NSW HSC 4 Unit Mathematics Examination 1984

1. **(i)** Show that **(a)** $4\int_e^{e^4} x\log_e x\ dx = 7e^8 - e^2$, **(b)** $\int_e^{e^4} \frac{dx}{x\log_e x} = 2\log_e 2$.

(ii) Evaluate **(a)** $\int_0^{\sqrt{3}} \frac{x+1}{x^2+1}\ dx$, **(b)** $\int_{-4}^{4} \frac{x+6}{\sqrt{x+5}}\ dx$, **(c)** $\int_{-\pi/6}^{\pi/6} \sin 4x\cos 2x\ dx$.

2. **(i)** Sketch the graphs of **(a)** $(x+3)(y-2) = 1$, **(b)** $x^2 + y^2 + 1 = 2(x+y)$.

(ii) Consider the ellipse E whose equation is $\frac{x^2}{4} + \frac{y^2}{2} = 1$.

(a) Show that the equation of E may be given in the parametric form $x = 2\cos\theta, y = \sqrt{2}\sin\theta$.

(b) Assuming that the perimeter p of E is given by the formula $p = 2\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta$, show that $p = 2\sqrt{2}\int_0^{\pi} \sqrt{2 - \cos^2\theta}\ d\theta$.

(c) Use seven evenly spaced ordinates from $\theta = 0$ to $\theta = \pi$ and Simpson's rule to estimate p. Round off your answer to two decimal places.

3. **(i)** Calculate the modulus and argument of the product of the roots of the equation $(5+3i)z^2 - (1-4i)z + (8-2i) = 0$.

(ii) Let $A = 1+i, B = 2-i$. Draw sketches to show the loci specified on the Argand diagram by

(a) $\arg(z - A) = \pi/4$, **(b)** $|z - A| = |z - B|$.

(iii) Show that the point representing $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ on the Argand diagram lies on the circle of radius one with centre at the point which represents 1.

(iv) R is a positive real number and z_1, z_2 are complex numbers. Show that the points on the Argand diagram which represent respectively the numbers z_1, z_2, $\frac{z_1 - iRz_2}{1 - iR}$, form the vertices of a right-angled triangle.

4. **(i)** Show that, if $(r\cos\theta, r\sin\theta)$, $\left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$ lie on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with centre O, then $\frac{1}{r^2} + \frac{1}{s^2} = \frac{1}{a^2} - \frac{1}{b^2}$.

Deduce that, if P, Q are points on the hyperbola such that OP is perpendicular to OQ, then

$\frac{1}{OP^2} + \frac{1}{OQ^2}$ is independent of the position of P and Q.

Give a sketch of a hyperbola and points O, P, Q related in this way.

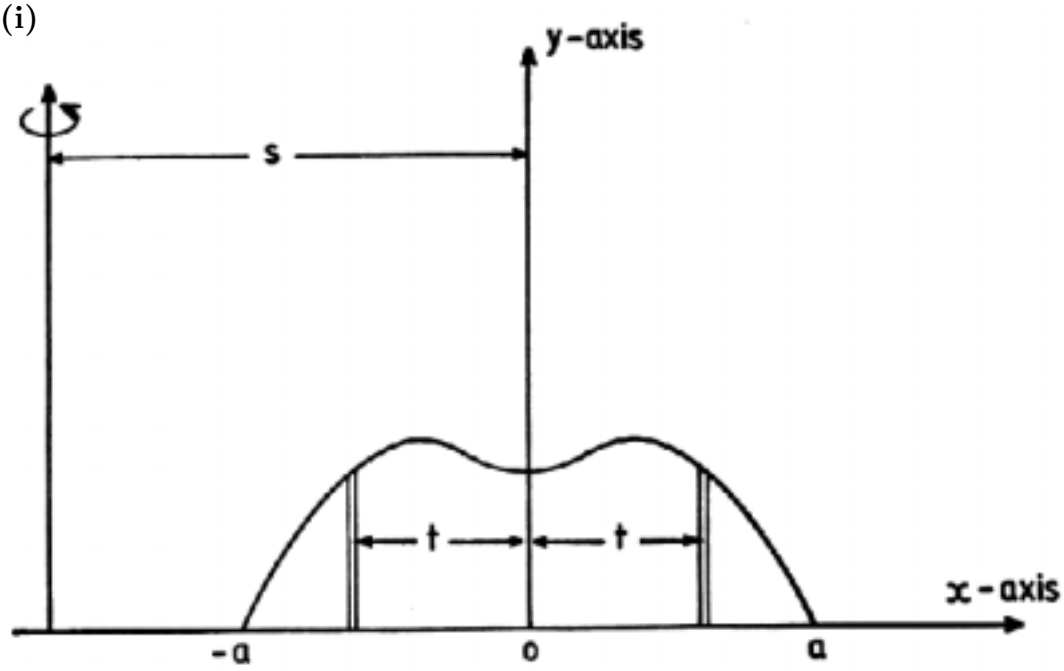
(ii) The line $L : ax + by = 1$ meets the rectangular hyperbola $H : xy = c^2$ in two distinct points, $X_1(x_1, y_1), X_2(x_2, y_2)$. $X_0(x_0, y_0)$ is the mid-point of X_1X_2.

(a) Find a quadratic equation whose roots are x_1, x_2.

(b) Show that the equation of L may be written $\frac{x}{x_0} + \frac{y}{y_0} = 2$.

(c) Deduce that every line L' (other than an asymptote) through the centre of a rectangular hyperbola bisects all chords parallel to a certain direction and describe how this direction is related to the direction of L'.

5. (i)



The diagram shows the area A between the smooth curve $y = f(x), -a \leq x \leq a$, and the x-axis. (Note that $f(x) \geq 0$ for $-a \leq x \leq a$ and $f(-a) = f(a) = 0$). The area A is rotated about the line $x = -s$ (where $s > a$) to generate the volume V. This volume is to be found by slicing A into thin vertical strips, rotating these to obtain cylindrical shells, and adding the shells. Two typical strips of width δt whose centre lines are distance t from the y-axis are shown.

(a) Show that the indicated strips generate shells of approximate volume $2\pi f(-t)(s-t)\delta t, 2\pi f(t)(s+t)\delta t$, respectively.

(b) Assuming that the graph of f is symmetrical about the y-axis, show that $V = 2\pi sA$.

(ii) Assuming the results of part **(i)**, solve the following problems.

(a) A doughnut shape is formed by rotating a circular disc of radius r about an axis in its own plane at a distance of s $(s > r)$ from the centre of the disc. Find the volume of the doughnut.

(b) The shape of a certain party jelly can be represented by rotating the area between the curve $y = \sin x, 0 \leq x \leq \pi$, and the x-axis about the line $x = -\pi/4$. Find the volume generated.

6. Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection (upward inclination to the horizon-

tal), α, but with different velocities U, V metres per second $U < V$.

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(a) Show that, while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time. Hence, express the horizontal distance from P to the foot of the wall in terms of h, α.

(b) Show that $V(\tan\alpha + \tan\beta) = 2U\tan\alpha$, and deduce that, if $\beta = \frac{1}{2}\alpha$, then $U < \frac{3}{4}V$.

7. (i) In how many ways can the five letters of the word CONIC be arranged in a line so that the two (indistinguishable) C's are separated by at least one other letter?

(ii) It is given that x, y, z are positive numbers. Prove that

(a) $x^2 + y^2 \geq 2xy$,

(b) $x^2 + y^2 + z^2 - xy - yz - zx \geq 0$.

Multiply both sides of the inequality **(b)** by $(x + y + z)$ to obtain

(c) $x^3 + y^3 + z + 3 \geq 3xzy$.

Deduce from **(c)**, or prove otherwise, that

(d) $(x + y + z)(x^{-1} + y^{-1} + z^{-1}) \geq 9$.

Suppose that x, y, z satisfy the additional constraint that $x + y + z$ is equal to 1.

Is it true that the minimum value of the expression $x^{-1} + y^{-1} + z^{-1}$ is equal to 9? Justify your answer.

8. (i) Write down expressions for $\sin(\alpha+\beta), \cos(\alpha+\beta)$ in terms of $\sin\alpha, \cos\alpha, \sin\beta, \cos\beta$.

Deduce that $\tan(\alpha + \beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$,

and $\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\gamma\tan\alpha-\tan\beta\tan\gamma}$.

By means of the substitution $t = \tan\theta$, transform the equation
$\sin 4\theta + a\sin 2\theta + b\cos 2\theta + b = 0$
into a cubic equation in t. (a, b are real constants, $a \neq 2$). Suppose the roots of the transformed equation are $\tan\alpha, \tan\beta, \tan\gamma$. Show that $\alpha + \beta + \gamma$ is a multiple of π.

(ii) A woman travelling along a srraight flat road passes three points at intervals of 200 m. From these points she observes the angle of elevation of the top of the hill to the left of the road to be respectively $30°, 45°$, and again $45°$. Find the height of the hill.