NSW HSC 4 Unit Mathematics Examination 1981

1. **(i)** Evaluate: **(a)** $\int_0^{\pi} x \sin x \; dx$; **(b)** $\int_2^4 \frac{dx}{x^2-4x+8}$; **(c)** $\int_{-1}^{1} \frac{4+x^2}{4-x^2} \; dx$.

(ii) Let n be a positive integer, and let $I_n = \int_1^2 (\log_e x)^n \; dx$. Prove that: $I_n = 2(\log_e 2)^n - nI_{n-1}$. Hence evaluate $\int_1^2 (\log_e x)^4 \; dx$ as a polynomial in $\log_e 2$.

2. **(i)** Sketch the curve $y = 2 + \frac{1}{x^2-1}$, $(x \neq \pm 1)$, showing the location and nature of all satationary points and the equations of all asymptotes.

(ii) **(a)** Factorise $1 + x + x^2 + x^3$.

(b) Prove that the equation $\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + c = 0$ has no real root if $c > \frac{7}{12}$. How many real roots are there if $c \leq \frac{7}{12}$?

3. The ellipse E has cartesian (x, y) equation $\frac{x^2}{4} + \frac{y^2}{3} = 1$. Write down its eccentricity, the co-ordinates of its foci S and S', and the equation of each directrix. Sketch the curve and indicate on your diagram the foci and directrices. P is an arbitrary point on E.

(i) Prove that the sum of the distances $SP, S'P$ is independent of P.

(ii) Derive the equation of the normal line to E at P, and prove that it bisects the angle SPS'.

4. For the complex number $z = x + iy$:

(i) Find the curve in the Argand diagram for which $\Re(z^2) = 3$, and sketch the curve.

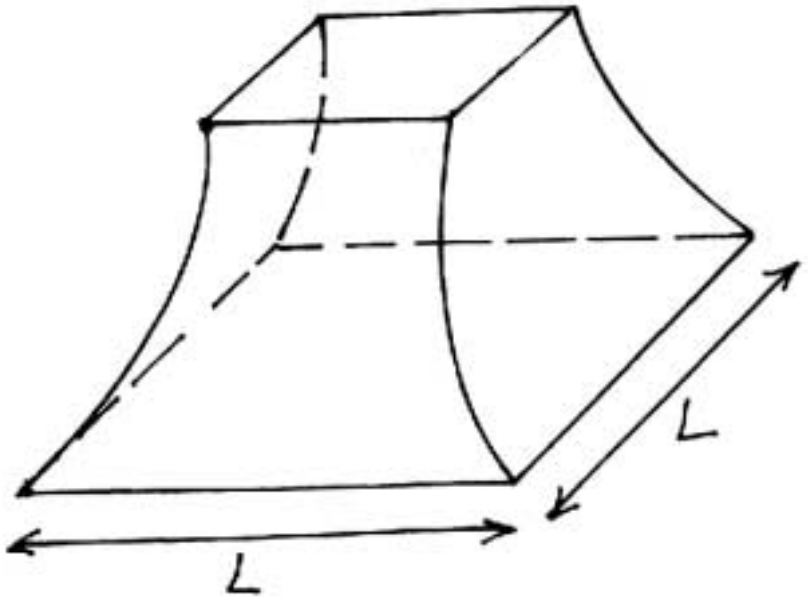
(ii) Find the curve such that $\Im(z^2) = 4$.

(iii) Solve completely the equation $z^2 = 3 + 4i$.

(iv) The region R of the Argand diagram consists of the set of all values of z such that both $0 < \Re(z^2) < 3$ and $0 < \Im(z^2) < 4$. Draw a sketch of the region R, showing the coordinates of the vertices.

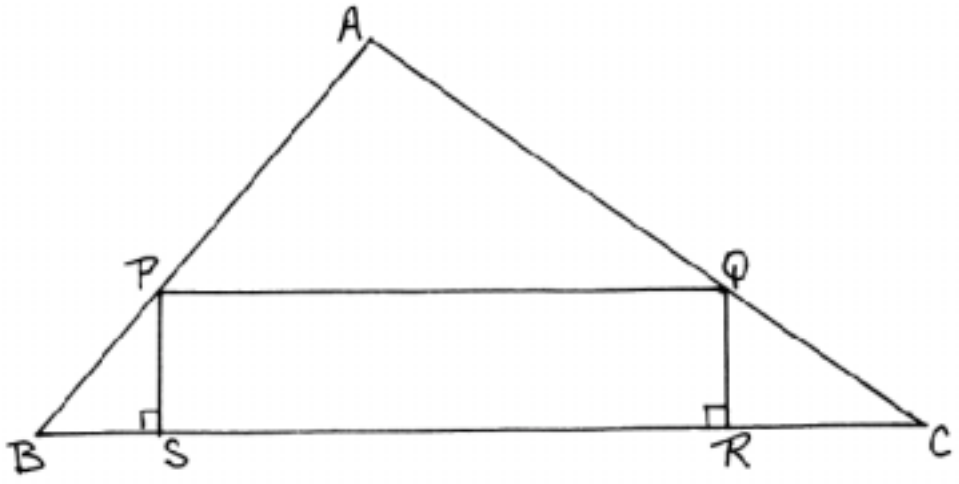
5. **(i)** By reference to an appropriate diagram, or otherwise, show that for $a > b > 0$, $\int_0^b \sqrt{a^2 - x^2} \; dx = \frac{1}{2} b\sqrt{a^2 - b^2} + \frac{1}{2} a^2 \sin^{-1} \frac{b}{a}$.

(ii)



A stone building of height H metres has the shape of a flat-topped square 'pyramid' with curved sides as shown in the figure. The cross-section at height h metres is a square with sides parallel to the sides of the base and of length $l(h) = \frac{L}{\sqrt{h+1}}$, where L is the side length of the square base in metres. Find the volume of the building given that $H = L = 30$. Give your answer correct to the nearest cubic metre.

(ii)



In the figure, PQ is parallel to BC, and $PQRS$ is a rectangle. Prove that the maximum area of $PQRS$ is half the area of the triangle ABC.

6. A point P is moving in a circular path around a centre O. Define the angular velocity of P with respect to O at time t.

Derive expressions for the tangential and normal components of the acceleration of P at time t.

A particle of mass m, attached by a light rod to a pivot point C, moves with constant speed in a horizontal circle whose centre O is distant h metres below C. Show that the time T seconds taken for one revolution of the particle is given by $T = 2\pi\sqrt{h/g}$ where g metres per second per second is the acceleration due to gravity.

Discuss the effect, if any, on the motion of the particle if

(i) Its mass is reduced; **(ii)** Its speed is doubled.

7. (i) Find all x such that $\sin x = \cos 5x$ and $0 < x < \pi$.

(ii) Nine persons gather to play football by forming two teams of four to play each other, the remaining person acting as referee. In how many different ways can the teams be formed?

If two particular persons are not to be in the same team, how many ways are there then to choose the teams?

(iii) Prove that, when x, y, z are real and not all equal,

$$x^2 + y^2 + z^2 > yz + zx + xy,$$

and deduce that, when also $x + y + z = 1$, then $yz + zx + xy < \frac{1}{3}$.

8. (i) (a) Given that ω is a complex root of the equation $x^3 = 1$, show that ω^2 is also a root of this equation.

(b) Show that $1 + \omega + \omega^2 = 0$, and $1 + \omega^2 + \omega^4 = 0$.

(c) Let α, β be real numbers. Find, in its simplest form, the cubic equation whose roots are $\alpha + \beta, \alpha\omega + \beta\omega^{-1}, \alpha\omega^2 + \beta\omega^{-2}$.

(ii) Using induction, show that for each positive integer n there are unique positive integers p_n and q_n such that $(1 + \sqrt{2})^n = p_n + q_n\sqrt{2}$.

Show also that $p_n^2 - 2q_n^2 = (-1)^n$.