NSW HSC 4 Unit Mathematics Examination 1986

1. **(i)** Evaluate $\int_0^1 \frac{x}{\sqrt{2-x}}\,dx$.

(ii) Use integration by parts to show that $\int_0^1 \tan^{-1} x\,dx = \frac{1}{4}\pi - \frac{1}{2}\log_e 2$.

(iii) Find numbers A, B, C such that $\frac{x^2}{4x^2-9} \equiv A + \frac{B}{2x-3} + \frac{C}{2x+3}$. Hence evaluate $\int_0^1 \frac{x^2}{4x^2-9}\,dx$.

(iv) Using the substitution $t = \tan(\frac{1}{2}\theta)$, or otherwise, show that

$$\int_0^{\pi/3} \frac{1}{1+\sin\theta}\,d\theta = \sqrt{3} - 1.$$

2. The functions $S(x), C(x)$ are defined by the formulae $S(x) = \frac{1}{2}(e^x - e^{-x})$, and $C(x) = \frac{1}{2}(e^x + e^{-x})$.

(i) **(a)** Verify that $S'(x) = C(x)$,

(b) Show that $S(x)$ is an increasing function for all real x.

(c) Prove that $\{C(x)\}^2 = 1 + \{S(x)\}^2$.

(ii) **(a)** $S(x)$ has an inverse function, S^{-1}, for all values of x. Briefly justify this statement.

(b) Let $y = S^{-1}(x)$. Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

(c) Hence, or otherwise, show that $S^{-1}(x) = \log_e\left[x + \sqrt{1+x^2}\,\right]$.

(d) Show that $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}} = \log_e\left\{\frac{2+\sqrt{5}}{1+\sqrt{2}}\right\}$.

3. **(i)** The point $P(x_1, y_1)$ lies on the hyperbola of equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(a) Find the equation of the normal at P.

(b) The normal at P meets the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. Show that $NG : ON = b^2 : a^2$, where O is the origin $(0, 0)$.

(ii) **(a)** Show that if a is a multiple root of the polynomial equation $f(x) = 0$ then $f(a) = f'(a) = 0$.

(b) The polynomial $\alpha x^{n+1} + \beta x^n + 1$ is divisible by $(x-1)^2$. Show that $\alpha = n$, and $\beta = -(1+n)$.

(c) prove that $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ has no multiple roots for any $n \geq 1$.

4. **(i)** Given that $z_1 = 3 - i, z_2 = 2 + 5i$, express in the form $a + ib$, where a, b are real,

(a) $(\overline{z_1})$ **(b)** z_1/z_2 **(c)** $|(\overline{z_1}/z_2)|$

(ii) Given that for the complex number $z, |z| = 2, \arg z = 2\pi/5$, write in the form $a + ib$, where a, b are real:

(a) z **(b)** z^7.

(iii) (a) Draw a sketch of the region of the Argand diagram consisting of the set of all values of z for which $1 \leq |z| \leq 4$ and $\pi/4 \leq \arg z \leq 3\pi/4$.

(b) (*α*) The curve in the Argand diagram for which $|z - 2| + |z - 4| = 10$ is an ellipse. Write down the coordinates of the centre, and the lengths of the major and minor axes of this ellipse.

(*β*) On a separate Argand diagram, show the region for which z satisfies the inequalities $z + \overline{z} \leq 6$ or $|z - 2| + |z - 4| \leq 10$.

5. (i) (a) The coordinates of the vertices of a triangle ABC are $(0, 2), (1, 1), (-1, 1)$ respectively and H is the point $(0, H)$. The line through H, parallel to the x-axis meets AB and AC at X and Y. Find the length of XY.

(b) The region enclosed by the triangle ABC, defined in **(a)** above, is rotated about the x-axis. Find the volume of the solid of revolution so formed.

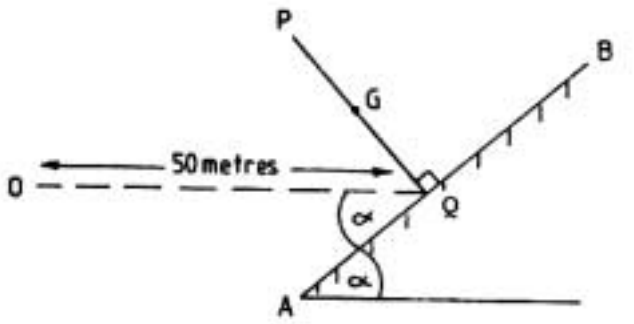
(ii) A particle travels in a straight line away from a fixed wall. At time t secs its acceleration $a\,\text{m.s}^{-2}$ is given by $a = t \sin t$.

Determine the velocity $v\,\text{m.s}^{-1}$, and the distance from the wall x metres, as functions of t, given that $v = V$ and $x = 0$, at $t = 0$.

6. (a) A cyclist is travelling with constant speed v metres per second around a circular track of radius r metres. Draw a diagram of the forces on the cyclist, and show that the cyclist must lean inwards towards the centre of the circle at an angle θ, where $\tan\theta = v^2/(rg)$ and θ is the angle to the vertical of the line from the mass centre to the point of wheel contact.

(b) The figure represents the front view of a cyclist PQ, with mass centre G, riding on a circular competition track AB which is banked at an angle α to the horizontal.

The track is so designed that when the cyclist is travelling at a constant speed of 40 km per hour around a horizontal circle of centre O and radius $OQ = 50$ metres, then PQ and AB are perpendicular. Assuming the value of g to be $10\,\text{m.s}^{-1}$, show that $\alpha = 14°$, to the nearest degree.

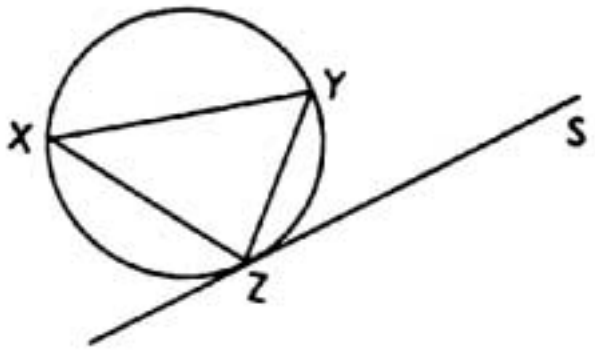


(c) Also calculate the force of the wheels on the track at Q in the direction AB, given that a cyclist of total mass 80 kg is travelling around the same circle at 50 km per hour.

7. **(i)** Find all x such that $\cos 2x = \sin 3x$, and $0 \leq x \leq \pi/2$.

(ii) Appropriate diagrams should accompany each of your solutions to this section.

(a) In the figure, ZS is the tangent to the circle at Z, and X, Y are any two points on the circle. By drawing the diameter through Z, or otherwise, prove that $\angle YZS = \angle ZXY$.



(b) In the given figure, XY produced meets ZS at P. The lengths PX, PY and PZ are x, y, z respectively. Prove that $z^2 = xy$.

(c) Two unequal circles intersect at L, M. The common tangent AB touches the circles at A, B. Prove that $\angle LM$ produced bisects AB.

8. **(i)** Let α, β, γ be the roots of the cubic equation $x^3 + px^2 + q = 0$, where p, q are real. The equation $x^3 + ax^2 + bx + c = 0$ has roots $\alpha^2, \beta^2, \gamma^2$. Find a, b, c as functions of p, q.

(ii) A committee of 4 women and 3 men are to be seated at random around a circular table with 7 seats. What is the probability that all the women will be seated together?

(iii) The function $f(x)$ is given, for $x > 0$, by $f(x) = 2\log_e x - \frac{x^2-1}{x}$.

(a) Show that the only zero of $f(x)$ occurs at $x = 1$.

(b) Let $g(x) = \frac{x \log_e x}{x^2-1}$, for $x > 0$ and $x \neq 1$. Show that $0 < g(x) < \frac{1}{2}$.