

(vi) Calculate the gradient of SP and SQ.

(vii) Determine whether $\angle PSQ$ is a right angle.

(b) The graph shows part of the curve whose parametric equations are
$x = t^2 + t - 2, y = t^3 - t, t \geq 0$.



(i) Find the values of t corresponding to points O and E on the curve.

(ii) The volume V of the solid formed by rotating the shaded area about the y axis is to be calculated using cylindrical shells.

Express V in the form
$V = 2\pi \int_a^b f(t)\ dt$.

Specify the limits of integration a and b and function $f(t)$. You may leave $f(t)$ in unexpanded form.
Do NOT evaluate this integral.

4. (a) Find α and β, given that $z^3 + 3z + 2i = (z - \alpha)^2(z - \beta)$.

(b) Alex decides to go bungy-jumping. This involves being tied to a bridge at a point O by an elastic cable of length l metres, and then falling vertically from rest from this point.

After Alex free-falls l metres, she is slowed down by the cable, which exerts a force, in newtons, of Mgk times the distance greater than l that she has fallen (where M is her mass in kilograms, g m/s^2 is the constant acceleration due to gravity, and k is a constant).

Let x m be the distance Alex has fallen, and let v m/s be her speed at x. You may assume that her acceleration is given by $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

(i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g$ when $x \leq l$,

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x - l)$ when $x > l$.

(ii) Show that $v^2 = 2gl$ when Alex first passes $x = l$.

(iii) Show that $v^2 = 2gx - kg(x - l)^2$ for $x > l$.

(iv) Show that Alex's fall is halted first at $x = l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \left(\frac{1}{k}\right)^2}$.

(v) Suppose $\frac{1}{k} = \frac{l}{4}$.

Show that O must be at least $2l$ metres above any obstruction on Alex's path.

5. (a) Let $f(x) = \frac{(x-2)(x+1)}{5-x}$ for $x \neq 5$.

(i) Show that $f(x) = -x - 4 + \frac{18}{5-x}$.

(ii) Explain why the graph of $y = f(x)$ approaches that of $y = -x - 4$ as $x \to \infty$ and as $x \to -\infty$.

(iii) Find the values of x for which $f(x)$ is positive, and the values of x for which $f(x)$ is negative.

(iv) Using part **(i)**, show that the graph of $y = f(x)$ has two stationary points. (There is no need to find the y coordinates of the stationary points.)

(v) Sketch the curve $y = f(x)$. Label all asymptotes, and show the x intercepts.

(b) A jar contains w white and r red jellybeans. Three jellybeans are taken at random from the jar and eaten.

(i) Write down an expression, in terms of w and r, for the probability that these 3 jellybeans were white.

Gary observed that if the jar had initially contained $(w + 1)$ white and r red jellybeans, then the probability that the 3 eaten jellybeans were white would have been double that in part **(i)**.

(ii) Show that $r = \frac{w^2 - w - 2}{5 - w}$.

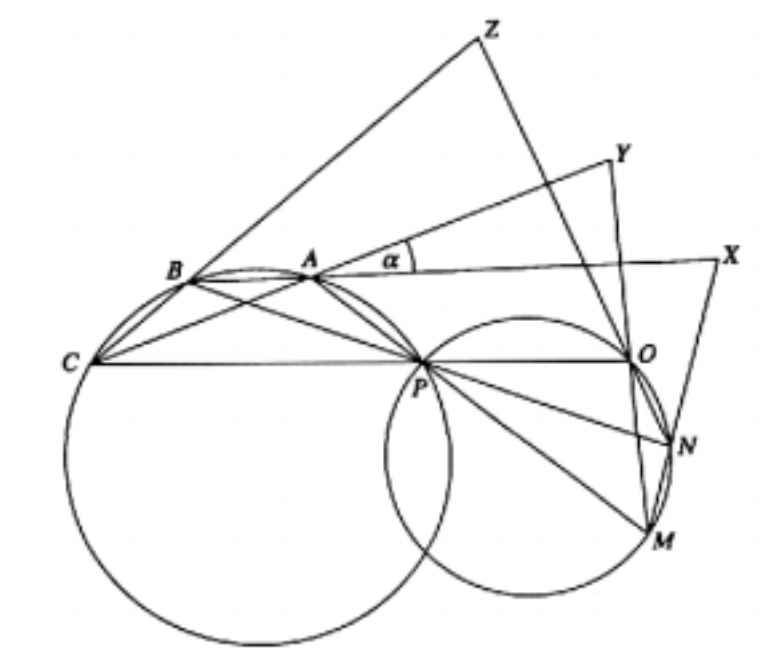
(iii) Using part **(a) (v)**, or otherwise, determine all possible numbers of white and red jellybeans.

6. (a) (i) Given that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, explain why
$\int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx < \int_0^{\frac{\pi}{2}} e^{-2x/\pi}\, dx$.

(ii) Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\, dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx$.

(iii) Hence show that $\int_0^{\pi} e^{-\sin x}\, dx < \frac{\pi}{e}(e - 1)$.

(b)



Circles $PABC$ and $PMNO$ intersect at P, and APM, BPN, and CPO are straight lines. BA and MN produced meet at X, CA and MO produced meet at Y, and CB and NO produced meet at Z, as in the diagram. Let $\angle YAX = \alpha$.

(i) Prove that $\angle BPC = \alpha$.

(ii) Prove that $\angle OMN = \alpha$.

(iii) Prove that $XYAM$ is a cyclic quadrilateral.

(iv) Prove that $\angle XYM = \angle BCP$.

(v) Prove that X, Y, and Z are collinear.

7. (a) (i) It is known that if $f'(x) \geq 0$ and $f(0) = 0$, then $f(x) \geq 0$ for $x > 0$.

Show that $\sin x - x + \frac{x^3}{6} \geq 0$ for $x > 0$, and hence show that $\frac{\sin x}{x} \geq 1 - \frac{x^2}{6}$ for $x > 0$.

Let the points $A_0, A_1, A_2, \ldots, A_{n-1}$ represent the nth roots of unity on an Argand diagram, where A_k represents $\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$.

Let P be the regular polygon $A_0 A_1 \cdots A_{n-1}$.

(ii) Show that the area of P is $\frac{n}{2} \sin \frac{2\pi}{n}$.

(iii) Using part **(i)**, or otherwise, show that for all $n \geq 26, P$ covers more than 99% of the unit circle.

(b) For all integers $n \geq 1$, let $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$.

That is: $t_1 = \frac{1}{2}$

$t_2 = \frac{1}{3} + \frac{1}{4}$

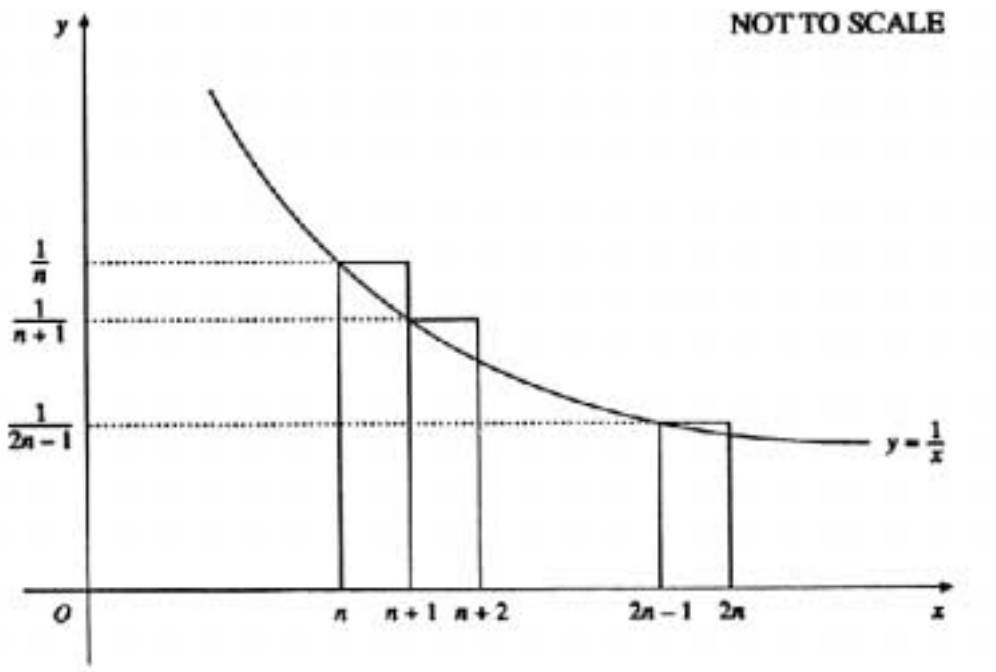
$t_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

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(i) Show that $t_n + \frac{1}{2n} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$

(ii)



The diagram above shows the graph of the function $y = \frac{1}{x}$ for $n \leq x \leq 2n$.

Use the diagram to show that $t_n + \frac{1}{2n} > \ln 2$.

[Note that it can similarly be shown that $t_n < \ln 2$.]

(iii) For all integers $n \geq 1$, let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$.

That is: $s_1 = 1 - \frac{1}{2}$

$s_2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$s_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

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Prove by mathematical induction that $s_n = t_n$.

(iv) Hence find, to three decimal places, the value of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9999} - \frac{1}{10000}$.

8. (a) Suppose $a > 0, b > 0, c > 0$.

(i) Prove that $a^2 + b^2 \geq 2ab$.

(ii) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(iii) Given $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
prove that $a^3 + b^3 + c^3 \geq 3abc$.

In parts **(iv)** and **(v)**, assume $x > 0, y > 0, z > 0$.

(iv) By making suitable substitutions into **(iii)**, show that $x + y + z \geq 3(xyz)^{1/3}$.

(v) Suppose $(1 + x)(1 + y)(1 + z) = 8$. Prove that $xyz \leq 1$.

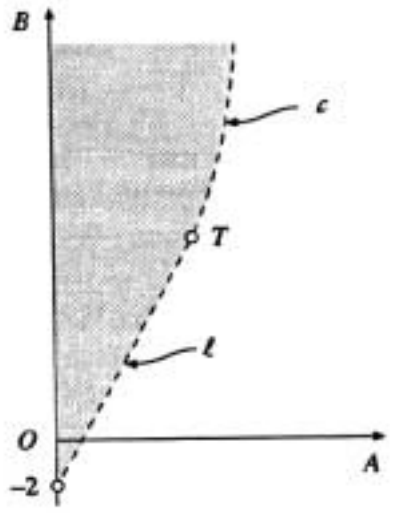
(b) Let $x = \alpha$ be a root of the quartic polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$, where A and B are real. Note that α may be complex.

(i) Show that $\alpha \neq 0$.

(ii) Show that $x = \alpha$ is a root of $Q(x) = x^2 + \frac{1}{x^2} + A\big(x + \frac{1}{x}\big) + B$.

(iii) With $u = x + \frac{1}{x}$, show that $Q(x)$ becomes $R(u) = u^2 + Au + (B - 2)$.

(iv) For certain values of A and B, $P(x)$ has no real roots. Let $\mathcal{D}$ be the region of the AB plane where $P(x)$ has no real roots and $A \geq 0$.



The region $\mathcal{D}$ is shaded in the figure. Specify the bounding straight-line segment ℓ and the curved segment c. Determine the coordinates of T.