NSW HSC 4 Unit Mathematics Examination 1989

1. (a) Evaluate $|2+3i|$.

(b) Given that a and b are real numbers, express in the form $x+iy$, where x and y are real:

(i) $(a+bi)(\overline{5+i})$; **(ii)** $\frac{a+bi}{3+4i}$.

(c) Find the complex square roots of $10-24i$, giving your answers in the form $x+iy$, where x and y are real.

(d) On an Argand diagram shade in the region containing all points representing complex numbers z such that $2 \leqq \Re(z) \leqq 4$ and $-1 \leqq \Im(z) \leqq 3$.

(e) Find in modulus-argument form all complex numbers z such that $z^3 = -1$ and plot them on an Argand diagram.

(f) On separate diagrams draw a neat sketch of the locus specified by:

(i) $\arg[z-(1+\sqrt{3}i)] = \frac{\pi}{3}$; **(ii)** $z^2 - z^{-2} = 16i$.

2. (a) Evaluate: **(i)** $\int_0^{\sqrt{3}} \frac{2}{x^2+9}\,dx$; **(ii)** $\int_1^3 x^2 \ln x\,dx$; **(iii)** $\int_{\sqrt{3}}^2 \sqrt{4-x^2}\,dx$.

(b) (i) Write $\frac{4x^2-5x-7}{(x-1)(x^2+x+2)}$ in the form $\frac{A}{x-1} + \frac{Bx+C}{x^2+x+2}$.

(ii) Hence evaluate $\int_{-1}^0 \frac{4x^2-5x-7}{(x-1)(x^2+x+2)}\,dx$.

3. (a) The ellipse $\mathcal{E} : \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse $\mathcal{E}$ indicating its foci S, S' and its dirctrices.

(ii) Show that the tangent at $P(x_1, y_1)$ on the ellipse $\mathcal{E}$ has equation $9x_1x + 25y_1y = 225$.

(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through S. Show that $4(y_2-y_1) = x_1y_2 - x_2y_1$.

(iv) It is also known that $Q(x_2, y_2)$ lies on $\mathcal{E}$. Show that the tangents at P and Q on the ellipse intersect on the directrix corresponding to S.

(v) Find the equation of the normal to $\mathcal{E}$ at P and decide under what circumstances, if any, it passes through S or S'.

(b) A public opinion survey of a certain parliamentary proposition finds 47% of the population in favour, 38% opposed and 15% undecided. Three persons are selected at random. Using the expansion

$$(p+q+r)^3 = p^3+q^3+r^3+3p^2q+3q^2r+3pq^2+3qr^2+3rp^2+3r^2p+6pqr$$

or otherwise, find the probability that:

(i) one person is in favour, one opposed and one is undecided;

(ii) exactly two persons are opposed;

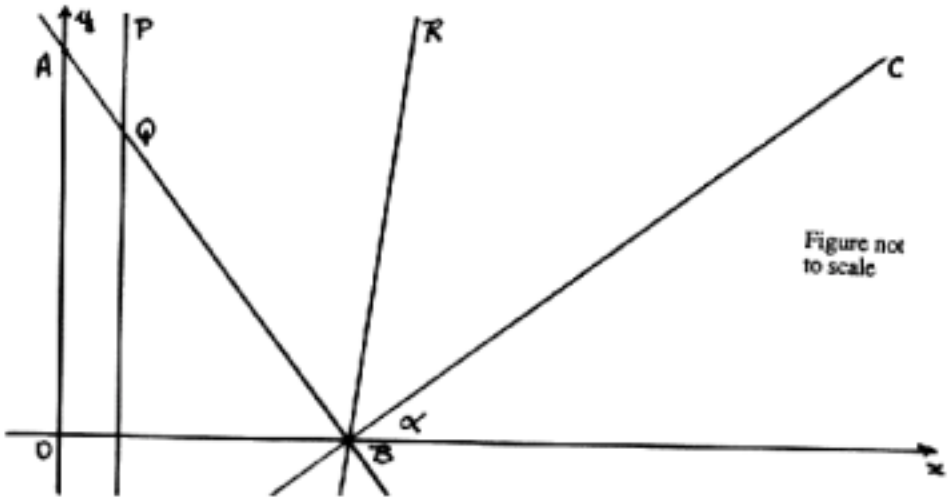
(iii) at least two persons are of the same opinion, either in favour, or opposed or undecided.

4. (a) (i) Write an expression for $\sin(\alpha+\beta), \cos(\alpha+\beta)$ in terms of $\sin\alpha, \sin\beta, \cos\alpha$ and $\cos\beta$.

(ii) Show that $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

(iii) Hence find $\tan\left(\alpha+\frac{\pi}{4}\right)$ in terms of $\tan\alpha$.

(b)



The lines AB and BC in the diagram above have equations $3y = -4x + 20$ and $4y = 3x - 15$ respectively and meet at $B(5, 0)$. BC makes an angle α with the x axis. The line PQ has equation $x = 1$ and meets the line AB in Q. BR is the bisector of $\angle ABC$.

(i) Show that AB is perpendicular to BC and then copy the diagram.

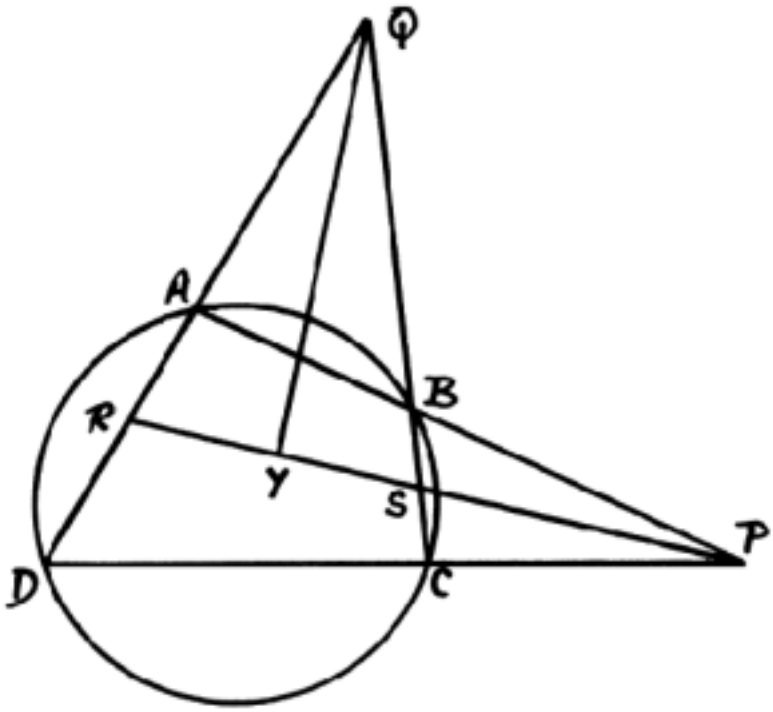
(ii) Use **(a)** to show that BR has equation $y = 7x - 35$.

(iii) The bisector of $\angle PQB$ has slope $1/3$ and meets BR at S. Calculate the co-ordinates of S.

(iv) Draw SM and SN perpendicular to AB and BC, meeting AB at M and BC at N, respectively. Prove that $SM = SN$.

(v) Show that S is the centre of a circle tangential to PQ, AB and BC and write down the equation of the circle.

5. (a)



Let $ABCD$ be a cyclic quadrilateral; AB and DC produced meet at P; DA and CB produced meet at Q as in the diagram.

Let PR be the internal bisector of $\angle APD$ meeting AD at R and BC at S.

Let QY be the internal bisector of $\angle DQC$ meeting PR at Y as in the diagram.

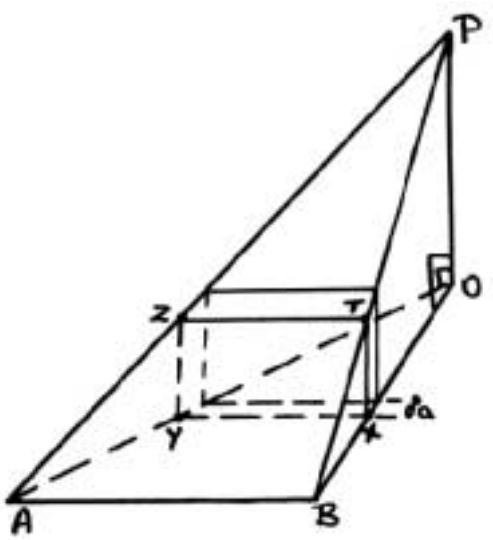
Copy the diagram and prove that: **(i)** $\angle QRS = \angle QSR$; **(ii)** $QY \perp PR$.

(b) Let ABO be an isosceles triangle, $AO = BO = r,\ AB = b$.

Let $PABO$ be a triangular pyramid with height $OP = h$ and OP perpendicular to the plane of ABO as in the diagram.

Consider a slice $\mathcal{S}$ of the pyramid of width δa as in the diagram.

The slice $\mathcal{S}$ is perpendicular to the plane of ABO at XY with $XY||AB$ and $XB = a$. Note that $XT||OP$.



(i) Show that the volume of $\mathcal{S}$ is $\left(\frac{r-a}{r}\right)b\left(\frac{ah}{r}\right)\delta a$, when δa is small. (You may assume that the slice is approximately a rectangular prism of base $XYZT$ and height δa.)

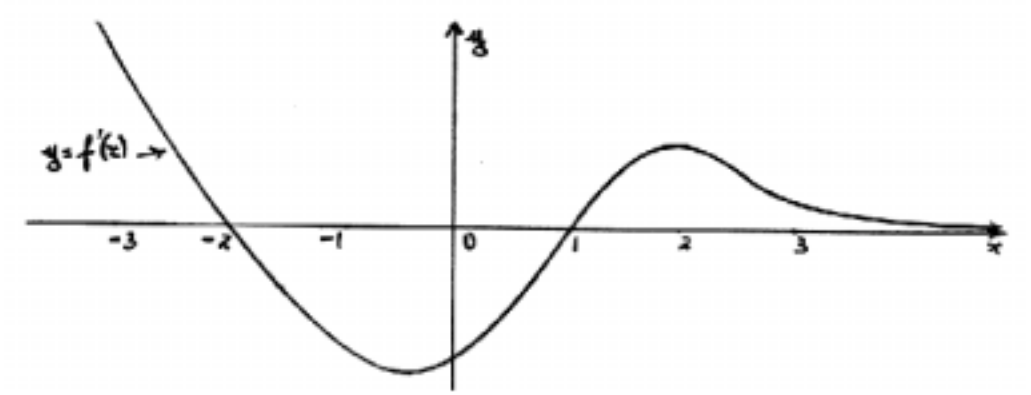
(ii) Hence show that the pyramid $PABO$ has volume $\frac{1}{6}hbr$.

(iii) Suppose now that $\angle AOB = \frac{2\pi}{n}$ and that n identical pyramids $PABO$ are arranged about O as centre with common vertical axis OP to form a solid $\mathcal{C}$. Show that the volume V_n of $\mathcal{C}$ is given by $V_n = \frac{1}{3}r^2hn\sin\frac{\pi}{n}$.

(iv) Note that when n is large, the solid $\mathcal{C}$ approximates a right circular cone. Using the fact that $\frac{\sin x}{x} \to 1$ as $x \to 0$, find $\lim_{n\to\infty} V_n$.

Hence verify that a right circular cone of radius r and height h has volume $\frac{1}{3}\pi r^2h$.

6. (a)



The function $f(x)$ has derivative $f'(x)$ whose graph appears in the diagram. You are given that $f'(-2) = f'(1) = 0$, $f'(x) \to \infty$ as $x \to -\infty$ and $f'(x) \to 0$ as $x \to \infty$.

(i) Sketch the graph of $f(x)$ showing its behaviour at its stationary points. You are given that $f(0) = 0$ and $f(3) > 0$.

(ii) Describe the behaviour of $f(x)$ as $x \to \pm\infty$.

(b) (i) Sketch the graph of $g(x) = x^4 - 4x^3 + 4x^2 - \frac{1}{2}$ showing that it has four real zeros.

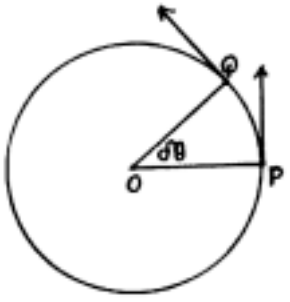
(ii) On different diagrams, sketch the curves:

(α) $y = |g(x)|$; **(β)** $y^2 = g(x)$.

(iii) (α) Indicate the nature of the curve $y = g(x)$ at a zero of $g(x)$.

(β) Calculate the slope of the curve $y^2 = g(x)$ at any point x and describe the nature of the curve at a zero of $g(x)$.

7. (a) A particle of mass m kg moves in a horizontal circle with centre O and radius r metres, with uniform speed v metres per second.

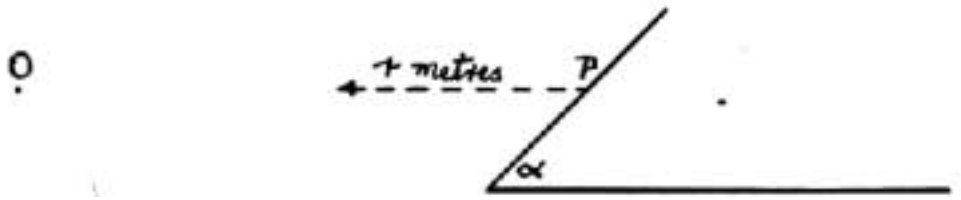


At time t, the particle is at point P while at time $t + \delta t$, it is at point Q with $\angle POQ = \delta\theta$ as in the diagram.

(i) Calculate the complement of the velocity at Q in the direction PO.

(ii) Hence show that the particle is subject to a force of $\frac{mv^2}{r}$ newtons directed towards O.

(iii) Suppose that the circle lies on a track banked at an angle α to the horizontal as in the figure below.



Draw a diagram of all the forces on the moving particle P and show that the resultant of these forces is normal to the track precisely when $\tan\alpha = \frac{v^2}{rg}$. Here g m.s^{-2} is the acceleration due to gravity.

(b) (i) Find real numbers a and b such that

$$x^4 + x^3 + x^2 + x + 1 = (x^2 + ax + 1)(x^2 + bx + 1).$$

(ii) Given that $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a solution of $x^4 + x^3 + x^2 + x + 1 = 0$, find the exact value of $\cos\frac{2\pi}{5}$.

8. (a) Find all values θ with $0 \leqq \theta \leqq 2\pi$ such that $\sin\theta - \sqrt{3}\cos\theta = 1$.

(b) The difference between a real number r are the greatest integer less than or equal to r is called the fractional part of $r, F(r)$. Thus $F(3.45) = 0.45$. Note that for all real numbers $r, 0 \leqq F(r) < 1$.

(i) Let $a = 2136\log_{10} 2$.

Given that $F(a) = \ 7.0738\cdots \times 10^{-5}$

observe that $F(2a) = 14.1476\cdots \times 10^{-5}$

$F(3a) = 21.2214\cdots \times 10^{-5}$

(α) Use your calculator to show that $\log_{10} 1.989 < F(4223a) < \log_{10} 1.990$.

(β) Hence calculate an integer M such that the ordinary decimal representation of 2^M begins with 1989. Thus 2^M=1989 ... .

(ii) Let r be a real number and let m and n be non-zero integers with $m \neq n$.

(α) Show that if $F(mr) = 0$, then r is rational.

(β) Show that if $F(mr) = F(nr)$, then r is rational.

(iii) Suppose that b is an irrational number. Let N be a positive integer and consider the fractional parts $F(b), F(2b), \ldots, F((N+1)b)$.

(α) Show that these $N+1$ numbers $F(b), \ldots, F((N+1)b)$ are all distinct.

(β) Divide the interval $0 \leqq x < 1$ into N subintervals each of length $1/N$ and show that there must be integers m and n with $m \neq n$ and $1 \leqq m,\ n \leqq N+1$ such that $F((m-n)b) < 1/N$.

(iv) Given that $\log_{10} 2$ is irrational, choose any integer N such that $1/N < \log_{10} \frac{1990}{1989}$; note that in **(i)** , $F(a) < \log_{10} \frac{1990}{1989}$.

Use **(iii)** to decide whether there exists another integer M such that $2^M = 1989\ldots$.