NSW HSC 4 Unit Mathematics Examination 1992

1. (a) Find: **(i)** $\int \tan\theta \sec^2\theta \; d\theta$ **(ii)** $\int \frac{2x+6}{x^2+6x+1} \; dx$.

(b) Evaluate $\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{dx}{\sqrt{(x-1)(3-x)}}$ by using the substitution $u = x - 2$.

(c) Evaluate $\int_0^1 \frac{5(1-t)}{(t+1)(3-2t)} \; dt$.

(d) (i) Find $\int xe^x \; dx$ **(ii)** Evaluate $\int_0^1 2x^3 e^{x^2} \; dx$.

2. (a) The points A and B represent the complex numbers $3 - 2i$ and $1 + i$ respectively.

(i) Plot the points A and B on an Argand diagram and mark the point P such that $OAPB$ is a parallelogram.

(ii) What complex number does P represent?

(b) Let $z = a + ib$ where $a^2 + b^2 \neq 0$.

(i) Show that if $\Im(z) > 0$ then $\Im\left(\frac{1}{z}\right) < 0$.

(ii) Prove that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.

(c) Describe and sketch the locus of those points z such that:

(i) $|z - i| = |z + i|$ **(ii)** $|z - i| = \sqrt{2}\,|z + i|$.

(d) It is given that $1 + i$ is a root of $P(z) = 2z^3 - 3z^2 + rz + s$ where r and s are real numbers.

(i) Explain why $1 - i$ is also a root of $P(z)$.

(ii) Factorize $P(z)$ over the real numbers.

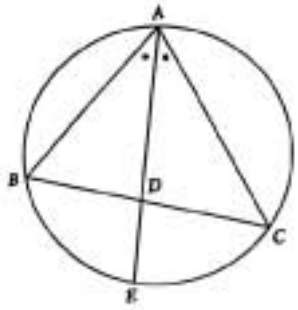
3. (a) The ellipse $\mathcal{E}$ has equation $\frac{x^2}{100} + \frac{y^2}{75} = 1$.

(i) Sketch he curve $\mathcal{E}$, showing on your diagram the coordinates of the foci and the equation of each directrix.

(ii) Find the equation of the normal to the ellipse at the point $P(5, 7.5)$.

(iii) Find the equation of the circle that is tangential to the ellipse at P and $Q(5, -7.5)$.

(b) In the diagram, the bisector AD of $\angle BAC$ has been extended to intersect the circle ABC at E.



Copy the diagram.

(i) Prove that the triangles ABE and ADE are similar.

(ii) Show that $AB.AC = AD.AE$.

(iii) Prove that $AD^2 = AB.AC - BD.DC$.

4. (a) Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. (You are not asked to find the primitive functions.)

(i) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \; d\theta = 0$ **(ii)** $\int_0^{\pi} \sin^7\theta \; d\theta = 0.$

(iii) $\int_{-1}^{1} e^{-x^2} \; dx = 0$ **(iv)** $\int_0^{\frac{\pi}{2}} (\sin^8\theta - \cos^8\theta) \; d\theta = 0$

(v) For $n = 1, 2, 3, \ldots, \int_0^1 \frac{dt}{1+t^n} \leq \int_0^1 \frac{dt}{1+t^{n+1}}$.

(b) Let $f(x) = \ln(1+x) - \ln(1-x)$ where $-1 < x < 1$.

(i) Show that $f'(x) > 0$ for $-1 < x < 1$.

(ii) On the same diagram sketch $y = \ln(1+x)$ for $x > -1$.

$y = \ln(1-x)$ for $x < 1$.

and $y = f(x)$ for $-1 < x < 1$.

Clearly label the three graphs.

(iii) Find an expression for the inverse function $y = f^{-1}(x)$.

5. (a) The solid $\mathcal{S}$ is a rectangular prism of dimensions $a \times a \times 2a$ from which right square pyramids of base $a \times a$ and height a have been removed from each end. The solid $\mathcal{T}$ is a wedge that has been obtained by slicing a right circular cylinder of radius a at 45° through a diameter AB of its base.

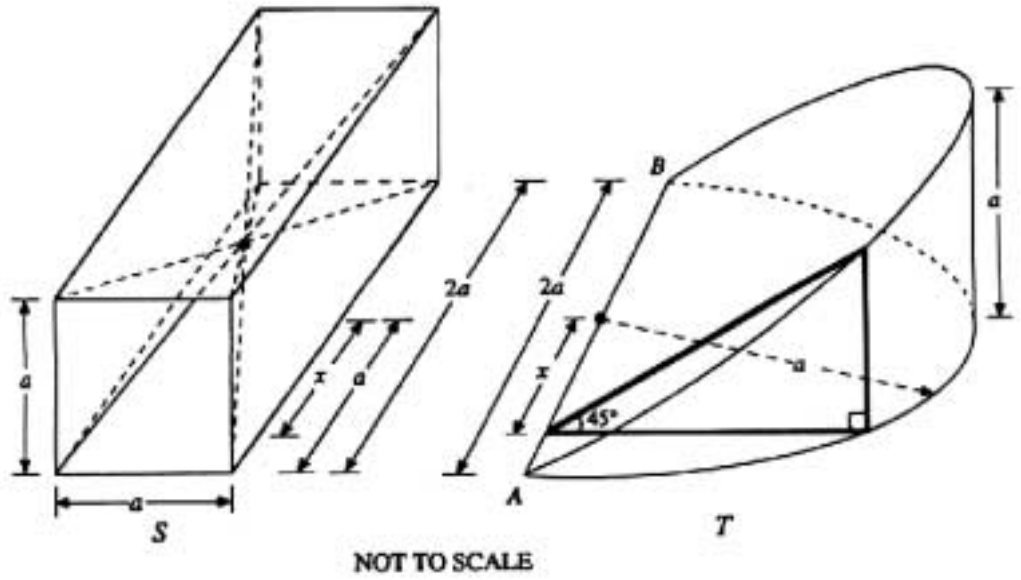
Consider a cross-section of $\mathcal{S}$ which is parallel to its square base at distance x from its centre, and a corresponding cross-section of $\mathcal{T}$ which is perpendicular to AB and at distance x from its centre.

(i) The triangular cross-section of $\mathcal{T}$ is shown on the diagram on the next page. Show that it has area $\frac{1}{2}(a^2 - x^2)$.

(ii) Draw the cross-section of $\mathcal{S}$ and calculate its area.

(iii) Express the volumes of $\mathcal{S}$ and $\mathcal{T}$ as definite integrals.

(iv) What is the relationship between the volumes of $\mathcal{S}$ and $\mathcal{T}$? (There is no need to evaluate either integral.)



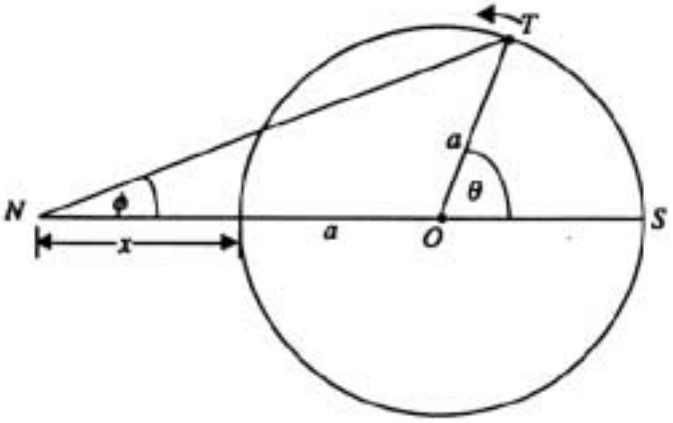
(b) An object is fired vertically upwards with initial speed 400m/s from the surface of the Earth.

Assume that the acceleration due to gravity at height x above the Earth's surface is $10/\left[1+\frac{x}{R}\right]^2$ m/s^2 where the radius of the Earth, $R = 6.4 \times 10^6$ m.

(i) Show that $\frac{d}{dx}(\frac{1}{2}v^2) = -10/\left[1+\frac{x}{R}\right]^2$ where v is the speed of the object at height x. (Neglect air resistance.)

(ii) Calculate the maximum height the object reaches. Give your answer to the nearest metre.

6. (a)



The diagram shows a model train $\mathcal{T}$ that is moving around a circular track, centre O and radius a metres. The train is moving at a constant speed of u m/s. The point N is in the same plane as the track and is x metres from the nearest point on the track. The line NO produced meets the track at S.

Let $\angle TNS = \phi$ and $\angle TOS = \theta$ as in the diagram.

(i) Express $\frac{d\theta}{dt}$ in terms of a and u.

(ii) Show that $a\sin(\theta - \phi) - (x + a)\sin\phi = 0$. and deduce that $\frac{d\phi}{dt} = \frac{u\cos(\theta-\phi)}{(x+a)\cos\phi + a\cos(\theta-\phi)}$

(iii) Show that $\frac{d\phi}{dt} = 0$ when NT is tangential to the track.

(iv) Suppose that $x = a$.

Show that the train's angular velocity about N when $\theta = \frac{\pi}{2}$ is $\frac{3}{5}$ times the angular velocity about N when $\theta = 0$.

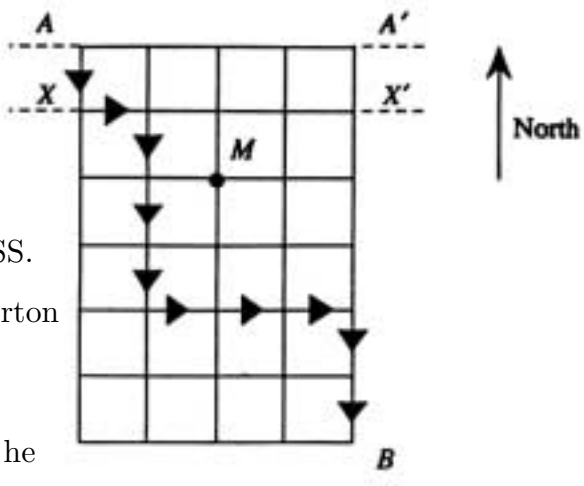
(b) Let n be an integer with $n \geq 2$.

(i) For $i = 1, 2, \ldots, n$ suppose x_i is a real number satisfying $0 < x_i < \pi$.

Use mathematical induction to show that there exist real numbers $a_1, a_2, \ldots, a_n$ such that $|a_i| \leq 1$ for $i = 1, 2, \ldots, n$, and such that $\sin(x_1 + x_2 + \cdots + x_n) = a_1\sin x_1 + a_2\sin x_2 + \cdots + a_n\sin x_n$.

(ii) Deduce that $\sin nx \leq n\sin x$ whenever $0 < x < \pi$.

7. (a) The diagram shows the road grid of a city.



Ayrton drives exactly 10 blocks from his home, A, to his workplace, B, which is 6 blocks south (S) and 4 blocks east (E). The route on the diagram is SESSSEEESS.

(i) By how many different routes can Ayrton drive to work?

(ii) By how many different routes can Ayrton drive to work on those days that he wishes to stop at the shop marked M?

(iii) The street marked AA' is made one-way westward. How many different routes can Ayrton follow if he cannot drive along AA'?

(iv) Suppose that instead of AA' the street marked XX' is made one-way westward. How many different routes can Ayrton follow if he cannot drive along XX'?

(b) Suppose that $z^7 = 1$ where $z \neq 1$.

(i) Deduce that $z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0$.

(ii) By letting $x = z + \frac{1}{z}$ reduce the equation in **(i)** to a cubic equation in x.

(iii) Hence deduce that $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{8}$.

8. (a) Consider the function $f(x) = e^x\left(1 - \frac{x}{10}\right)^{10}$.

(i) Find the turning points of the graph of $y = f(x)$.

(ii) Sketch the curve $y = f(x)$ and label the turning points and any asymptotes.

(iii) From your graph deduce that $e^x \leq \left(1 - \frac{x}{10}\right)^{-10}$ for $x < 10$.

(iv) Using **(iii)**, show that $\left(\frac{11}{10}\right)^{10} \leq e \leq \left(\frac{10}{9}\right)^{10}$.

(b) Let n be a positive integer and let x be any positive approximation to $\sqrt{n}$. Choose y so that $xy = n$.

(i) Prove that $\frac{x+y}{2} \geq \sqrt{n}$.

(ii) Suppose that $x > \sqrt{n}$.

Show that $\frac{x+y}{2}$ is a closer approximation to $\sqrt{n}$ than x is.

(iii) Suppose $x < \sqrt{n}$.

How large must x be in terms of n for $\frac{x+y}{2}$ to be a closer approximation to $\sqrt{n}$ than x is?