NSW HSC 4 Unit Mathematics Examination 1993

1. (a) Evaluate $\int_3^8 \frac{x}{(x+1)-\sqrt{x+1}}\,dx$ by using the substitution $x+1=u^2$.

(b) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ by using the substitution $t=\tan\frac{\theta}{2}$.

(c) Evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

(d) (i) Find real numbers a, b and c such that $\frac{4x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$.

(ii) Hence find $\int \frac{4x+3}{(x^2+1)(x+2)}\,dx$.

2. (a) (i) On an Argand diagram shade in the region determined by the inequalities

$$2 \le \Im(z) \le 4 \text{ and } \frac{\pi}{6} \le \arg z \le \frac{\pi}{4}.$$

(ii) Let z_0 be the complex number of maximum modulus satisfying the inequalities of **(i)**. Express z_0 in the form $a+ib$.

(b) Let θ be a real number and consider $(\cos\theta + i\sin\theta)^3$.

(i) Prove $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^3\theta$.

(ii) Find a similar expression for $\sin 3\theta$.

(c) Find the equation, in Cartesian form, of the locus of the point z if $\Re\left(\frac{z-4}{z}\right)=0$.

(d) By substituting appropriate values of z_1 and z_2 into the equation $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$ show that $\frac{\pi}{4} = \tan^{-1}2 - \tan^{-1}\frac{1}{3}$.

(e) Let P, Q and R represent the complex numbers w_1, w_2, and w_3 respectively. What geometric properties characterize triangle PQR if $w_2 - w_1 = i(w_3 - w_1)$? Give reasons for your answer.

3. (a) Consider the point $P\left(ct, \frac{c}{t}\right)$, where $t \neq \pm 1$, which lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent to the hyperbola at P is $x + t^2y = 2ct$.

(ii) Let the tangent to the hyperbola at P intersect the coordinate axes at A and B. Show that $PA = PB$.

(iii) Let the normal to the hyperbola at P meet the axes of symmetry of the hyperbola at C and D. Show that $PC = PD = PA$.

[You may assume that the equation of the normal is $t^3x - ty = c(t^4 - 1)$.]

(iv) Sketch a graph of the hyperbola showing the results proved so far.

(v) Explain why A, B, C and D must be the vertices of a square.

(b) Let $R(x_0, y_0), P(x_1, y_1)$, and $Q(x_2, y_2)$ be points on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

(i) If d is the distance between the points R and P, show that

$$-\frac{d^2}{2} = x_0x_1 + y_0y_1 + g(x_0 + x_1) + f(y_0 + y_1) + c.$$

(ii) Suppose Q is also distance d from R. Explain why the equation of the chord PQ is

$$-\frac{d^2}{2} = x_0x + y_0y + g(x_0 + x) + f(y_0 + y) + c.$$

4. (a) Let $f(x) = \frac{1-x}{x}$. On separate diagrams sketch the graphs of the following functions.

For each graph label any asymptote.

(i) $y = f(x)$ **(ii)** $y = f(|x|)$ **(iii)** $y = e^{f(x)}$ **(iv)** $y^2 = f(x)$.

Discuss the behaviour of the curve of **(iv)** at $x = 1$.

(b) Suppose k is a constant greater than 1. Let $f(x) = \frac{1}{1+(\tan x)^k}$ where $0 \le x \le \frac{\pi}{2}$.

[You may assume $f\left(\frac{\pi}{2}\right) = 0$.]

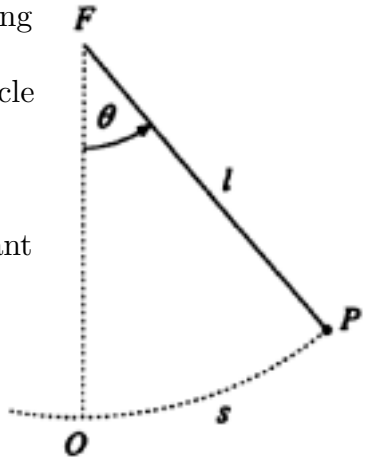
(i) Show that $f(x) + f\left(\frac{\pi}{2} - x\right) = 1$ for $0 \le x \le \frac{\pi}{2}$.

(ii) Sketch $y = f(x)$ for $0 \le x \le \frac{\pi}{2}$.

[There is no need to find $f'(x)$ but assume $y = f(x)$ has a horizontal tangent at $x = 0$. Your graph should exhibit the property of **(b) (i)**.]

(iii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^k}$.

5. (a) The diagram shows a simple pendulum consisting of a particle P, of mass m kg, which is attached to a fixed point F by a string of length l metres. The particle P moves along a circular arc in a fixed vertical plane through F. The point O is the lowest point of the arc, $\angle OFP = \theta$, and the arc length $OP = s$ metres. The time t is measured in seconds and g m/s^2 is the constant gravitational acceleration.



(i) Show that the tangential acceleration of P is given by $\frac{d^2s}{dt^2} = l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$, where $\dot{\theta} = \frac{d\theta}{dt}$.

(ii) Show that the equation of motion of the pendulum is $l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = -g\sin\theta$.

(iii) Suppose the pendulum is given an initial angular velocity of $\sqrt{\frac{g}{l}}$ radians/second at $\theta = 0$.

Show that $\frac{1}{2}l\dot{\theta}^2 = g\left(\cos\theta - \frac{1}{2}\right)$.

Hence deduce that the maximum value of θ attained by the pendulum is $\frac{\pi}{3}$.

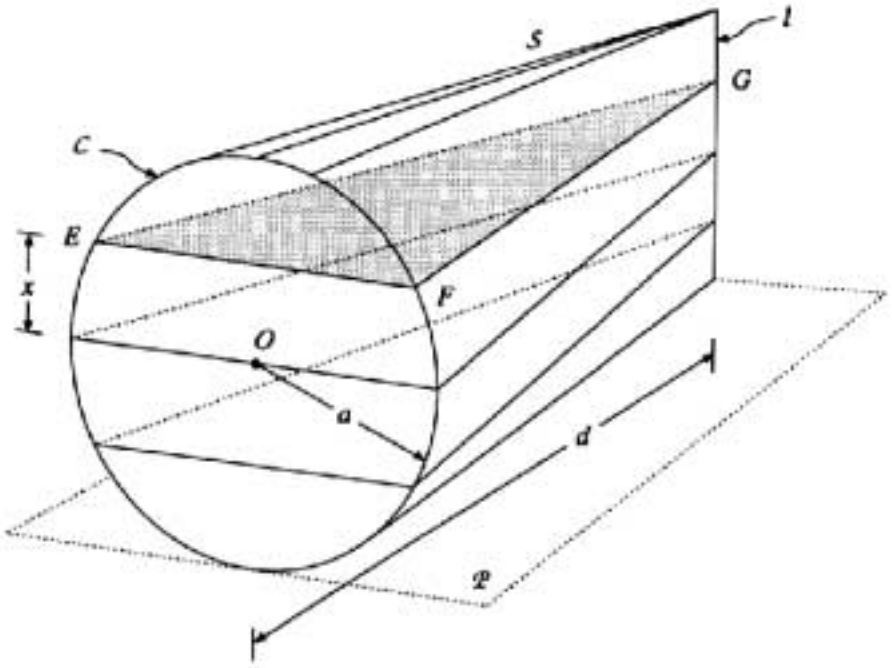
(iv) Suppose that on the initial upward swing the angular velocity is better approximated from the equation $\frac{1}{2}l\dot{\theta}^2 = g\left(\cos\theta - \frac{1}{2}\right) - \frac{g}{10}(2\sin\theta - \theta)$.

Use one application of Newton's method to find the maximum value of θ attained by the pendulum. Take $\theta = \frac{\pi}{3}$ as the first approximation.

(b) (i) If a is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(a) = 0$.

(ii) Find all roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$ if two of the roots are equal.

6. (a)

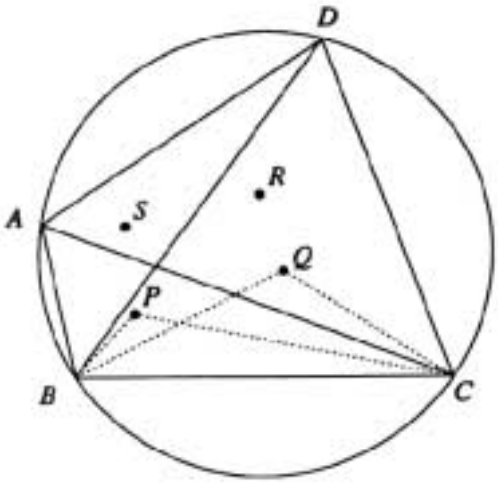


The solid $\mathcal{S}$ is generated by moving a straight edge so that it is always parallel to a fixed plane $\mathcal{P}$. It is constrained to pass through a circle $\mathcal{C}$ and line segment l. The circle $\mathcal{C}$, which forms a base for $\mathcal{S}$, has radius a and the line segment l is distance d from $\mathcal{C}$. Both $\mathcal{C}$ and l are perpendicular to $\mathcal{P}$ and sit on $\mathcal{P}$ in such a way that the perpendicular to $\mathcal{C}$ at its centre O bisects l.

(i) Calculate the area of the triangular cross-section EFG which is parallel to $\mathcal{P}$ and distance x from the centre O of $\mathcal{C}$.

(ii) Calculate the volume of $\mathcal{S}$.

(b) In the diagram $ABCD$ is a cyclic quadrilateral and P, Q, R and S are the incentres of triangles ABC, BCD, CDA and DAB respectively. The incentre of a triangle is the point of intersection of the bisectors of its three angles. Thus, for example, BP bisects $\angle ABC$ and CP bisects $\angle ACB$; similarly BQ bisects $\angle DBC$ and CQ bisects $\angle DCB$.



(i) Copy the diagram

(ii) Prove that $\angle PBQ (= \angle PBC - \angle QBC) = \frac{1}{2}\angle ABD$.

(iii) Prove that $\angle PCQ = \angle PBQ$ and hence explain why $BCQP$ must be a cyclic quadrilateral.

(iv) Prove that $\angle SPQ = \angle BAS + \angle BCQ$.

(v) Deduce that $SPQR$ is a rectangle.

7. (a) In the next 7 days, called day 1, day 2, ... , day 7, Esther and George must each take 3 days in a row off work. They choose their consecutive 3 days randomly and independently of each other.

(i) Show that the probability that they both have day 1 off together is $\frac{1}{25}$.

(ii) What is the probability that day 2 is the first day that they both have off together?

(iii) Find the probability that Esther and George have at least one day off together.

(b) For $n = 1, 2, 3, \ldots$, let $s_n = 1 + \sum_{r=1}^{n} \frac{1}{r!}$.

(i) Prove by mathematical induction that $e - s_n = e \int_0^1 \frac{x^n}{n!} e^{-x}\, dx$.

(ii) From **(i)**, deduce that $0 < e - s_n < \frac{3}{(n+1)!}$ for $n = 1, 2, 3, \ldots$.

[Remember that $e < 3$ and $e^{-x} \leq 1$ for $x \geq 0$.]

(iii) By using the inequality of **(ii)**, deduce that $(e - s_n)n!$ is not an integer for $n = 2, 3, 4, \ldots$.

(iv) Show that there cannot exist positive integers p and q such that $e = \frac{p}{q}$.

8. (a) Let the points $A_1, A_2, \ldots, A_n$ represent the nth roots of unity, $w_1, w_2, \ldots, w_n$, and suppose P represents any complex number z such that $|z| = 1$.

(i) Prove that $w_1 + w_2 + \cdots + w_n = 0$.

(ii) Show that $PA_i^2 = (z - w_i)(\overline{z} - \overline{w}_i)$ for $i = 1, 2, \ldots, n$.

(iii) Prove that $\sum_{i=1}^{n} PA_i^2 = 2n$.

(b) Let $f(x) = 1 + x^2$ and let x_1 be a real number. For $n = 1, 2, 3, \ldots$, define $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

[You may assume that $f'(x_n) \neq 0$.]

(i) Show that $|x_{n+1} - x_n| \geq 1$ for $n = 1, 2, 3, \ldots$.

(ii) Graph the function $y = \cot\theta$ for $0 < \theta < \pi$.

(iii) Using your graph from **(ii)**, show that there exists a real number θ_n such that $x_n = \cot\theta_n$ where $0 < \theta_n < \pi$.

(iv) Deduce that $\cot\theta_{n+1} = \cot 2\theta_n$ for $n = 1, 2, 3, \ldots$.

[You may assume that $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.]

(v) Find all points x_1 such that, for some $n, x_1 = x_{n+1}$.