NSW HSC 4 Unit Mathematics Examination 1983

1. (i) Draw a careful sketch of the curve $y = \frac{x^2}{x^2-1}$, indicating clearly any vertical or horizontal asymptotes, turning points or inflexions.

(ii) A function $f(x)$ is known to approach 0 as x appoaches ∞ and $-\infty$. Its derivative is given by $f'(x) = e^{-x^2}(x-1)^2(2-x)$.

From this information, describe the behaviour of $f(x)$ as x increases from $-\infty$ to $+\infty$. Include in your description an indication of those x where $f(x)$ is respectively increasing or decreasing and any points where f has a maximum or minimum value. Also explain why $f(x)$ must be positive for all real x.

Draw a sketch of a function $f(x)$ satisfying the given conditions.

2. (i) Find: **(a)** $\int_0^{\sqrt{3}} \frac{x+12}{x^2+9}\, dx$ **(b)** $\int_1^2 60x^3(1+x^2)^4\, dx$

(c) $\int_0^T xe^{-(x/2)}\, dx$, where T is a positive number.

(ii) Find the partial fraction decomposition of $\frac{16x}{x^4-16}$. Hence show that $\int_4^6 \frac{16x}{x^4-16}\, dx = \log_e(4/3)$, and also evaluate $\int_0^1 \frac{16x}{x^4-16}\, dx$.

3. Use mathematical induction to prove that for any real $\theta, \cos 6\theta + i \sin 6\theta = (\cos\theta + i\sin\theta)^6$.

Find the six sixth roots of -1, expressing each in the form $x + iy$ with x, y real.

Find also the four roots of the equation $z^4 - z^2 + 1 = 0$, and indicate their positions on an Argand diagram.

4. (i) Determine the (real) values of λ for which the equation $\frac{x^2}{4-\lambda} + \frac{y^2}{2-\lambda} = 1$ defines respectively an ellipse and an hyperbola.

Sketch the curve corresponding to the value $\lambda = 1$.

Describe how the shape of this curve changes as λ increases from 1 towards 2. What is the limiting position of the curve as 2 is approached?

(ii) P is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre O. A line drawn from O, parallel to the tangent to the ellipse at P, meets the ellipse at Q.

Prrove that the area of the triangle OPQ is independent of the position of P.

5. (i) An egg-timer has the shape of an hour-glass and can be described mathematically as being obtained by rotating the curve $y = x + 6x^3, -1/\sqrt{2} \le x \le 1/\sqrt{2}$, about the y-axis.

Use the method of decomposition into cylindrical shells to calculate its volume, correct to three significant figures.

(ii) A plane curve is defined implicitly by the equation $x^2 + 2xy + y^5 = 4$.

This curve has a horizontal tangent at the point $P(X, Y)$. Show that X is the unique real root of the equation $X^5 + X^2 + 4 = 0$, and that $-2 < X < -1$.

6. An object of irregular shape and of mass 100 kilograms is found to experience a resistive force, in newtons, of magnitude one-tenth the square of its velocity in metres per second when it moves through the air.

If the object falls from rest under gravity, assumed constant of value 9.8 $\mathrm{m.s}^{-2}$, calculate

(i) its terminal velocity;

(ii) the minimum height, to the nearest metre, of the release point above ground, if it attains a speed of 80% of its terminal velocity before striking the ground.

7. **(i)** A city council consists of 6 Liberal and 5 Labor aldermen, from whom a committee of 5 members is chosen at random. What is the probability that the Liberals have a majority on the committee?

(ii) Let α, β, γ be the roots of the equation $x^3 + qx + r = 0$, where $r \neq 0$. Obtain as functions of q, r in their simplest forms, the coefficients of the cubic equations whose roots are:

(a) $\alpha^2, \beta^2, \gamma^2$; **(b)** $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$; **(c)** $\alpha^{-2}, \beta^{-2}, \gamma^{-2}$.

(iii) Given that $x + y = s$, prove that, for $x > 0, y > 0, s > 0$, $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{s}$, and that $\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{8}{s^2}$.

8. **(i)** Given $\sin x \sin y = \frac{1}{2}(\cos A - \cos B)$, find A, B in terms of x, y. Hence prove that for any positive integer n,

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \sin^2 nx / \sin x.$$

(ii) In a triangle ABC, the point X on BC is such that AX bisects $\angle BAC$. Use the sine rule to prove that $AB/AC = BX/XC$.

In the figure, XY represents a vertical flagstaff of length a placed on top of a vertical tower AY of height b. An observer is at a point O, which is a vertical height h above B, and a horizontal distance d from A. Given that $\angle XOY = \angle YOA$, show that $(a-b)d^2 = (a+b)b^2 - 2b^2h - (a-b)h^2$.

