NSW HSC 4 Unit Mathematics Examination 1980

Question 1.

(i) Decompose $\frac{4}{1-x^4}$ into partial fractions and hence show that

$$\int_0^{1/2} \frac{4}{1-x^4}\, dx = \log 3 + 2\tan^{-1}\tfrac{1}{2}.$$

(ii) α, β, γ are non-zero and the roots of the cubic equation

$$x^3 + px + q = 0.$$

Find the equation whose roots are $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$.

Question 2.

(i) The function $f(x)$, defined for all real x, is called <u>even</u> if $f(x) = f(-x)$, for all x, and <u>odd</u> if $f(x) = -f(-x)$, for all x.

(a) Determine which of these functions is even, odd, or neither:

$$x;\ \frac{\cos x}{1+x^2};\ xe^x;\ xe^{-x^2}.$$

(b) Show that if $f(x)$ is even and is differentiable for all x, then $f'(x)$ is odd.

(c) If $f(x)$ is odd and is continuous for all x, and $b > a > 0$, use a sketch to explain why $\int_{-a}^{b} f(x)\, dx = \int_a^b f(x)\, dx$.

(ii) The polynomial $P(x)$ is given by $P(x) = x^5 - 5cx + 1$, where c is a real number.

(a) By considering the turning points, prove that if $c < 0, P(x)$ has just one real root, which is negative.

(b) Prove that $P(x)$ has three distinct real roots if and only if

$$c > \left(\frac{1}{4}\right)^{4/5}.$$

Question 3.

Assume that for any integers a, b and c, if $a|c$, $\quad b|c$ and if a and b are relatively prime, then $(ab)|c$.

Given that n is a positive integer:

(i) show that $n(n-1)(n+1)$ is always divisible by 6;

(ii) show that $n(n-1)(n+1)$ is divisible by 30 unless n is of the form $n = 5m \pm 2$ for some integer m;

(iii) deduce that $n^5 - n$ is divisible by 30;

(iv) hence, or otherwise, show that n^5 always has the same units digit as n.

Question 4.

From an origin on a horizontal plane, a particle is projected upwards with speed V and angle of elevation α. Show that the equation of the particle's path is

$$y = x \tan \alpha - \frac{gx^2 \sec^2 \alpha}{2V^2}.$$

where x, y are horizontal, vertical co-ordinates in the plane of the path of the particle and g is the acceleration due to gravity. Hence, or otherwise, deduce that the range on the horizontal plane is $\frac{V^2 \sin 2\alpha}{g}$.

A vertical wall is a distance d from the origin, and the plane of the wall is perpendicular to the plane of the particle's trajectory.

(i) Show that if $d < V^2/g$, the particle will strike the wall provided that

$$\beta < \alpha < \frac{\pi}{2} - \beta,$$

where $\beta = \frac{1}{2} \sin^{-1}(gd/V^2)$.

(ii) Show also that the maximum height the particle can reach on the wall is

$$\frac{V^4 - g^2 d^2}{2gV^2}.$$

Question 5.

(i) The complex number w is given by $w = -1 + \sqrt{3}i$.

(a) Show that $w^2 = 2\overline{w}$.

(b) Evaluate $|w|$ and $\arg w$.

(c) Show that w is a root of the equation $w^3 - 8 = 0$.

(ii) Sketch the region of the Argand diagram whose points z satisfy the inequalities $|z - \overline{z}| \leq 4$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$.

(iii) The complex number z is a function of the real number t given by the rule

$$z = \frac{t - i}{t + i}, \quad 0 \leq t \leq 1.$$

Evaluate $|z|$ and hence describe the locus of z in the Argand diagram as t varies from 0 to 1.

Question 6.

(i) (a) Write down the sum of n terms of the geometric series

$$1 + x + x^2 + x^3 + \cdots$$

and hence show that this series is convergent if and only if $|x| < 1$.

(b) State the comparison test for convergence of a series. Illustrate your answer by constructing a series and proving that it converges or diverges by comparing it with the geometric series.

(ii) (a) Find A, B such that

$$\frac{1}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}.$$

(b) The sum S_n of the first n terms of a series is

$$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}.$$

Show that

$$S_n = 1 - \frac{1}{n+1}.$$

Hence prove that the series converges, and find its sum.

Deduce that the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

converges, and that its sum is less than 2.

Question 7.

(i) Evaluate

(a) $\int_0^{\pi/2} x \cos x \; dx$

(b) $\int_0^1 \frac{x^2}{x+1} \; dx$

(c) $\int_0^{\pi/2} \frac{1-\tan x}{1+\tan x} \; dx$

(ii) Use the substitution $x = a - y$ to show that

$$\int_0^a f(x) \; dx = \int_0^a f(a-x) \; dx.$$

Hence, or otherwise, evaluate

(a) $\int_0^1 x(1-x)^{20} \; dx$

(b) $\int_0^{\pi/2} \cos^2 x \; dx$.

Question 8.

(i) Evaluate $\int_1^x \frac{du}{\sqrt{u}}$ for $x > 1$. Assuming that $f(x) \leq g(x)$ for all x in $a \leq x \leq b$, explain briefly why

$$\int_a^b f(x) \; dx \leq \int_a^b g(x) \; dx,$$

whenever both integrals exist. Hence, given that for $x > 0$

$$\log x = \int_1^x \frac{du}{u},$$

prove that $\frac{\log x}{x} \to$ as $x \to \infty$. Deduce that $x \log x \to 0$ as $x \to 0$.

(ii) Draw a careful sketch of the curve defined by

$$y = \begin{cases} x^2 (\log x)^2 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

paying attention to the slope of the curve and the turning points.

Question 9.

For each of the matrices $\mathbf{M}$ given below describe exactly the effect of the linear transformation of the Cartesian (x, y) plane which maps the point with position vector $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ to the point with position vector $\mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, where

$$\mathbf{r}' = \mathbf{M}\mathbf{r}.$$

In particular, state whether the transformation is singular or non-singular, and in the non-singular cases determine the inverse matrix $\mathbf{M}^{-1}$.

(i) $\mathbf{M} = \mathbf{M}_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) $\mathbf{M} = \mathbf{M}_2 = \begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{-3}{2} \\ \frac{3}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$

(iii) $\mathbf{M} = \mathbf{M}_3 = \begin{bmatrix} \frac{-3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$

(iv) $\mathbf{M} = \mathbf{M}_4 = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

(v) $\mathbf{M} = \mathbf{M}_5 = \mathbf{M}_2\mathbf{M}_3$.

Question 10.

Determine the exact ranges of values (if any) of the non-negative real number r for which the equation

$$x^2 + 2(r-1)xy + (2-r^2)y^2 = 1$$

represents respectively a circle, an ellipse, an hyperbola, and a rectangular hyperbola with respect to Cartesian (x, y) co-ordinates.

Draw a careful sketch of the curve corresponding to the value $r = 2$, showing all important features. Identify those values (if any) of r for whichy the equation represents any other type of locus and sketch the locus for each such value.