NSW HSC 4 Unit Mathematics Examination 1982

1. **(i)** Prove that the curve: $y = \frac{4x}{1+x^2}$ has a minimum at $A(-1,-2)$, a maximum at $B(1,2)$ and a point of inflexion at $O(0,0)$. Sketch the curve.

(ii) The cubic curve $y = ax^3+bx^2+cx+d$ also has a minimum at A and a maximum at B. Obtain the values of the coefficients a, b, c, d, and deduce that O is also a point of inflexion of this curve.

(iii) Prove that the two curves have only the three points A, B, O, in common.

2. **(i)** Find the indefinite integrals: **(a)** $\int x(1+x^2)^6\ dx$;

(b) $\int (x - \frac{1}{x-2})^2\ dx$, for $x > 2$.

(ii) Evaluate: $\int_0^{\pi/4} x \sin 2x\ dx$.

(iii) Let $I_n = \int_0^{\pi/4} \tan^n x\ dx$, where n is an integer, and $n \geq 3$.

Show that $I_n + I_{n-2} = \frac{1}{n-1}$. Hence evaluate I_7.

3. **(i)** For the complex number $z = 1 - \sqrt{3}i$, find $|z|, \arg z$. Also express each of $\overline{z}, z^2, 1/z$, and $\sqrt{z}$ in the form $a + ib$, where a, b are real numbers. Plot $z, \overline{z}, z^2, 1/z$ and both square roots of z on an Argand diagram, labelling them respectively with the letters A, B, C, D, E, F.

(ii) Sketch the region R in the Argand diagram consisting of those points z for which:

$|\arg z| < \frac{\pi}{3}, z + \overline{z} < 4$, and $|z| > 2$.

(iii) Solve the equation: $z^2 + 4z - 1 + 12i = 0$.

4. With respect to axes Ox, Oy, the line $x = 1$ is a directrix, and the point $(2,0)$ a focus of a conic of eccentricity $\sqrt{2}$.

(a) Find the equation of the conic, show that it is a rectangular hyperbola, and sketch the curve, indicating its asymptotes, foci and directrices.

(b) Find the equation of the normal to the curve at any point P on it.

(c) The normal to the curve at P meets Ox, Oy in $(X,0), (0,Y)$, respectively, and T is the point (X,Y). Show that as P varies on the curve, T always lies on the hyperbola $x^2 - y^2 = 8$.

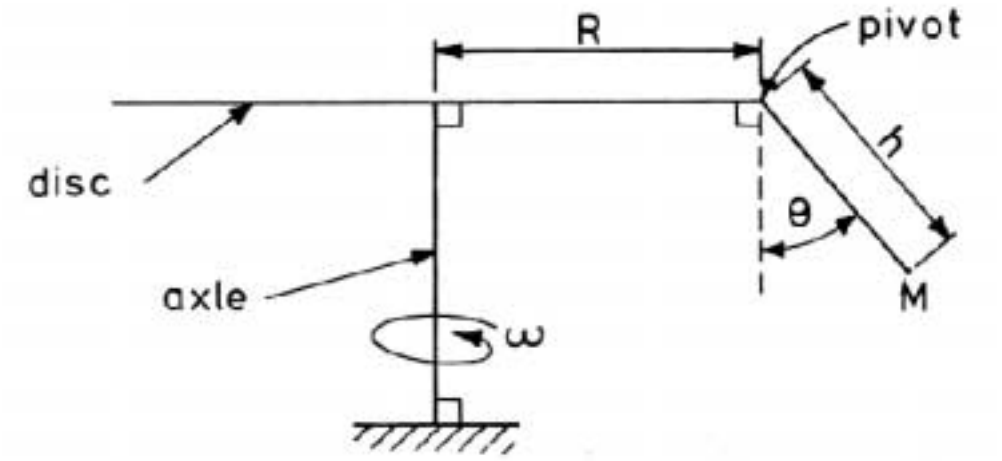
5. **(i)** Write down the volume of a right circular cylindrical shell of height h with inner and outer radii r, R, respectively. Hence show that for $\delta r = R - r$, this volume is $2\pi rh\ \delta r$, when δr is sufficiently small for $(\delta r)^2$ to be neglected.

The region bounded by the curves $y = \frac{1}{x+1}$, and $y = \frac{1}{x+2}$ between the ordinates at $x = 0$ amd $x = 2$ is rotated about the y axis, forming a solid of volume V units.

Show that: $V = 2\pi \int_0^2 \frac{x}{(x+1)(x+2)}\, dx$, and find V correct to three sig. fig.

(ii) A point P lies on the on the curve whose equation is $x^4 + y^4 = 1$. Prove that the distance of P from the origin is at most $2^{\frac{1}{4}}$.

6. A "chair-o-plane" at a fairground consists of seats hung from pivots attached the the rim of a horizontal circular disc, which is rotated by a motor driving its vertical axle. As the speed of rotation increases, the seats swing out. Represent a seat by a point mass M kg suspended by a weightless rod h metres below a pivot placed R metres from the axis of rotation. Assume that when the disc rotates about its axle with constant angular velocity ω radians per second, there is an equilibrium position in which the rod makes an angle θ with the vertical, as shown in the figure.



(a) Show that ω, θ satisfy the relation $(R + h\sin\theta)\omega^2 = g\tan\theta$.

(b) Use a graphical means to show that, for given ω there is just one value of θ in the range $0 \le \theta \le \frac{\pi}{2}$ which satisfies this relation.

(c) Given $R = 6, h = 2, \theta = 60°$, and assuming that $g = 10\,\text{m.s}^{-2}$, find the speed of M relative to the ground.

7. (i) Find all x such that: $\cos x + \sin x = 1 + \sin 2x$, and $0 \le x \le 2\pi$.

(ii) Use De Moivre's theorem to express $\cos 5\theta, \sin 5\theta$ in powers of $\sin\theta$ and $\cos\theta$. Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$.

Deduce that: $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

8. (i) Given that $a_n = \sqrt{2 + a_{n-1}}$ for integers $n \ge 1$, and that $a_0 = 1$, prove that, for $n \ge 1, \sqrt{2} < a_n < 2$.

(ii) Eight players enter a knock-out singles tennis tournament in which each of the four first round winners plays one second round game to decide who enters the final.

Assuming that all players are equally likely to win a game, show that the probability that two particular entrants play each other in the tournament is $\frac{1}{4}$.

Also show that if sixteen persons enter the tournament, then the probability that the two players meet is $\frac{1}{8}$.

Prove that for a similar knock-out tournament for 2^n players, the probability that two players meet is 2^{1-n}.