NSW HSC 4 Unit Mathematics Examination 1985

1. (i) Find: **(a)** $\int_0^{3\pi} x\cos x\ dx$; **(b)** $\int_0^1 \frac{1}{x^2+4x+5}\ dx$.

(ii) Find real numbers A, B, C such that

$$\frac{x}{(x-1)^2(x-2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Hence show that $\int_0^{\frac{1}{2}} \frac{x}{(x-1)^2(x-2)}\ dx = 2\log_e\left(\frac{3}{2}\right) - 1$.

(iii) Use the substitution $x = a - t$, where a is a constant, to prove that

$$\int_0^a f(x)\ dx = \int_0^a f(a-t)\ dt.$$

Hence, or otherwise, show that $\int_0^1 x(1-x)^{99}\ dx = \frac{1}{10100}$.

2. (i) (a) Find the turning points of the cubic polynomial $p(x) = x^3 - x^2 - 5x - 1$, and without attempting to solve the equation, show that the equation $p(x) = 0$ has three distinct real roots, two of which are negative.

(b) Sketch the graph of $p(x)$.

(c) Starting with the approximation $x = 0$, use one application of Newton's method to estimate a root of the equation $p(x) = 0$.

(d) What initial approximation would you use to estimate the positive root of $p(x) = 0$ by Nowton's method? State briefly your reasons for this choice.

(ii) (a) Sketch the function $g(x) = xe^{-x}$, for $x \geq -1$.

(b) Given $g(x)$ as in **(a)** above, the function $f(x)$ is given by the rule

$$f(x) = \begin{cases} g(x-2), & x \geq 1, \\ g(-x), & x \leq 1. \end{cases}$$

Find the zeros of this function, and the maximum and minimum values.

Draw a sketch of the graph of $y = f(x)$.

3. (i) Reduce the complex expression $\frac{(2-i)(8+3i)}{(3+i)}$ to the form $a + ib$, where a, b are real numbers.

(ii) The complex number z is given by $z = -\sqrt{3}_i$.

(a) Write down the values of $\arg z$ and $|z|$.

(b) Hence, or otherwise, show that $z^7 + 64z = 0$.

(iii) On the Argand diagram, let $A = 3 + 4i, B = 9 + 4i$.

(a) Draw a clear sketch to show the important features of the curve defined by

$$|z - A| = 5.$$

Also, for z on this curve, find the maximum value of $|z|$.

(b) On a separate diagram, draw a clear sketch to show the important feartures of the curve defined by

$$|z - A| + |z - B| = 12.$$

For z on this curve, find the greatest value of $\arg z$.

4. (i) The normal at the point P on the parabola $4ay = x^2$ intersects the y axis at Q. The directrix intersects the y axis at Y, and S is the focus of the parabola. The mid-point of QS is R. Show that $YR^2 - RP^2 = 4a^2$.

(ii) (a) Show that for all values of θ the point $P(3\cos\theta, 2\sin\theta)$ lies on an ellipse, and find its equation.

(b) Find the equation of the tangent to this ellipse at P.

(c) Show that the point $Q(-3\sin\theta, 2\cos\theta)$ also lies on the same ellipse.

(d) The tangents at P, Q to the ellipse intersect at T. Find the coordinates of T, and verify that T lies on the curve $4x^2 + 9y^2 = 72$.

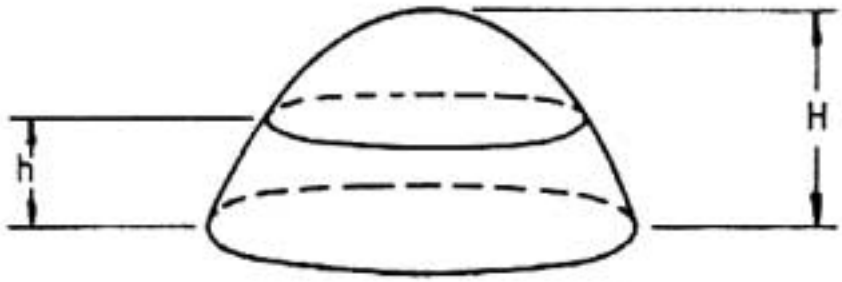
5. (i) A thin wire of length L is cut into two pieces, out of which a circle and a closed square are to be formed, so that the sum of the areas of the circle and square so formed is a minimum. Show that this minimum value is

$$\frac{L^2}{4(\pi + 4)}$$

(ii) (a) Using the substitution $x = a\sin\theta$, or otherwise, verify that $\int_0^a (a^2 - x^2)^{\frac{1}{2}} dx = \frac{1}{4}\pi a^2$.

(b) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

(c)



The diagram shows a mould of height H. At height h above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} =$

λ^2, where $\lambda = 1 - \frac{h^2}{H^2}$, and x, y are appropriate coordinates in the plane of the cross-section. Show that the volume of the mound is $\frac{8\pi abH}{15}$.

6. A particle of mass 10 kg is found to experience a resistive force, in newtons, of one-tenth of the square of its velocity in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of u metres per second, and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again. Assuming the value of g is $10\,\mathrm{m.s}^{-2}$,

(a) find the time the particle takes to reach A from O;

(b) show that the height OA is $50\log_e[1 + 10^{-3}u^2]$ metres;

(c) show that the particle's velocity $w\,\mathrm{m.s}^{-1}$ when it reaches O again is given by $w^2 = u^2(1 + 10^{-3}u^2)^{-1}$.

7. (i) Given that $\sin x + \sin y = 2\sin A\cos B$, find values for A and B in terms of x and y.

Solve the equation $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$, giving all solutions in the interval $0 \leq \theta \leq 2\pi$.

(ii) (a) A, B, C are three points lying on a given circle, and P is another point in the same plane. Write down two different angle tests to determine whether A, B, C, P are concyclic (i.e., P also lies on the given circle).

(b) In an acute-angled triangle with vertices L, M, N, the foot of the perpendicular from L to MN is P, and the foot of the perpendicular from N to LM is Q. The lines LP, QN intersect at H.

($\boldsymbol{\alpha}$) Draw a clear diagram showing the given information.

($\boldsymbol{\beta}$) Prove that $\angle PHM = \angle PQM$.

($\boldsymbol{\gamma}$) Prove that $\angle PHM = \angle LNM$.

($\boldsymbol{\delta}$) Produce MH to meet LN at R. Prove that $MR \perp LN$.

(c) What general result about triangles is proved in **(b)**?

8. (i) (a) In how many ways can 4 persons be grouped into two pairs to play a set of doubles tennis?

(b) The eight members of a tennis club meet to play two simultaneous sets of doubles tennis on two separate but otherwise identical courts. In how many different ways can the members of the club be selected for these two sets of tennis?

(ii) (a) Show that for $k \geq 0, 2k + 3 > 2\sqrt{(k+1)(k+2)}$.

(b) Hence prove that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\big[\sqrt{n+1} - 1\big].$$

(c) Is the statement that, for all positive integers N,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10} \text{ true?}$$

Give reaons for your answer.