NSW HSC 4 Unit Mathematics Examination 1991

1. (a) Find: **(i)** $\int \frac{t-1}{t^3}\,dt$ **(ii)** $\int \frac{e^x}{e^{2x}+9}\,dx$, using the substitution $u = e^x$.

(b) (i) Evaluate $\int_0^1 \frac{5}{(2t+1)(2-t)}\,dt$.

(ii) By using the substitution $t = \tan\frac{\theta}{2}$ and **(i)**, evaluate $\int_0^{\pi/2} \frac{d\theta}{3\sin\theta+4\cos\theta}$.

(c) Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$ where n is a non-negative integer.

(i) Show that $I_n = (n-1)\int_0^{\pi/2} \sin^{n-2} x\cos^2 x\,dx$ when $n \geqq 2$.

(ii) Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ when $n \geqq 2$.

(iii) Evaluate I_4.

2. (a) Plot on an Argand diagram the points P, Q and R which correspond to the complex numbers $2i\sqrt{3} - i$, and $-\sqrt{3} - i$, respectively.

Prove that P, Q and R are the vertices of an equilateral triangle.

(b) Let $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_2 = \cos\theta_2 + i\sin\theta_2$, where θ_1 and θ_2 are real.

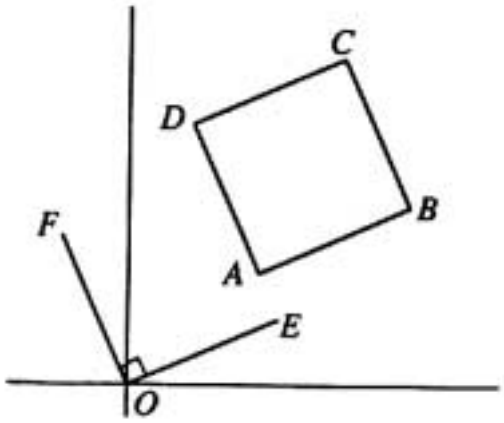
Show that: **(i)** $\frac{1}{z_1} = \cos\theta_1 - i\sin\theta_1$

(ii) $z_1 z_2 = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$.

(c) (i) Find all pairs of integers x and y such that $(x+iy)^2 = -3-4i$.

(ii) Using **(i)**, or otherwise, solve the quadratic equation $z^2 - 2z + (3+i) = 0$.

(d) In the Argand diagram, $ABCD$ is a square and OE and OF are parallel and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively.



(i) Explain why the point E corresponds to $w_2 - w_1$.

(ii) What complex number corresponds to the point F?

(iii) What complex number corresponds to the vertex D?

3. (a) The hyperbola $\mathcal{H}$ has equation $xy = 4$.

(i) Sketch $\mathcal{H}$ and indicate on your diagram the positions and coordinates of all points at which $\mathcal{H}$ intersects the axes of symmetry.

(ii) Show that the equation of the tangent to $\mathcal{H}$ at $P(2t, 2/t)$, where $t \neq 0$, is $x + t^2 y = 4t$.

(iii) If $s \neq 0$ and $s^2 \neq t^2$, show that the tangents to $\mathcal{H}$ at P and $Q(2s, 2/s)$ intersect at $M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$.

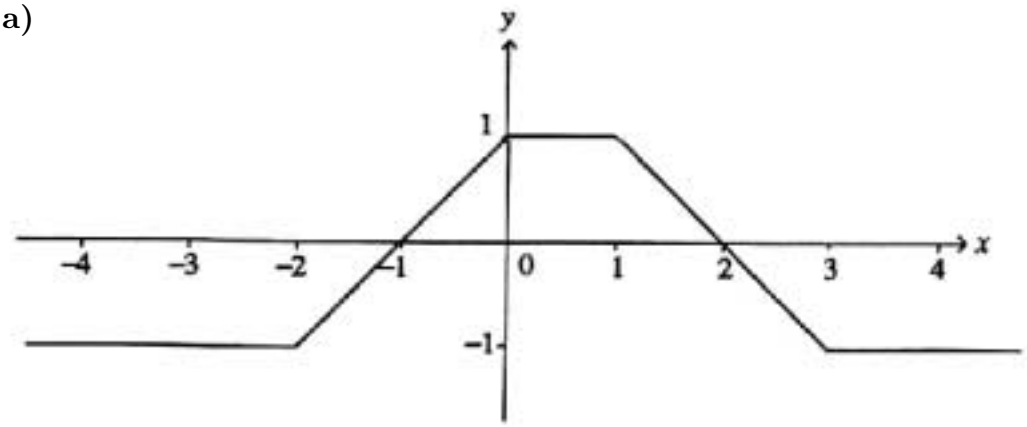
(iv) Suppose that in **(iii)** the parameter $s = -1/t$. Show that the locus of M is a straight line through, but excluding, the origin.

(b) Let $C_1 \equiv x^2 + 3y^2 - 1, C_2 \equiv 4x^2 + y^2 - 1$, and let λ be a real number.

(i) Show that $C_1 + \lambda C_2 = 0$ is the equation of a curve through the points of intersection of the ellipses $C_1 = 0$ and $C_2 = 0$.

(ii) Determine the values of λ for which $C_1 + \lambda C_2 = 0$ is the equation of an ellipse.

4. (a)



The diagram is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = -f(x)$ **(ii)** $y = |f(x)|$ **(iii)** $y = f(|x|)$ **(iv)** $y = \sin^{-1}(f(x))$.

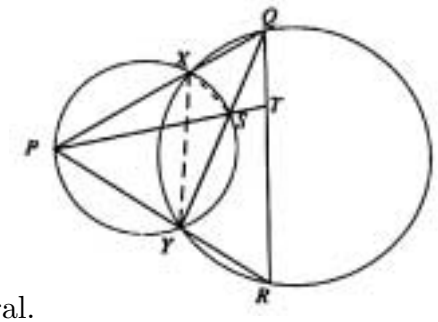
(b) The even function g is defined by $g(x) = \begin{cases} 4e^{-x} - 6e^{-2x} & \text{for } x \geqq 0; \\ g(-x) & \text{for } x < 0. \end{cases}$

(i) Show that the curve of $y = g(x)$ has a maximum turning point at $(\ln 3, 2/3)$.

(ii) Sketch the curve $y = g(x)$ and label the turning points, any points of inflexion, asymptotes, and points of intersection with the axes.

(iii) Discuss the behaviour of the curve $y = g(x)$ at $x = 0$.

5. (a) In the diagram, the circles $XPYS$ and $XYRQ$ intersect at the points X and Y, and PXQ, PYR, QSY, PST and QTR are straight lines.



(i) Explain why $\angle STQ = \angle YRQ + \angle YPS$.

(ii) Show that $\angle TRQ + \angle YPS + \angle SXQ = \pi$.

(iii) Deduce that $STQX$ is a cyclic quadrilateral.

(iv) Let $\angle QPY = \alpha$ and $\angle PQY = \beta$. Show that $\angle STQ = \alpha + \beta$.

(b) A drinking glass having the form of a right circular cylinder of radius a and height h, is filled with water. The glass is slowly tilted over, spilling water out of it, until it reaches the position where the water's surface bisects the base of the glass. Figure 1 shows this position.

Figure 1

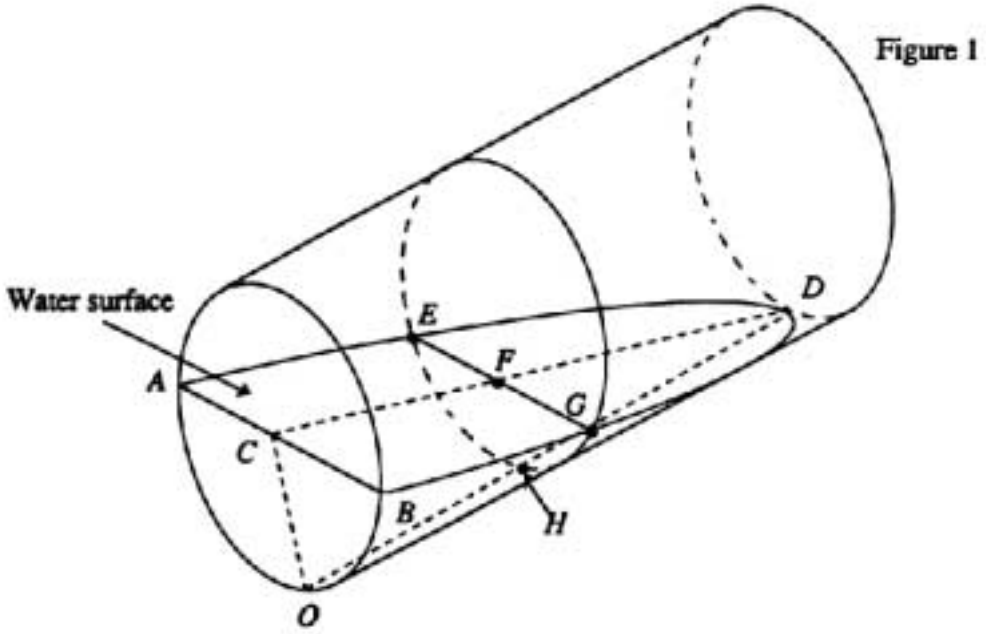
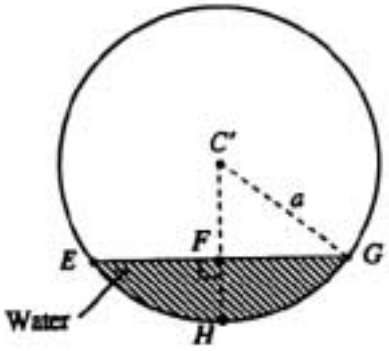
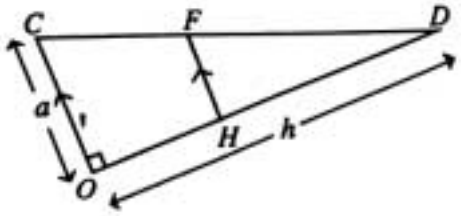


Figure 2



Note: $EG \perp C'H$ at F

Figure 3



Note: $FH \parallel CO$, $CO = a$, and $OD = h$

In Figure 1, AB is a diameter of the circular base with centre C, O is the lowest point on the base, and D is the point where the water's surface touches the rim of the glass.

Figure 2 shows a cross-section of the titled glass parallel to its base. The centre of this circular section is C' and EFG shows the water level. The section cuts the lines CD and OD of Figure 1 in F and H respectively.

Figure 3 shows the section COD of the tilted glass.

(i) Use Figure 3 to show that $FH = \frac{a}{h}(h - x)$, where $OH = x$.

(ii) Use Figure 2 to show that $C'F = \frac{ax}{h}$ and $\angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$.

(iii) Use **(ii)** to show that the area of the shaded segment EGH is $a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1 - \left(\frac{x}{h}\right)^2}\right]$.

(iv) Given that $\int \cos^{-1}\theta \; d\theta = \theta\cos^{-1}\theta - \sqrt{1-\theta^2}$, find the volume of water in the tilted glass of Figure 1.

6. (a) A nine-member Fund Raising Committee consists of four students, three teachers and two parents. The Committee meets around a circular table.

(i) How many different arrangements of the nine members around the table are possible if the students sit together as a group and so do the teachers, but no teacher sits next to a student?

(ii) One student and one parent are related. Given that all arrangements in **(i)** are equally likely, what is the probability that these two members sit next to each other?

(b) When a jet aircraft touches down, two different retarding forces combine to bring it to rest. If the aircraft has mass M kg and speed v m/s there is a constant frictional force of $M/4$ newtons and a force of $Mv^2/108$ newtons due to the reverse thrust of the engines. The reverse thrust of the engines does not take effect until 20 seconds after touchdown.

Let x be the distance in metres of the jet from its point of touchdown and let t be the time in seconds after touchdown.

(i) Show that $\frac{d^2x}{dt^2} = -\frac{1}{4}$ for $0 < t < 20$ and that for $t > 20$, and until the jet stops, $\frac{d^2x}{dt^2} = -\frac{1}{108}(27 + v^2)$.

(ii) Prove the identity $\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx}$.

(iii) If the jet's speed at touchdown is 60 m/s show that v=55 and $x = 1150$ at the instant the reverse thrust of the engines takes effect.

(iv) Show that when $t > 20, x = 1150 + 54\{\ln(27 + 55^2) - \ln(27 + v^2)\}$.

(v) Calculate how far from the touchdown point the jet comes to rest. Give your answer to the nearest metre.

7. (a) (i) By assuming that $\cos(A + B) = \cos A\cos B - \sin A\sin B$, prove the identity $\cos^3 A - \frac{3}{4}\cos A = \frac{1}{4}\cos 3A$.

(ii) Show that $x = 2\sqrt{2}\cos A$ satisfies the cubic equation $x^3 - 6x = -2$ provided $\cos 3A = -1/(2\sqrt{2})$.

(iii) Using **(ii)**, find the three roots of the equation $x^3 - 6x + 2 = 0$. Give your answers to four decimal places.

(b) Let $x = \alpha$ be a root of the quartic polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$ where $(2 + B)^2 \neq 4A^2$.

(i) Show that α cannot be $0, 1$ or -1.

(ii) Show that $x = 1/\alpha$ is a root.

(iii) Deduce that if α is a multiple root, then its multiplicity is 2 and $4B = 8 + A^2$.

8. (a) Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } 0 < x \leqq \pi/2; \\ 1 & \text{for } x = 0. \end{cases}$

(i) Find the derivative of f for $0 < x < \pi/2$ and prove that f' is negative in this interval.

(ii) Sketch the graph of $y = f(x)$ for $0 \leqq x \leqq \pi/2$ and deduce that $\sin x \geqq 2x/\pi$ in this interval.

(b) (i) Suppose $x > 0, y > 0$. Prove that $\frac{1}{x} + \frac{1}{y} \geqq \frac{4}{s}$ where $s = x + y$.

Hint: Consider $f(x) = \frac{1}{x} + \frac{1}{s-x}$.

(ii) Suppose $x_i > 0$ for $i = 1, 2, \ldots, n$ where $n \geqq 2$.

Show by mathematical induction, or otherwise, that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geqq \frac{n^2}{s}$ where s denotes the sum $x_1 + x_2 + \cdots + x_n$.