NSW HSC 4 Unit Mathematics Examination 1987

1. **(i)** Prove that: **(a)** $\int_1^2 \frac{t+1}{t^2}\ dt = \frac{1}{2} + \log_e 2$; **(b)** $\int_4^6 \frac{4\ dx}{(x-1)(x-3)} = 2\log_e\left(\frac{9}{5}\right)$.

(ii) **(a)** Use the substitution $x = \frac{2}{3}\sin\theta$ to prove that $\int_0^{2/3} \sqrt{4-9x^2}\ dx = \pi/3$.

(b) Hence, or otherwise, find the area enclosed by the ellipse $9x^2 + y^2 = 4$.

(iii) **(a)** Given that $I_n = \int_0^{\pi/2} \cos^n x\ dx$, prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$, where n is an integer and $n \geq 2$.

(b) Hence evaluate $\int_0^{\pi/2} \cos^5 x\ dx$.

2. The hyperbola h has equation $9x^2 - 16y^2 = 144$.

(a) Write down its eccentricity, the coordinates of its foci S and S', the equation of each directrix and the equation of each asymptote. Sketch the curve and indicate on your diagram the foci, directrices and asymptotes.

(b) $P(x_1, y_1)$ is an arbitrary point on h.

(α) Prove that the equation of the tangent ℓ at P is: $9x_1x - 16y_1y = 144$.

(β) Find the coordinates of the point G at which ℓ cuts the x-axis.

(γ) Hence prove that $\frac{SP}{S'P} = \frac{SG}{S'G}$.

3. A function $f(x)$ is defined by $f(x) = \frac{\log_e x}{x}$ for $x > 0$.

(a) Prove that the graph of $f(x)$ has a relative turning point at $x = e$ and a point of inflexion at $x = e^{3/2}$.

(b) Discuss the behaviour of $f(x)$ in the neighbourhood of $x = 0$ and for large values of x.

(c) Hence draw a clear sketch of $f(x)$ indicating on it all these features.

(d) Draw separate sketches of the graphs of:

(α) $y = \left|\frac{\log_e x}{x}\right|$; **($\beta$)** $y = \frac{x}{\log_e x}$. [Hint: There is no need to find any further derivatives to answer this part.]

(e) What is the range of the function $y = \frac{x}{\log_e x}$?

4. **(i)** Find the complex square roots of $7 + 6\sqrt{2}i$ giving your answers in the form $x + iy$, where x and y are real.

(ii) Let $z_1 = 4 + 8i$ and $z_2 = -4 - 8i$.

(a) Draw a neat sketch of the locus specified by $|z - z_1| = |z - z_2|$.

(b) Show that the locus specified by $|z - z_1| = 3|z - z_2|$ is a circle. Give its centre and radius.

(iii) (a) Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively. Find the complex number represented by B.

(b) The square is now rotated about O through $45°$ in an anticlockwise direction to $OA'B'C'$. Find the complex numbers represented by the points A', B' and C'.

5. (i) (a) On a number plane shade in the region representing the inequality $(x-2R)^2+y^2 \leqq R^2$.

(b) Show that the volume of a right circular cylindrical shell of height h with inner and outer radii x and $x+\delta x$ respectively is $2\pi xh\,\delta x$ when δx is sufficiently small for terms involving $(\delta x)^2$ to be neglected.

(c) The region $(x-2R)^2+y^2 \leqq R^2$ is rotated about the y-axis forming a solid of revolution called a torus. By summing volumes of cylindrical shells show that the volume V of the torus is given by: $V = 4\pi^2R^3$.

(ii) Five letters are chosen from the letters of the word CRICKET. These five letters are then placed alongside one another to form a five letter arrangement. Find the number of distinct five letter arrangements which are possible, considering all possible choices.

6. A particle of unit mass moves in a straight line against a rsistance numerically equal to $v+v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q>0$.

(a) Show that v is related to the displacement x by the formula $x = \tan^{-1}\left(\frac{Q-v}{1+Qv}\right)$.

(b) Show that the time t which has elapsed when the particle is travelling with velocity v is given by $t = \frac{1}{2}\log_e\left(\frac{Q^2(1+v^2)}{v^2(1+Q^2)}\right)$.

(c) Find v^2 as a function of t.

(d) Find the limiting values of v and x as $t \to \infty$.

7. (i) ABC is an isosceles triangle with $AB = AC$. Let Q be a point on the base BC between B and C. AQ produced meets the circle through the points A, B, C at P.

(a) Prove that triangle BQP is similar to triangle AQC.

(b) Show that $BP.CQ = PQ.AC$.

(c) Prove that $\frac{1}{BP}+\frac{1}{CP} = \frac{1}{PQ}\cdot\frac{BC}{AC}$

(ii) (a) Prove using mathematical induction that for $n \geqq 1$,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \leqq 2-\frac{1}{n}.$$

(b) Prove that $1.45 \leqq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{99^2} \leqq 1.99$.

8. (i) Write down the general solution of the equation $\sin 2\theta + \cos 5\theta = 0$.

(ii) (a) A polynomial $R(x)$ is given by $R(x) = x^7 - 1$. Let $\rho \neq 1$ be that complex root of $R(x) = 0$ which has the smallest positive argument. Show that:

($\boldsymbol{\alpha}$) $R(x) = (x-1)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$,

($\boldsymbol{\beta}$) $1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 = 0$.

(b) Let $\theta = \rho + \rho^2 + \rho^4$ and $\phi = \rho^3 + \rho^5 + \rho^6$.

($\boldsymbol{\alpha}$) Prove that $\theta + \phi = -1$ and $\theta\phi = 2$.

($\boldsymbol{\beta}$) Show that $\theta = \frac{-1+i\sqrt{7}}{2}$ and $\phi = \frac{-1-i\sqrt{7}}{2}$.

(c) Given that $T(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$
$= \{(x-\rho)(x-\rho^2)(x-\rho^4)\}\{(x-\rho^3)(x-\rho^5)(x-\rho^6)\}$
write the polynomial $T(x)$ as a product of two cubics with coefficients involving θ, ϕ and rational numbers.