NSW HSC 4 Unit Mathematics Examination 1988

1. **(a)** Find the exact value of:-

(i) $\int_0^1 \frac{2x}{1+2x}\,dx$; **(ii)** $\int_1^e \frac{(\log_e x)^2}{x}\,dx$; **(iii)** $\int_0^{\frac{1}{2}} \cos^{-1} x\,dx$.

(b) Use the substitution $t = \tan\frac{\theta}{2}$ to find the exact value of $\int_0^{\pi/2} \frac{d\theta}{\sin\theta+2}$.

2. **(a)** Draw a neat sketch of the function $f(x) = (x-2)(6-x)$. State the co-ordinates of its vertex and of its points of intersection with both co-ordinate axes.

(b) Hence or otherwise draw a neat sketch of the function $g(x) = \frac{16}{(x-2)(6-x)}$. Clearly indicate on your sketch the equations of the vertical asymptotes and the co-ordinates of any stationary points.

(c) The lines with equations $x = 3$ and $x = 5$ cut the graph of $y = g(x)$ at P and Q respectively. Mark on your sketch the co-ordinates of P and Q. Shade the region R bounded by $y = g(x)$ and the line PQ.

(d) Prove that the area of R is $\left(\frac{32}{3} - 8\log_e 3\right)$ square units.

3. **(a)** On a perticular island, twenty per cent of all turtles survive for four weeks after hatching. Fifteen turtles hatch on the same day and are tagged for a study.

(i) **(*α*)** all fifteen turtles will survive the four weeks;

(*β*) none of the turtles survives the four weeks.

(ii) Write down expressions for the probability that:

(*α*) no more than three turtles survive the four weeks;

(*β*) at least three turtles survive the four weeks.

(b) The population P of a town increases at a rate proportional to the number by which the town's population exceeds 1000. This can be expressed by the differential equation $\frac{dP}{dt} = k(P - 1000)$ where t is the time in years and k is a constant.

(i) By differentiation show that $P = 1000 + Ae^{kt}$, where A is a constant, is a solution of this equation.

(ii) The population of the town was 2500 at the start of 1970 and 3000 at the start of 1985. Find its population at the start of the year 2000.

(iii) During which year will the population reach 4000?

4. **(a)** **(i)** Express $z = \sqrt{2} - i\sqrt{2}$ in a modulus-argument form.

(ii) Hence write z^{22} in the form $a + ib$, where a and b are real.

(b) (i) On an Argand diagram shade in the region R containing all points representing complex numbers z such that $1 < |z| < 2$ and $\frac{\pi}{4} < \arg z < \frac{\pi}{2}$.

(ii) In R mark with a dot the point K representing a complex number z. Clearly indicate on your diagram the points M, N, P and Q representing the complex numbers $\overline{z}, -z, \frac{1}{z}$ and $2z$ respectively.

(c) Show that the locus specified by $3|z - (4 + 4i)| = |z - (12 + 12i)|$ is a circle. Write down its radius and the co-ordinates of its centre. Draw a neat sketch of the circle.

5. The hyperbola H has equation $xy = 16$.

(a) Sketch this hyperbola and indicate on your diagram the positions and co-ordinates of all points at which the curve intersects the axes of symmetry.

(b) $P(4p, \frac{4}{p})$, where $p > 0$, and $Q(4q, \frac{4}{q})$, where $q > 0$, are two distince arbitrary points on H. Find the equation of the chord PQ.

(c) Prove that the equation of the tangent at P is $x + p^2y = 8p$.

(d) The tangents at P and Q intersect at T. Find the co-ordinates of T.

(e) The chord PQ produced passes through the point $N(0, 8)$.

(i) Find the equation of the locus of T.

(ii) Give a geometrical description of this locus.

6. A body of mass one kilogram is projected vertically upwards from the ground at a speed of 20 metres per second. The particle is under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the magnitude of the particle's velocity at that time.

In the following questions take the acceleration due to gravity to be 10 metres per second per second.

(a) While the body is travelling upwards the equation of motion is $\ddot{x} = -(10 + \frac{1}{40}v^2)$.

(i) Taking $\ddot{x} = v\frac{dv}{dx}$, calculate the greatest height reached by the particle.

(ii) Taking $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach this greatest height.

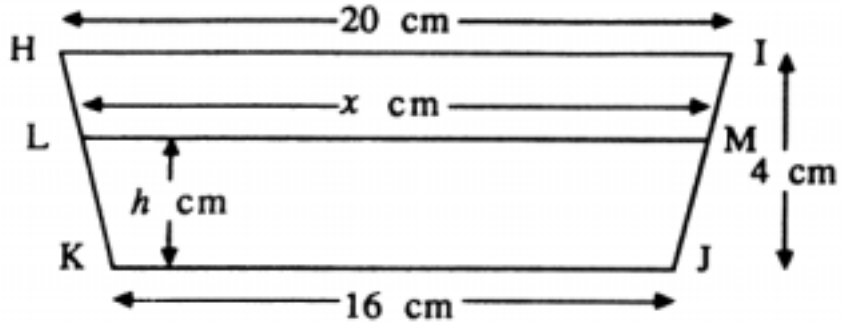
(b) Having reached its greatest height the particle falls to its starting point. The particle is still under the effect of both gravity and a resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$.

(i) Write down the equation of motion of the particle as it falls.

(ii) Find the speed of the particle when it returns to its starting point.

7. (a) (i)

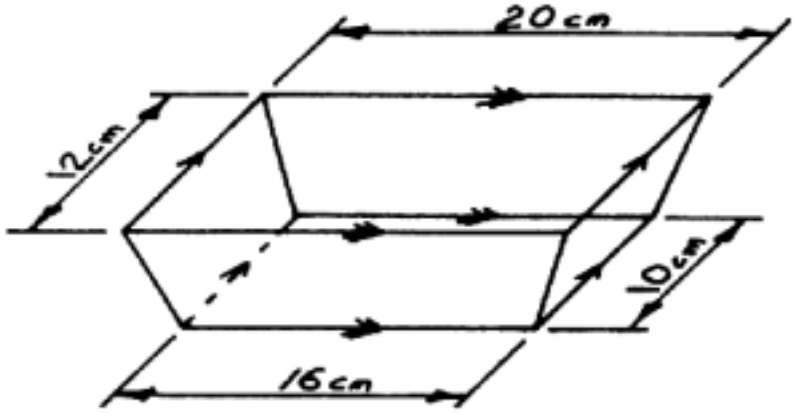
Figure not to scale



A trapezium $HIJK$ has parallel sides $KJ = 16$ cm and $HI = 20$ cm. The distance between these sides is 4 cm. L lies on HK and M lies on IJ such that LM is parallel to KJ. The shortest distance from K to LM is h cm and LM has length x cm. Prove that $x = 16 + h$.

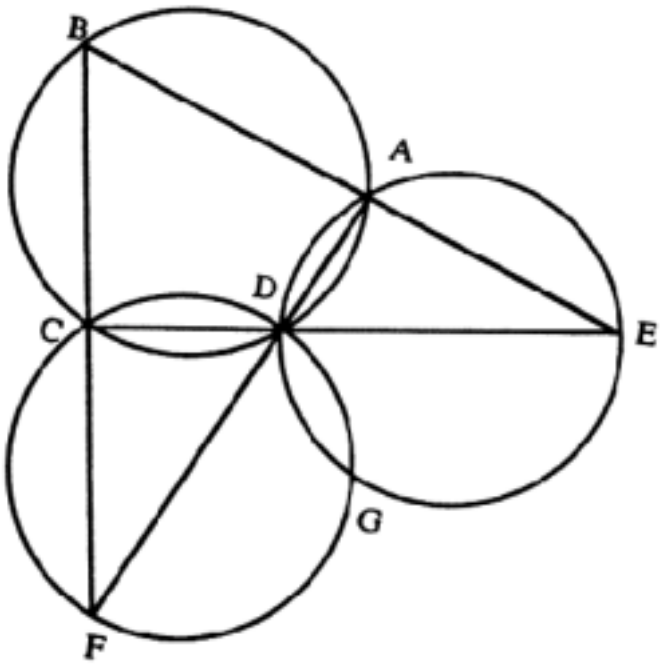
(ii)

Figure not to scale



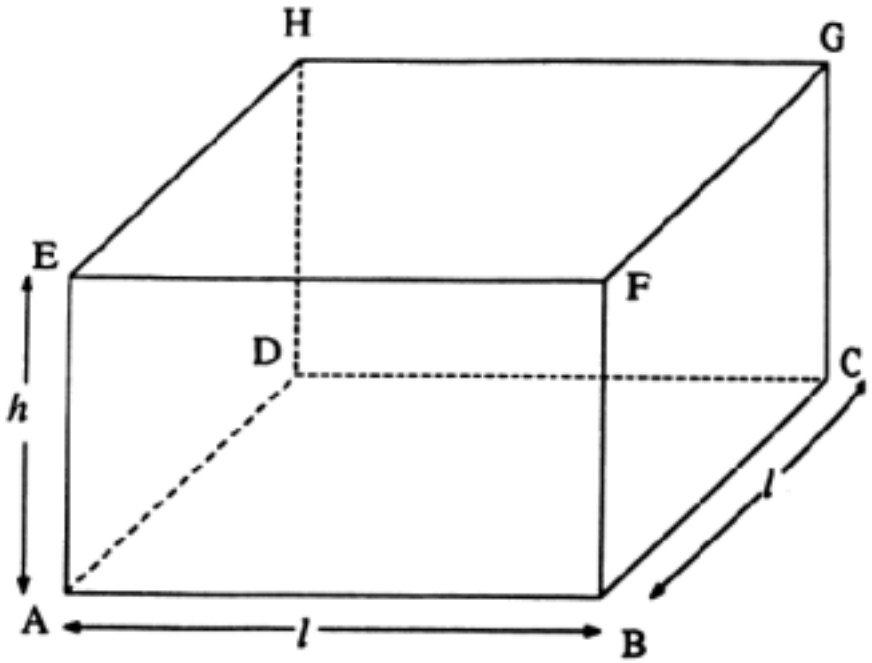
The diagram above is of a cake tin with a rectangular base with sides of 16 cm and 10 cm. Its top is also rectangular with dimensions 20 cm and 12 cm. The tin has depth 4 cm and each of its four side faces is a trapezium. Find its volume.

(b)



$ABCD$ is a cyclic quadrilateral. BA and CD are both produced and intersect at E. BC and AD produced intersect at F. The circles EAD, FCD intersect at G as well as at D. Prove that the points E, G and F are collinear.

8. (a)



A building is in the shape of a square prism with base edge ℓ metres and height h metres. It stands on level ground. A base diagonal AC is produced to a point

K. From K it is found that the angles of elevation of F and G are $30°$ and $45°$ respectively. Prove that $\frac{h}{\ell} = \frac{\sqrt{2}+\sqrt{10}}{4}$.

(b) Newton's method may be used to determine numerical approximations to the real roots of the equation $x^3 = 2$.

Let $x_1 = 2, x_2, x_3, \ldots, x_n, \ldots$ be a series of estimations obtained by iterative applications of Newton's method.

(i) Show that $x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$.

(ii) Show algebraically that $x_{n+1} - \sqrt[3]{2} = \frac{\left(x_n - \sqrt[3]{2}\right)^2\left(2x_n + \sqrt[3]{2}\right)}{3x_n^2}$.

(iii) Given that $x_n > \sqrt[3]{2}$, show that $x_{n+1} - \sqrt[3]{2} < \left(x_n - \sqrt[3]{2}\right)^2$.

(iv) Show that x_{12} and $\sqrt[3]{2}$ agree to at least 267 decimal places.