NSW HSC 4 Unit Mathematics Examination 1990

1. **(a)** Let $z = a + ib$, where a and b are real numbers and $a \neq 0$.

(i) Express $|z|$ and $\tan(\arg z)$ in terms of a and b.

(ii) Express $\frac{z}{3+5i}$ in the form $x + iy$, where x and y are real.

(b) **(i)** If $w = \frac{1+i\sqrt{3}}{2}$, show that $w^3 = -1$.

(ii) Hence calculate w^{10}.

(c) If $z = 5 - 5i$, write z, z^2 and $\frac{1}{\bar{z}}$ in modulus-argument form.

(d) Let u and v be two complex numbers, where $u = -2 + i$, and v is defined by $|v| = 3$ and $\arg v = \frac{\pi}{3}$.

(i) On an Argand diagram plot the points A and B representing the complex numbers u and v respectively.

(ii) Plot the points C and D represented by the complex numbers $u - v$ and iu, respectively.

Indicate any geometric relationships between the four point A, B, C, and D.

2. **(a)** Find the exact values of: **(i)** $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\, dx$ **(ii)** $\int_0^{\sqrt{2}} \sqrt{4-x^2}\, dx$.

(b) Find: **(i)** $\int \frac{dx}{(x+1)(x^2+2)}$ **(ii)** $\int \cos^3 x\, dx$, by writing $\cos^3 x = (1 - \sin^2 x)\cos x$, or otherwise.

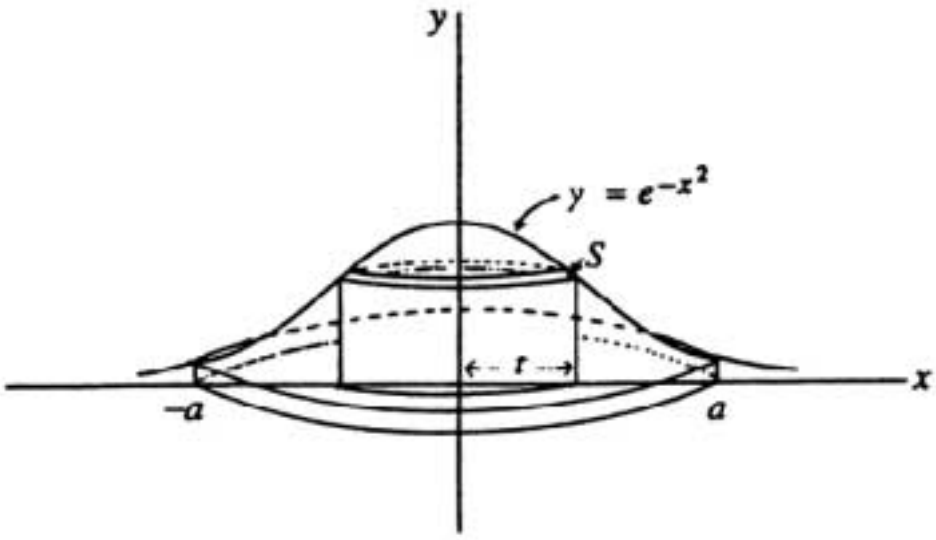
(c) Let $I_n = \int_0^x (1+t^2)^n\, dt, n = 1, 2, 3, \ldots$.

Use integration by parts to show that $I_n = \frac{1}{2n+1}(1+x^2)^n x + \frac{2n}{2n+1} I_{n-1}$.

Hint : Observe that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$.

3. The region under the curve $y = e^{-x^2}$ and above the x axis for $-a \leqq x \leqq a$ is rotated about the y axis to form a solid.

(i) Divide the resulting solid into cylindrical shells S of radius t as in the diagram.



Show that each shell S has approximate volume $\delta V = 2\pi t e^{-t^2}\,\delta t$ where δt is the thickness of the shell.

(ii) Hence calculate the volume of the solid.

(iii) What is the limiting value of the volume of the solid as a approaches infinity?

(b) Consider the functions f, g defined by $f(x) = \frac{x-1}{x+2}$ for $x \neq -2$,

$$g(x) = [f(x)]^2$$

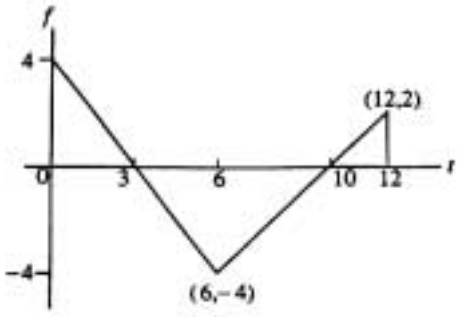
(i) Sketch the hyperbola $y = f(x)$, clearly labelling the horizontal and vertical asymptotes and the points of intersection with the x and y axes.

(ii) Find all turning points of $y = g(x)$.

(iii) Using the same diagram as used in **(i)**, sketch the curve $y = g(x)$ clearly labelling it.

(iv) On a separate diagram sketch the curve given by $y = g(-x)$.

4. (a) The diagram shows the graph of the function f,



where $f(t) = \begin{cases} 4 - \frac{4}{3}t, \\ \text{for } 0 \leqq t \leqq 6 \\ \\ t - 10, \\ \text{for } 6 \leqq t \leqq 12. \end{cases}$

The function F is defined for $0 \leqq x \leqq 12$ by $F(x) = \int_0^x f(t)\, dt$.

(i) Calculate $F(6)$ and $F(12)$.

(ii) Calculate those values of x for which $F(x) = 0$.

(iii) Find all turning points of F.

(b) Let $L_1 \equiv 4x - 5y + 1$ and $L_2 \equiv 2x + 3y - 5$.

(i) Find the point P of intersection of the two straight lines $L_1 = 0$ and $L_2 = 0$.

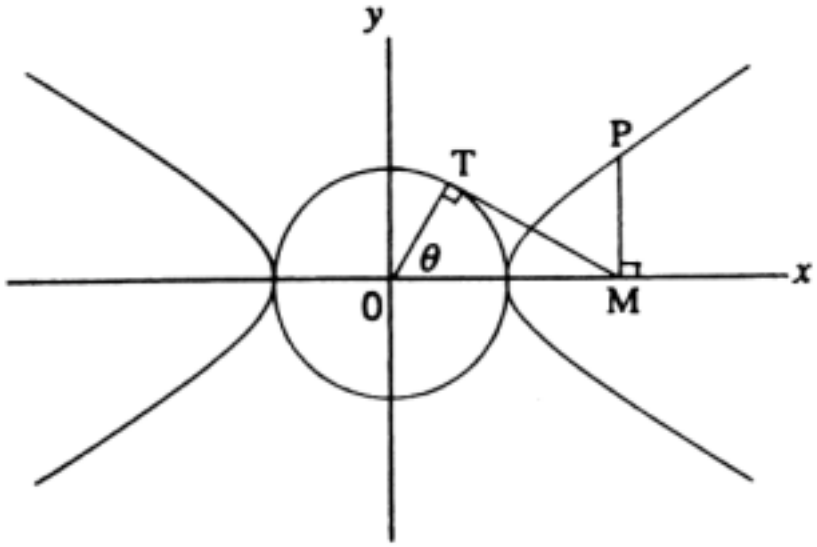
(ii) In the Cartesian plane draw the lines $L_1 = 0$ and $L_2 = 0, L_1 = 6, L_2 = 7$, marking the point P.

Explain why these four lines define a parallelogram.

(iii) If a and b are constants, not both zero, explain why $aL_1 + bL_2 = 0$ defines a straight line through P.

(iv) Using part **(iii)**, or otherwise, prove that the diagonal through P of the parallelogram defined in part **(ii)** has equation: $7L_1 - 6L_2 = 0$.

5. (a)



The figure shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$, where $a, b > 0$. The point T lies on the circle with $\angle TOx = \theta$, where $0 < \theta < \frac{\pi}{2}$.

The tangents to the circle at T meets the x axis at M; MP is perpendicular to Ox and P is a point on the hyperbola in the first quadrant.

(i) Show that P has coordinates $(a \sec\theta, b \tan\theta)$.

(ii) Suppose that Q is a point on the hyperbola with coordinates $(a \sec\phi, b \tan\phi)$.

If $\theta + \phi = \frac{\pi}{2}$, and $\theta \neq \frac{\pi}{4}$, show that the chord PQ has equation $ay = b(\cos\theta + \sin\theta)x - ab$.

(iii) Show that every such chord PQ passes through a fixed point and find its

coordinates.

(iv) Show that as θ approaches $\frac{\pi}{2}$ the chord PQ approaches a line parallel to an asymptote.

(b) (i) How many different five figure numbers can be formed from the digits 1,2,3,4,5 without repetition?

(ii) How many of these numbers are greater than 45321?

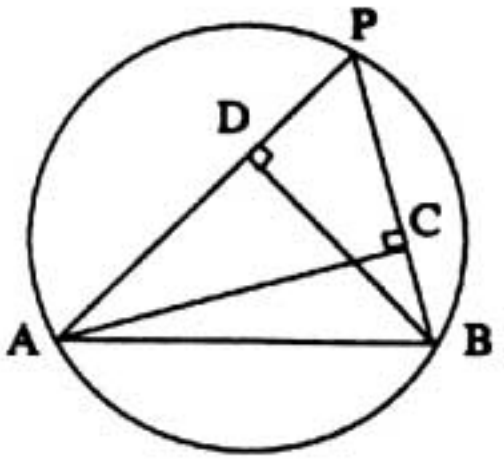
(iii) How many of these numbers are less than 45321?

6. (a) (i) Write down the relations which hold between the roots α, β, γ of the equation $ax^3 + bx^2 + cx + d = 0, (a \neq 0)$, and the coefficients a, b, c, d.

(ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$. You are given that the roots α, β, γ of this equation satisfy $\alpha = \beta + \gamma$. Use part **(i)** to find α.

(iii) Suppose that the equation $x^3 + px^2 + qx + r = 0$ has roots λ, μ, ν which satisfy $\lambda = \mu + \nu$. Show that $p^3 - 4pq + 8r = 0$.

(b) In the diagram, AB is a fixed chord of a circle, P a variable point in the circle and AC and BD are perpendicular to BP and AP respectively. Copy this diagram.



(i) Show that $ABCD$ is a cyclic quadrilateral on a circle with AB as diameter.

(ii) Show that the triangles PCD and APB are similar.

(iii) Show that as P varies, the segment CD has constant length.

(iv) Find the locus of the midpoint of CD.

7. (a) A mass of m kilograms falls from a stationary balloon at height h metres above the ground. It experiences air resistance during its fall equal to mkv^2, where v is its speed in metres per second and k is a positive constant.

Let x be the distance in metres of the mass from the balloon, measured positively as it falls.

(i) Show that the equation of motion of the mass is $\ddot{x} = g - kv^2$, where g is the acceleration due to gravity.

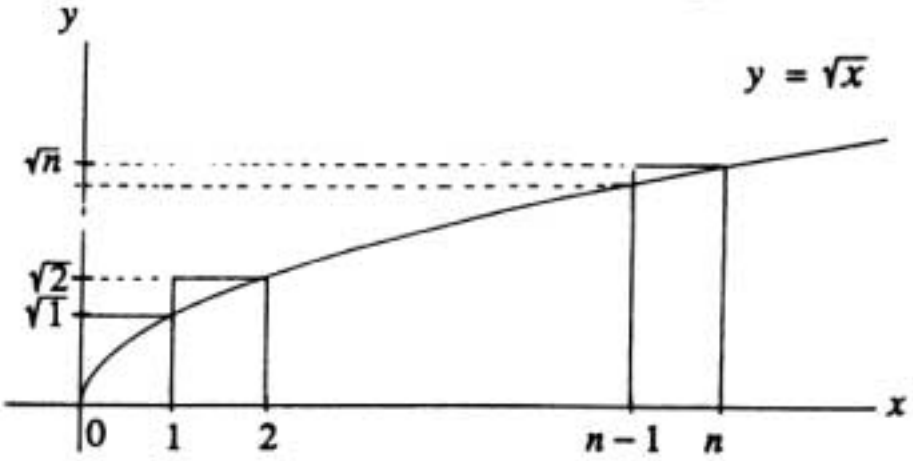
(ii) Find v^2 as a function of x. Hint: $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2) = v\frac{dv}{dx}$.

(iii) Find the velocity V as the mass hits the ground in terms of g, k and h.

(iv) Find the velocity of the mass as it hits the ground if air resistance is neglected.

(b)

figure not to scale



Consider the graph of the function $y = \sqrt{x}$.

(i) Show that this curve is increasing for all $x \geqq 0$.

(ii) Hence show that $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \geqq \int_0^n \sqrt{x}\, dx = \frac{2}{3}n\sqrt{n}$.

(iii) Use mathematical induction to show that $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \leqq \frac{4n+3}{6}\sqrt{n}$, for all integers $n \geqq 1$.

(iv) Use parts **(ii)** and **(iii)** to estimate $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{10\ 000}$ to the nearest hundred.

8. (a) You are given that $2\cos A \sin B = \sin(A + B) - \sin(A - B)$.

Let $S = 1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta$.

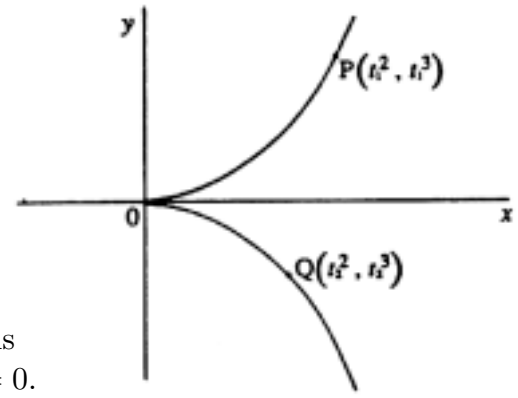
(i) Prove that $S \sin\frac{\theta}{2} = \sin\frac{7\theta}{2}$.

(ii) Hence show that if $\theta = \frac{2\pi}{7}$, then $1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta = 0$.

(iii) By writing S in terms of $\cos\theta$, prove that $\cos\frac{2\pi}{7}$ is a solution of the polynomial equation $8x^3 + 4x^2 - 4x - 1 = 0$.

(b) Consider the curve defined parametrically by $x = t^2, y = t^3$.

Let $P(t_1^2, t_1^3)$ and $Q(t_2^2, t_2^3)$ be two distinct points on the curve.



(i) Write down the equation of the curve in terms of x and y only.

(ii) Show that the equation of the chord PQ is given by $(t_1 + t_2)y - (t_1^2 + t_1t_2 + t_2^2)x + t_1^2t_2^2 = 0$.

(iii) Hence, or otherwise, show that the equation of the tangent to the curve at a point corresponding to t, where $t \neq 0$, is given by $2y - 3tx + t^3 = 0$.

(iv) Let $R(x_0, y_0)$ be a point in the plane such that $x_0^3 > y_0^2 > 0$. Prove that there are precisely 3 tangents from R to the curve and sketch this on a diagram.